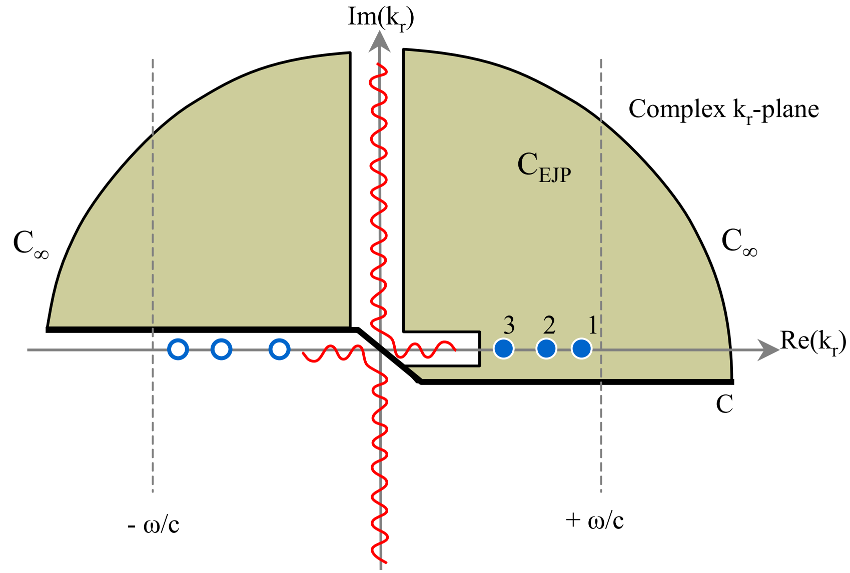
Computational Ocean Acoustics

- Ray Tracing
- Wavenumber Integration
- Normal Modes
- Parabolic Equation

Normal Modes

- Mathematical Derivation
  - Point and Line Sources in Waveguide (5.2)
    - Modal Expansion of Depth-Dependent Green's Function (5.3)
    - Ideal Waveguide (5.4)
  - Generalized Derivation (5.5)
    - Pekeris Waveguide
    - Virtual Modes
  - Deep Water Problem – The Munk Profile (5.6)
- Numerical Approaches
  - Finite Difference Methods (5.7.1)
  - Layer Methods (5.7.2)
  - Shooting Methods (5.7.3)
  - Root Finders (5.7.4)



Location of eigenvalues for the Pekeris problem using the EJP branch cut.

Contour Integral

$$\int_{-\infty}^{\infty} + \int_{C_\infty} + \int_{C_{EJP}} = 2\pi i \sum_{m=1}^{M} \mathrm{res}(k_{rm}) \ ,$$

$\mathrm{res}(k_{rm})$: residue of the mth pole enclosed by the contour.

$$p(r, z) = \frac{i}{2} \sum_{m=1}^{M} \frac{p_1(z_<; k_{rm})\, p_2(z_>; k_{rm})}{\partial W(z_s; k_r)/\partial k_r|_{k_r=k_{rm}}} H_0^{(1)}(k_{rm} r)\, k_{rm} - \int_{C_{EJP}} ,$$

where k_{rm} is the mth zero of the Wronskian, ordered such that $\mathrm{Re}\{k_{r1}\} > \mathrm{Re}\{k_{r2}\} > \cdots$.

