Computational Ocean Acoustics

- Ray Tracing
- Wavenumber Integration
- Normal Modes
- Parabolic Equation

Normal Modes

- Perturbation Approaches
  - Attenuation (5.8)
  - Group Velocity (5.8.1)
- Modes for Range-Dependent Envir.
  - Coupled Modes (5.9)
  - One-way Coupled Modes
  - Adiabatic Modes

Modal Group Velocity

$$u_n(\omega) = \frac{d\omega}{dk_{rn}} \,.$$

$$u_n \simeq \frac{(\omega + \Delta\omega) - \omega}{k_{rn}(\omega + \Delta\omega) - k_{rn}(\omega)} \,.$$

Perturbation Formulation

$$K^2(z) = \frac{(\omega + \Delta\omega)^2}{c^2(z)} \simeq \frac{\omega^2}{c^2(z)} + \frac{2\,\Delta\omega\,\omega}{c^2(z)} \,.$$

$$\begin{aligned} K^2 &= K_0^2 + \epsilon K_1^2 \\ K_0^2 &= \omega^2/c^2, \\ K_1^2 &= 2\omega/c^2 \\ \epsilon &= \Delta\omega \end{aligned}$$

$$k_{r1}^2 = \int_0^D \frac{2\omega}{c^2(z)} \frac{\Psi_0^2(z)}{\rho(z)} \, dz \,.$$

Finite Difference Perturbation

$$k_r^2(\omega + \Delta\omega) \simeq k_{r0}^2(\omega) + \Delta\omega\, k_{r1}^2$$

$$\frac{k_r^2(\omega + \Delta\omega) - k_{r0}^2(\omega)}{\Delta\omega} \simeq k_{r1}^2 \,.$$

$$\frac{k_r^2(\omega + \Delta\omega) - k_{r0}^2(\omega)}{\Delta\omega} \rightarrow_{\Delta\omega\rightarrow 0} \frac{dk_r^2}{d\omega} \,.$$

$$\frac{d(k_r^2)}{d\omega} = 2k_r \frac{dk_r}{d\omega} = k_{r1}^2 \,.$$

Modal Group Slowness

$$\frac{dk_r}{d\omega} = \frac{k_{r1}^2}{2k_r} = \frac{\omega}{k_r} \int_0^D \frac{\Psi_0^2(z)}{\rho(z)\, c^2(z)} \, dz \,.$$

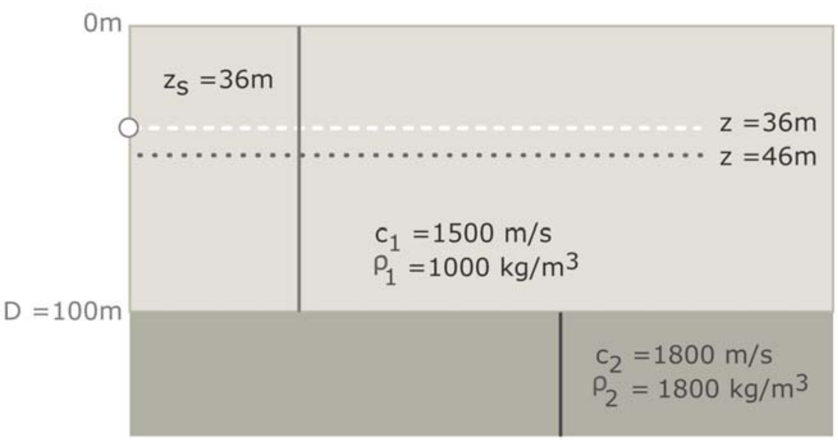
Modal Group Speed - Penetrable Bottom

$$u_n = \frac{d\omega}{dk_{rn}} = \boxed{\frac{k_{rn}}{\omega}} \left[\int_0^\infty \frac{\Psi_0^2(z)}{\rho(z)\, c^2(z)}\, dz \right]^{-1}$$

$\frac{1}{v_n}$

Modal
Phase Velocity

Pekeris Waveguide



[See Fig 2.28b in Jensen, Kuperman, Porter and Schmidt. *Computational Ocean Acoustics.* New York: Springer-Verlag, 2000.]

Modal Group Speed - Penetrable Bottom

$$u_n = \frac{d\omega}{dk_{rn}} = \frac{k_{rn}}{\omega}\left[\int_0^\infty \frac{\Psi_0^2(z)}{\rho(z)\, c^2(z)}\, dz\right]^{-1}$$

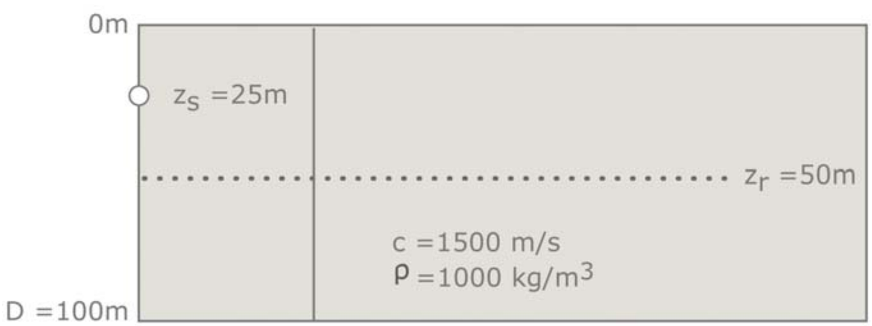
$\frac{1}{v_n}$

Isovelocity, Ideal Waveguide

Modal Phase Velocity

$$u_n = \frac{k_{rn} c^2}{\omega} = \frac{c^2}{v_n}$$

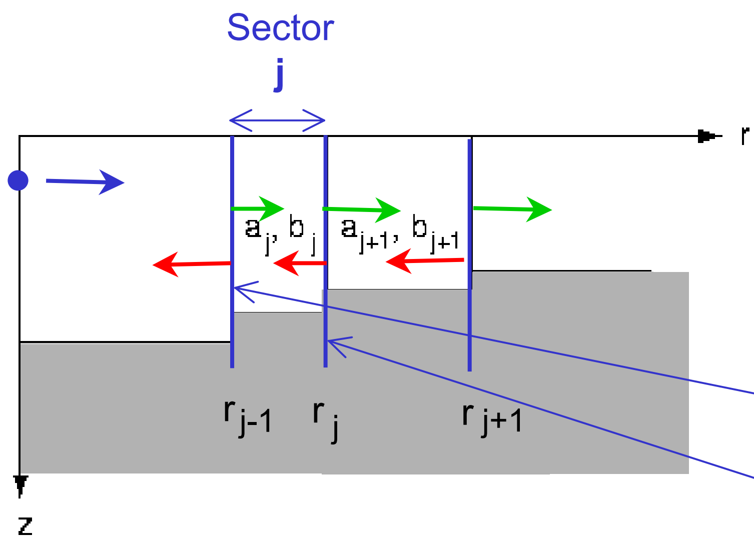
Ideal Waveguide



[See Jensen, Fig 2.22]

Normal Modes for Range-Dependent Environments

Coupled Modes



$$p^j(r,z) = \sum_{m=1}^{M} \left[a_m^j \, \widehat{H}1_m^j(r) + b_m^j \, \widehat{H}2_m^j(r) \right] \Psi_m^j(z) \, ,$$

Normalized Hankel Functions

$$\widehat{H}1_m^j(r) = \frac{H_0^{(1)}(k_{rm}^j r)}{H_0^{(1)}(k_{rm}^j r_{j-1})} \, , \qquad \text{Forward}$$

$$\widehat{H}2_m^j(r) = \frac{H_0^{(2)}(k_{rm}^j r)}{H_0^{(2)}(k_{rm}^j r_j)} \, , \qquad \text{Backward}$$

Asymptotic

Forward

$$\widehat{H}1_m^j(r) \simeq H1_m^j(r) = \sqrt{\frac{r_{j-1}}{r}} \, e^{ik_{rm}^j (r - r_{j-1})} \, ,$$

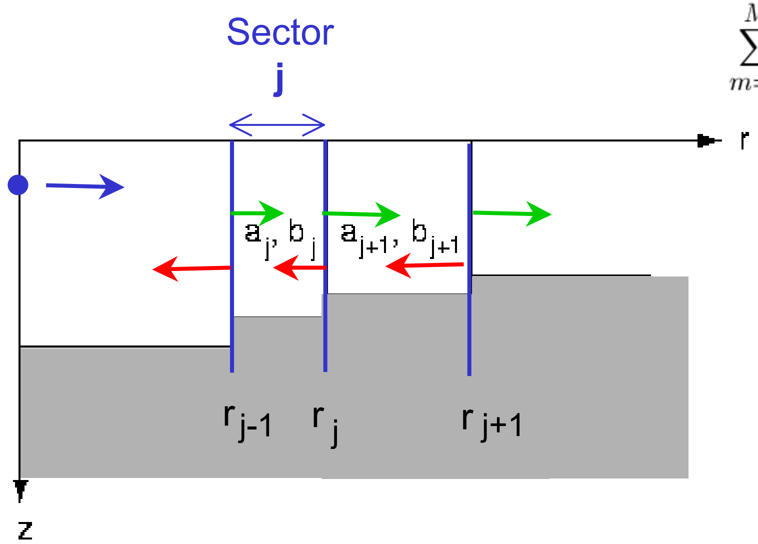
Backward

$$\widehat{H}2_m^j(r) \simeq H2_m^j(r) = \sqrt{\frac{r_j}{r}} \, e^{ik_{rm}^j (r_j - r)} \, .$$

Continuity of Pressure

jth interface

$$\sum_{m=1}^{M} \left(a_m^{j+1} + b_m^{j+1}\, H2_m^{j+1}(r_j)\right) \Psi_m^{j+1}(z) = \sum_{m=1}^{M} \left[a_m^j\, H1_m^j(r_j) + b_m^j\right] \Psi_m^j(z).$$