Branch Cut Selection

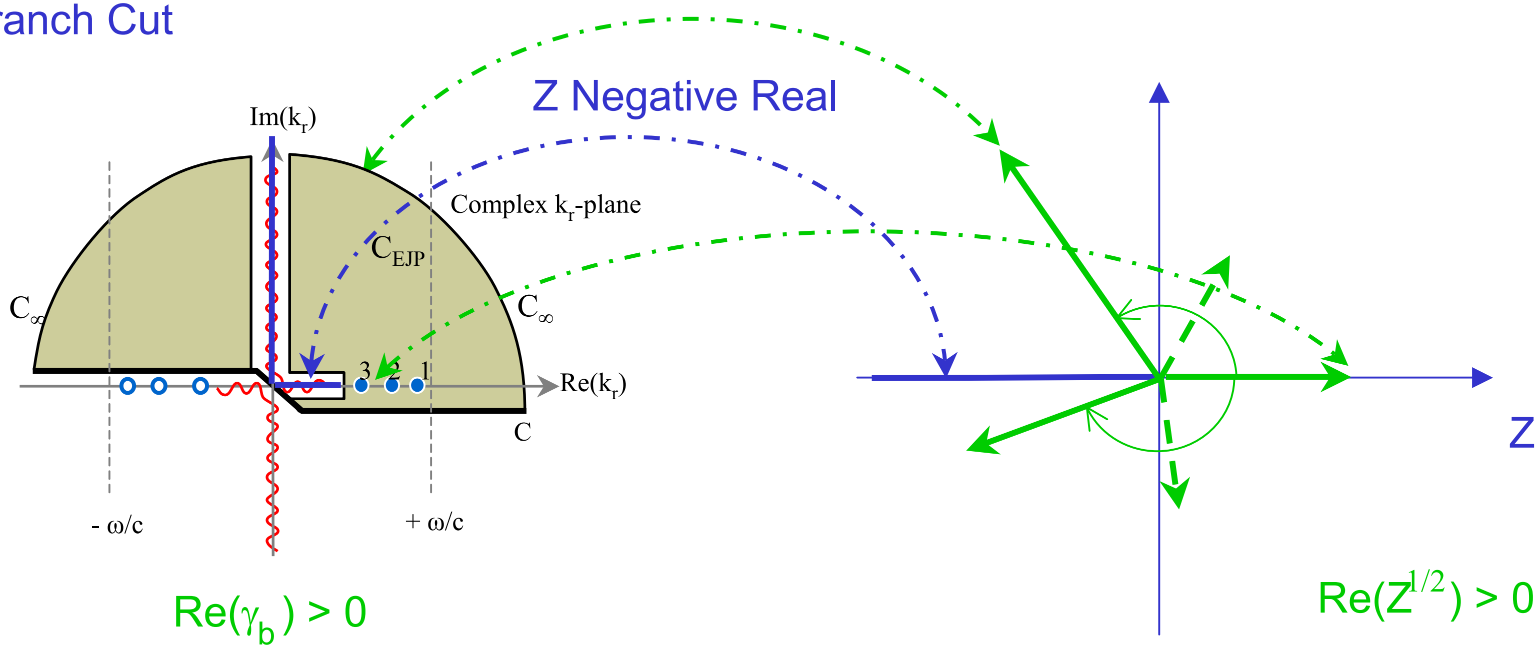
$$\gamma_b \equiv -ik_{z,b} = \sqrt{k_r^2 - \left(\frac{\omega}{c_b}\right)^2} \,,$$

Complex Square Root

$$Z = R \exp(i\theta) = R \exp(i\theta + n2\pi)$$

$$Z^{1/2} = R^{1/2} \exp(i\theta/2 + n\pi)$$

EJP Branch Cut



EJP Brach Cut: Bottom field decaying for all k_r => Physical Riemann Sheet

Branch Cut Selection

$$\gamma_b \equiv -ik_{z,b} = \sqrt{k_r^2 - \left(\frac{\omega}{c_b}\right)^2} ,$$

Complex Square Root

$$Z = R \exp(i\theta) = R \exp(i\theta+n2\pi)$$

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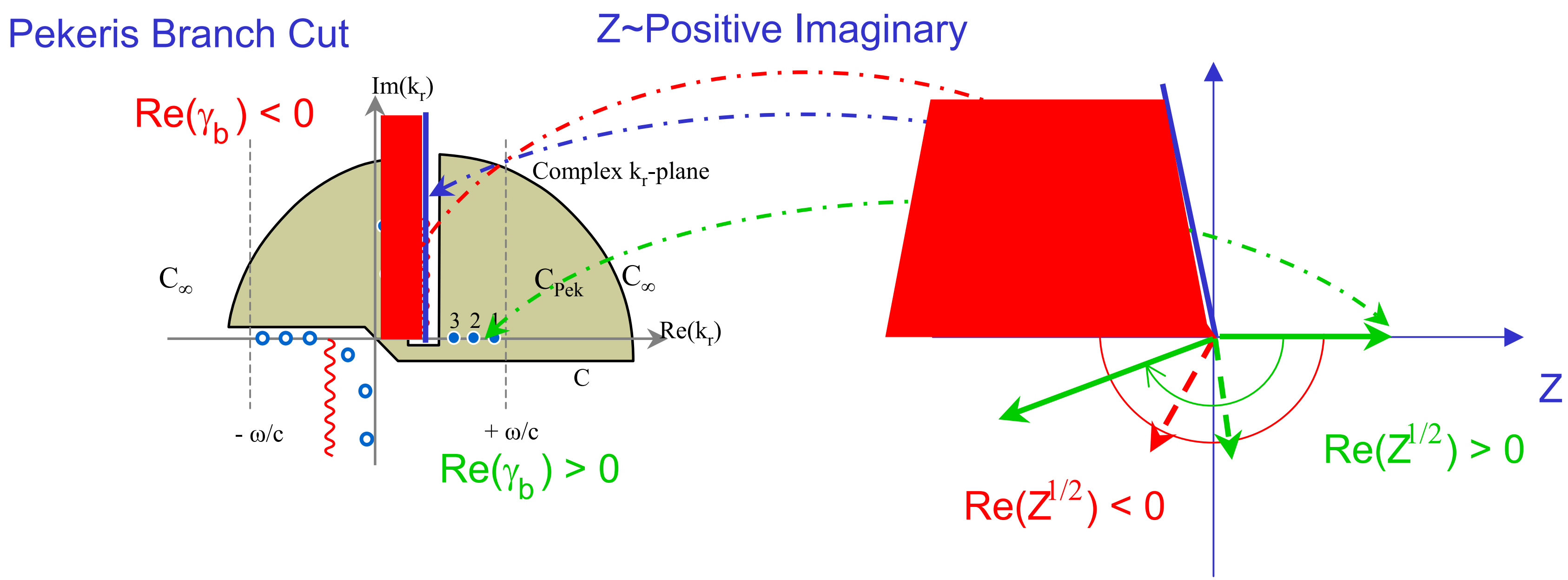
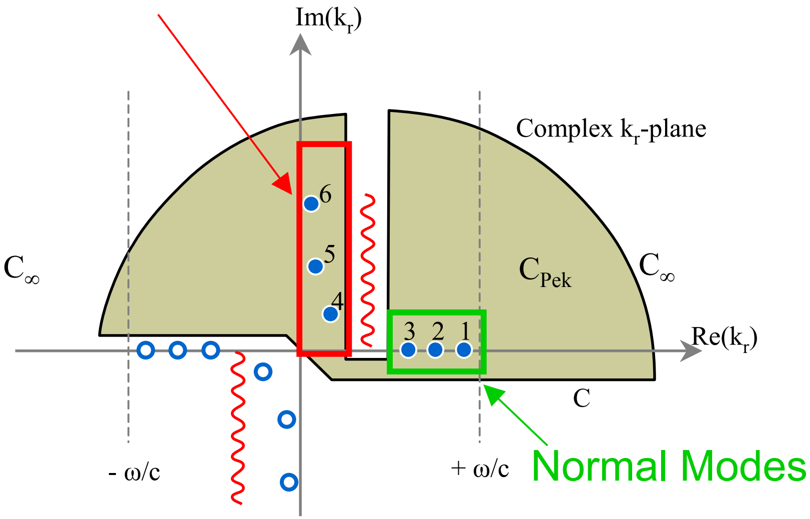


Pekeris Branch Cut: Uncovers Virtual Modes on un-physical Riemann Sheet

Pekeris Branch Cut



Location of eigenvalues for the Pekeris problem using the Pekeris branch cut.

[See Jensen, Fig 5.8.
Modes 1 and 4 are normal modes;
Modes 10 and 12 are virtual modes]

Pekeris waveguide Problem

$$\Psi(z) = A\sin(k_z z)\ ,$$

$$k_z = \sqrt{\left(\frac{\omega}{c}\right)^2 - k_r^2}\ .$$

Characteristic Equation

$$\tan(k_z D) = -\frac{i\rho_b\, k_z}{\rho\, k_{z,b}}\ ,$$

Modal Field Contribution

$$p = \left(e^{ik_{zm}z} + e^{-ik_{zm}z}\right) e^{ik_{rm}r}\ .$$

A Deep Water Problem: WKB Approximation

The Munk profile

$$c(z) = 1500.0\left[1.0 + \epsilon\left(\tilde{z} - 1 + e^{-\tilde{z}}\right)\right] .$$

$$\epsilon = 0.00737\,,$$

$$\tilde{z} = \frac{2\,(z - 1300)}{1300} .$$

[See Fig 5.9 and 5.10 in Jensen, Kuperman, Porter and Schmidt. *Computational Ocean Acoustics.* New York: Springer-Verlag, 2000.]

Ray-Mode Analogy

WKB approximation

$$\Psi(z) \simeq A \frac{e^{i \int_0^z k_z(z)\, dz}}{\sqrt{k_z(z)}} + B \frac{e^{-i \int_0^z k_z(z)\, dz}}{\sqrt{k_z(z)}} ,$$

where

$$k_z^2(z) = \frac{\omega^2}{c^2(z)} - k_r^2 .$$

Turning points

$$k_z^2(z) = 0$$

[See Jensen, Fig. 5.11]

Deep Ocean Waveguide

Modal Cycle Distance

$$L_m = \frac{-2\pi}{dk_{rm}/dm},$$

Finite Difference Form

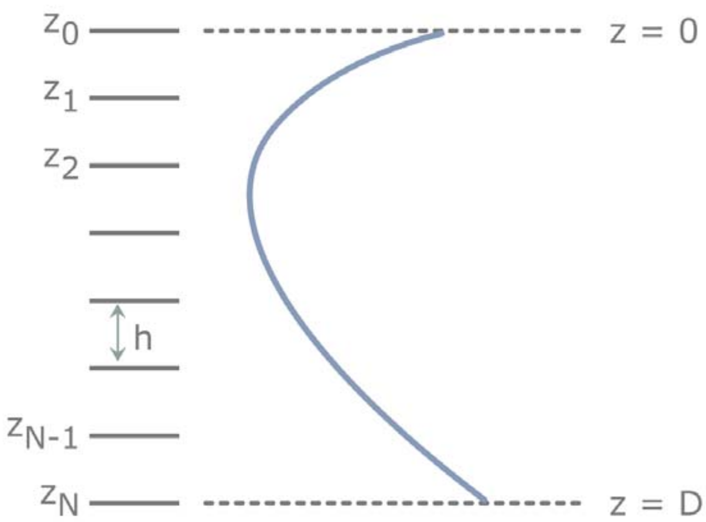
$$L_m \simeq \frac{2\pi}{k_{rm} - k_{r(m+1)}}.$$

[See Jensen, Fig. 5.12]

Mode 30

$$L_{30} = 57.4\text{km}$$

Numerical Approaches



Finite Difference Formulation

Depth-separated Helmholtz Equation - Source

$$(\mathbf{C} - k_r^2 \mathbf{I})\, \mathbf{x} = \mathbf{b}\ ,$$

Modal Eigenvalue Problem

$$(\mathbf{C} - k_r^2 \mathbf{I})\, \mathbf{x} = \mathbf{0}\ .$$

Algebraic Eigenvalue Problem

$$\det A(k_r^2) = 0\ .$$

Constant Density

$$\Psi''(z) + \left[\frac{\omega^2}{c^2(z)} - k_r^2\right]\Psi(z) = 0\,,$$

Taylor series expansion

$$\Psi_{j+1} = \Psi_j + \Psi'_j\,h + \Psi''_j\,\frac{h^2}{2!} + \Psi'''_j\,\frac{h^3}{3!} + \cdots$$

Forward difference approximation

$$\Psi'_j = \frac{\Psi_{j+1} - \Psi_j}{h} - \Psi''_j\,\frac{h}{2} + \cdots \simeq \frac{\Psi_{j+1} - \Psi_j}{h} + O(h)\,.$$

From governing equation

$$\Psi''(z) = -\left[\frac{\omega^2}{c^2(z)} - k_r^2\right]\Psi(z)\,.$$

$$\Psi'_j \simeq \frac{\Psi_{j+1} - \Psi_j}{h} + \left[\frac{\omega^2}{c^2(z_j)} - k_r^2\right]\Psi_j\,\frac{h}{2} + O(h^2)\,.$$

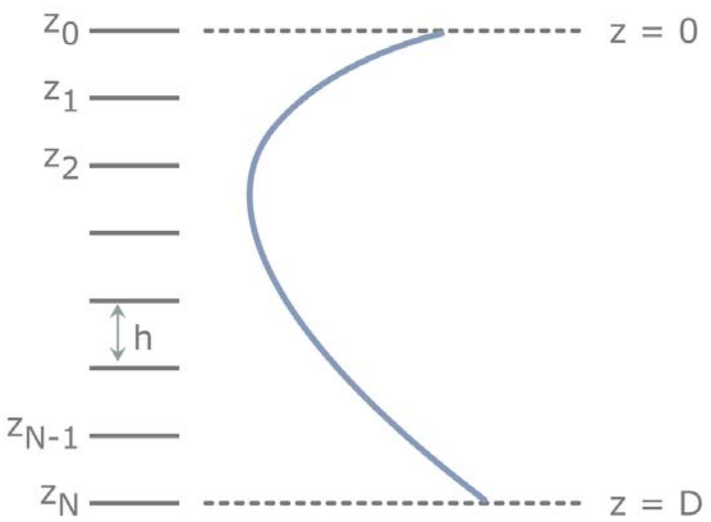
Backward Difference Approximation

$$\Psi_{j-1} = \Psi_j - \Psi'_j\,h + \Psi''_j\,\frac{h^2}{2!} - \Psi'''_j\,\frac{h^3}{3!} + \cdots \simeq \frac{\Psi_j - \Psi_{j-1}}{h} + O(h)\,,$$

$$\Psi'_j \simeq \frac{\Psi_j - \Psi_{j-1}}{h} - \left[\frac{\omega^2}{c^2(z_j)} - k_r^2\right]\Psi_j\,\frac{h}{2} + O(h^2)\,.$$

Centered Difference Approximation

$$\Psi''_j = \frac{\Psi_{j-1} - 2\Psi_j + \Psi_{j+1}}{h^2} + O(h^2)\,.$$



Continuous Modal Equations

$$\begin{aligned}
\Psi''(z) + \left[\frac{\omega^2}{c^2(z)} - k_r^2\right] \Psi(z) &= 0\,, \\
f^T(k_r^2)\,\Psi(0) + \frac{g^T(k_r^2)}{\rho}\,\frac{d\Psi(0)}{dz} &= 0\,, \\
f^B(k_r^2)\,\Psi(D) + \frac{g^B(k_r^2)}{\rho}\,\frac{d\Psi(D)}{dz} &= 0\,.
\end{aligned}$$

Discrete Modal Equations

$$\Psi_{j-1} + \left\{-2 + h^2\left[\frac{\omega^2}{c^2(z_j)} - k_r^2\right]\right\}\Psi_j + \Psi_{j+1} = 0\,, \qquad j = 1, \ldots N-1\,,$$

$$\frac{f^T}{g^T}\,\Psi_0 + \frac{1}{\rho}\left\{\frac{\Psi_1 - \Psi_0}{h} + \left[\frac{\omega^2}{c^2(0)} - k_r^2\right]\Psi_0\,\frac{h}{2}\right\} = 0\,,$$

$$\frac{f^B}{g^B}\,\Psi_N + \frac{1}{\rho}\left\{\frac{\Psi_N - \Psi_{N-1}}{h} - \left[\frac{\omega^2}{c^2(D)} - k_r^2\right]\Psi_N\,\frac{h}{2}\right\} = 0\,.$$

$$\frac{1}{h\rho}\,\Psi_{j-1} + \frac{-2 + h^2\left[\omega^2/c^2(z_j) - k_r^2\right]}{h\rho}\,\Psi_j + \frac{1}{h\rho}\,\Psi_{j+1} = 0\,.$$

Algebraic Eigenvalue Problem

$$\mathbf{A}(k_r^2)\,\mathbf{\Psi} = 0\,.$$

Field Vector

$$\mathbf{\Psi} = \Psi_0, \Psi_1, \ldots, \Psi_N$$

Tri-diagonal Coefficient Matrix

$$\mathbf{A} = \begin{bmatrix} d_0 & e_1 & & & & & \\ e_1 & d_1 & e_2 & & & & \\ & e_2 & d_2 & e_3 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & e_{N-2} & d_{N-2} & e_{N-1} & \\ & & & & e_{N-1} & d_{N-1} & e_N \\ & & & & & e_N & d_N \end{bmatrix},$$

$$\begin{aligned} d_0 &= \frac{-2 + h^2\left[\omega^2/c^2(z_0) - k_r^2\right]}{2\,h\rho} + \frac{f^T(k_r^2)}{g^T(k_r^2)}\,, \\ d_j &= \frac{-2 + h^2\left[\omega^2/c^2(z_j) - k_r^2\right]}{h\rho}\,, \qquad j = 1, \ldots N-1\,, \\ d_N &= \frac{-2 + h^2\left[\omega^2/c^2(z_N) - k_r^2\right]}{2\,h\rho} - \frac{f^B(k_r^2)}{g^B(k_r^2)}\,, \end{aligned}$$

$$e_j = \frac{1}{h\rho}, \qquad j = 1, \ldots N\,.$$

Free Surface

$$f/g \to \infty \Rightarrow \Psi_0 = 0$$

Solving the Modal Eigenvalue Problem

1. QR algorithm - designed for subsets of modes.
2. Sturm's method
   - Bi-section, Sturm sequences
   - Newton's Method, Sturm sequences
   - Inverse Iteration
   - Richardson Extrapolation

Sturm Sequences

Real Eigenvalue Problem

$$\mathbf{A}(k_r^2)\mathbf{\Psi} = (\mathbf{B} - \lambda \mathbf{I})\mathbf{\Psi} = \mathbf{0}$$

$$\lambda = \frac{h k_r^2}{\rho}$$

Tri-diagonal Matrix

$$\mathbf{B} = \begin{bmatrix} \alpha_1 & \beta_1 & & & & \\ \gamma_1 & \alpha_2 & \beta_2 & & & \\ & \gamma_2 & \alpha_3 & \beta_3 & & \\ & & \ddots & \ddots & \ddots & \\ & & & \gamma_{N-2} & \alpha_{N-1} & \beta_{N-1} \\ & & & & \gamma_{N-1} & \alpha_N \end{bmatrix},$$

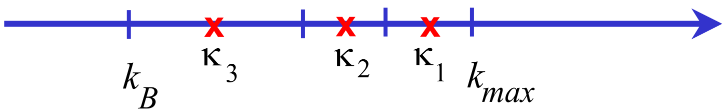
$$\alpha_j = \frac{-2 + (h\omega)^2 / c^2(z_j)}{h\rho}$$

$$\beta_j = \gamma_j = \frac{1}{h\rho}$$

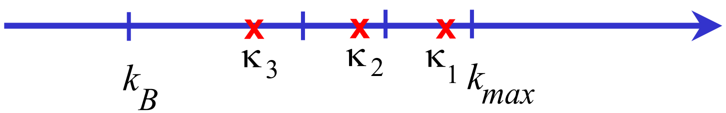
$$\gamma_{N-1} = \frac{2}{h\rho}$$

$$\alpha_N = \frac{-2 + (h\omega)^2 / c^2(z_j)}{h\rho} - 2f^B/g^B$$

Bi-section



Newton Iteration



Sturm Sequence

$$p_{-1}(\lambda) = 0$$

$$p_0(\lambda) = 1$$

$$p_k(\lambda) = (\lambda - \alpha_k) p_{k-1}(\lambda) - \gamma_{k-1}\beta_{k-1} p_{k-2}(\lambda), \; k = 1, 2, \ldots N$$

- Number of *eigevalues* $> \lambda$ equals number of *zero-crossings* in Sturm sequence $p_k(\lambda), k = 1, 2, .ldotsN$.
- If $p_N(\lambda) = 0$ then λ is an *eigenvalue* .

Inverse Iteration

$$(\mathbf{B}-\lambda_m\mathbf{I})\mathbf{v}_m = 0$$

Eigenvalue estimate

$$\kappa_m = \lambda_m - \epsilon \;\text{ where }\; 0 < |\lambda_m - \kappa_m| < min|\lambda_i - \kappa_m|,\; i \neq m$$

Starting Vector

$$\mathbf{w_0} = \mathbf{c_1 v_1} + \cdots \mathbf{c_n v_n},\;\; \mathbf{c_i} \neq \mathbf{0}$$

Recurrence

$$(\mathbf{B}-\kappa_m\mathbf{I})\mathbf{w_k} = \mathbf{w_{k-1}},\;\; \mathbf{k} = \mathbf{1},\mathbf{2}\cdots$$

$$\mathbf{w_1} = \sum_{\mathbf{i=1}}^{\mathbf{n}} \mathbf{c_i}(\mathbf{B}-\kappa_\mathbf{m}\mathbf{I})^{-1}\mathbf{v_i}$$

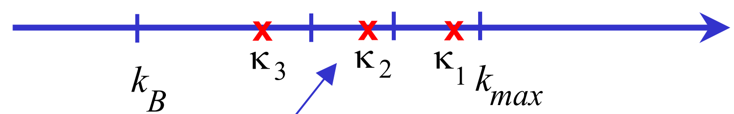
$$\text{Eigenvalues}\mu_i = \frac{1}{\lambda_i - \kappa_m} \Rightarrow$$

$$\mathbf{w_1} = \frac{c_1}{\lambda_1-\kappa_m}\mathbf{v}_1 + \cdots + \frac{c_i}{\lambda_i-\kappa_m}\mathbf{v}_i + \cdots + \frac{c_n}{\lambda_n-\kappa_m}\mathbf{v}_n$$

$$\mathbf{w_k} = \frac{1}{(\lambda_m-\kappa_m)^k}\left\{c_m\mathbf{v}_m + \sum_{i\neq m} c_i \left(\frac{\lambda_m-\kappa_m}{\lambda_i-\kappa_m}\right)^k \mathbf{v}_i\right\}$$

Asymptotics

$$\mathbf{w_k} \rightarrow \alpha\mathbf{v_m},\;\; \frac{\mathbf{w_{k;j}}}{\mathbf{w_{k-1;j}}} \rightarrow \frac{\mathbf{1}}{\lambda_\mathbf{m} - \kappa_\mathbf{m}}$$



Improved Discrete Problem Eigenvalues

Continuous Problem Eigenvalues

Richardson extrapolation.

$$k_r^2(h) = k_0^2 + b_2h^2 + b_4h^4 + \cdots,$$

Other Methods

Layer Method

- Analytical Solution in each layer
- Direct Global Matrix as for Wavenumber Integration
- Search for zeros of determinant.
- Modal amplitude through Wronskian

Numerov's method

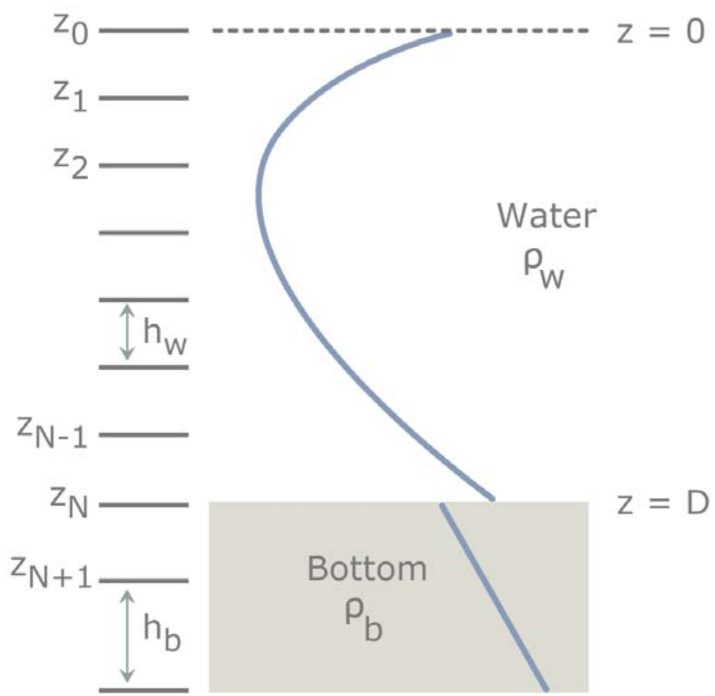
$$\Psi''(z) + \left[\frac{\omega^2}{c^2(z)} - k_r^2\right] \Psi(z) = 0\,,$$

$$\left(\frac{1}{h^2} + \frac{1}{12}\,k_{z,j-1}^2\right)\Psi_{j-1} + \left(-\frac{2}{h^2} + \frac{10}{12}\,k_{z,j}^2\right)\Psi_j + \left(\frac{1}{h^2} + \frac{1}{12}\,k_{z,j+1}^2\right)\Psi_{j+1} = 0,$$

$$k_{z,j}^2 = \frac{\omega^2}{c^2(z_j)} - k_r^2\,.$$

- Standard scheme: $O(h^2)$
- Numorov's method: $O(h^4)$. Twice CPU time

Treatment of Interfaces



Water

$$\Psi_{j-1}+\left\{-2+h_w^2\left[\frac{\omega^2}{c^2(z_j)}-k_r^2\right]\right\}\Psi_j+\Psi_{j+1}=0\,, \qquad j=1,\dots N-1\,,$$

Bottom

$$\Psi_{j-1}+\left\{-2+h_b^2\left[\frac{\omega^2}{c^2(z_j)}-k_r^2\right]\right\}\Psi_j+\Psi_{j+1}=0\,, \qquad j=N+1,\dots$$

Continuity of Pressure

$$\Psi_N=\Psi(D^-)=\Psi(D^+)$$

Continuity of Particle Velocity

$$\frac{d\Psi(D)/dz}{\rho_w}=\frac{d\Psi(D)/dz}{\rho_b}\,,$$

$$\left\{\frac{\Psi_N-\Psi_{N-1}}{h_w}-\left[\frac{\omega^2}{c^2(D^-)}-k_r^2\right]\Psi_N\,\frac{h_w}{2}\right\}\Big/\rho_w$$
$$=\left\{\frac{\Psi_{N+1}-\Psi_N}{h_b}+\left[\frac{\omega^2}{c^2(D^+)}-k_r^2\right]\Psi_N\,\frac{h_b}{2}\right\}\Big/\rho_b\,,$$

$$\frac{\Psi_{N-1}}{h_w\rho_w}+\frac{-\Psi_N+\left[\omega^2/c^2(D^-)-k_r^2\right]\Psi_N\,h_w^2/2}{h_w\rho_w}$$
$$+\frac{-\Psi_N+\left[\omega^2/c^2(D^+)-k_r^2\right]\Psi_N\,h_b^2/2}{h_b\rho_b}+\frac{\Psi_{N+1}}{h_b\rho_b}=0\,.$$

Mode Normalization

$$N_m = \int_0^D \frac{\Psi_m^2(z)}{\rho(z)}\,dz - \frac{1}{2k_{rm}} \left.\frac{d(f/g)^T}{dk_r}\right|_{k_{rm}} \Psi_m^2(0) + \frac{1}{2k_{rm}} \left.\frac{d(f/g)^B}{dk_r}\right|_{k_{rm}} \Psi_m^2(D).$$

$$\int_0^D \frac{\Psi_m^2(z)}{\rho(z)}\,dz \simeq \frac{D}{N}\left(\frac{1}{2}\phi_0 + \phi_1 + \phi_2 + \cdots + \phi_{N-1} + \frac{1}{2}\phi_N\right) ,$$

$$\phi_j = \frac{\Psi_j^2}{\rho(z_j)} .$$