Coupling Operator

$$\int (\cdot)\, \frac{\Psi_l^{j+1}(z)}{\rho_{j+1}(z)}\, dz\ ,$$

Orthogonality

$$\int \frac{\Psi_m^{j+1}(z)\, \Psi_l^{j+1}(z)}{\rho_{j+1}(z)}\, dz = \delta_{lm}\ ,$$

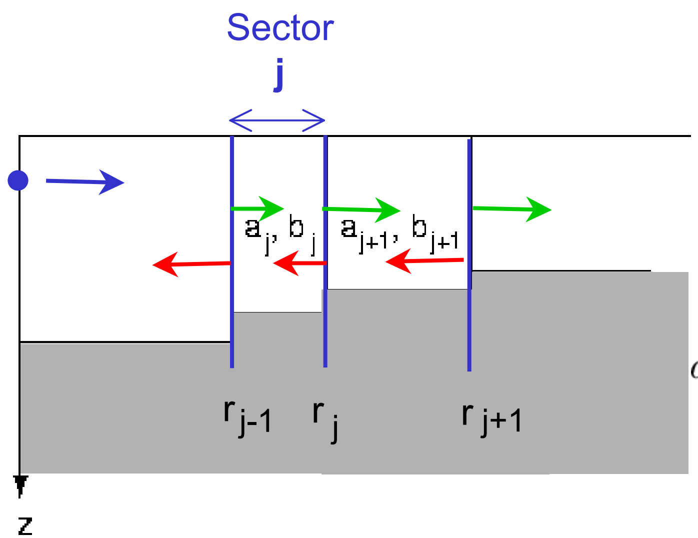
$$a_l^{j+1} + b_l^{j+1}\, H2_l^{j+1}(r_j) = \sum_{m=1}^{M} \left[a_m^j\, H1_m^j(r_j) + b_m^j\right] \tilde{c}_{lm}\ , \qquad l = 1, \ldots M\ ,$$

$$\tilde{c}_{lm} = \int \frac{\Psi_l^{j+1}(z)\, \Psi_m^j(z)}{\rho_{j+1}(z)}\, dz\ .$$

Matrix Notation

$$\mathbf{a}^{j+1} + \mathbf{H}_2^{j+1}\, \mathbf{b}^{j+1} = \widetilde{\mathbf{C}}^j \left(\mathbf{H}_1^j\, \mathbf{a}^j + \mathbf{b}^j\right)\ ,$$

Continuity of Radial Particle Velocity



$$\frac{1}{\rho_j}\frac{\partial p^j(r,z)}{\partial r} \simeq \frac{1}{\rho_j}\sum_{m=1}^{M} k_{rm}^j \left[a_m^j\, H1_m^j(r) - b_m^j\, H2_m^j(r)\right] \Psi_m^j(z)\ .$$

Coupling Operator

$$\int (\cdot)\, \Psi_l^{j+1}(z)\, dz\ ,$$

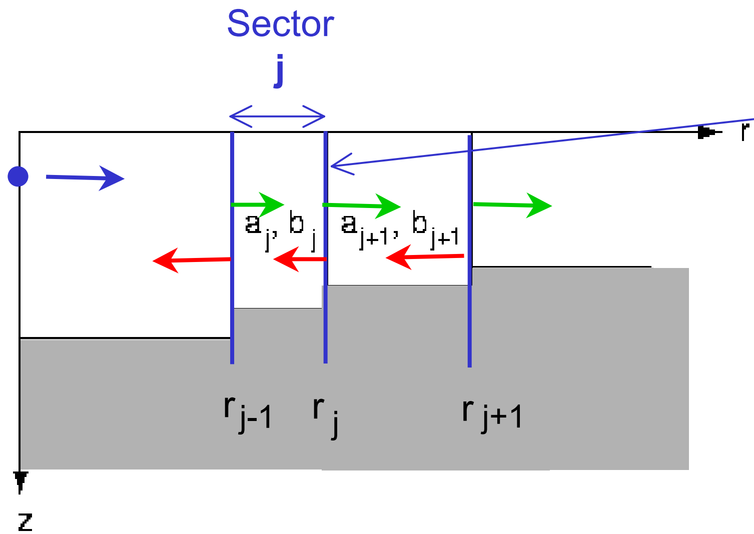
$$a_l^{j+1} - b_l^{j+1}\, H2_l^{j+1}(r_j) = \sum_{m=1}^{M} \left[a_m^j\, H1_m^j(r_j) - b_m^j\right] \hat{c}_{lm}, \qquad l = 1, \ldots M\ ,$$

$$\hat{c}_{lm} = \frac{k_{rm}^j}{k_{rl}^{j+1}} \int \frac{\Psi_l^{j+1}(z)\, \Psi_m^j(z)}{\rho_j(z)}\, dz\ .$$

Matrix Notation

$$\mathbf{a}^{j+1} - \mathbf{H}_2^{j+1}\, \mathbf{b}^{j+1} = \widehat{\mathbf{C}}^j \left(\mathbf{H}_1^j\, \mathbf{a}^j - \mathbf{b}^j\right)\ .$$

Combined Coupling Equations



$$\begin{bmatrix} \mathbf{a}^{j+1} \\ \mathbf{b}^{j+1} \end{bmatrix} = \begin{bmatrix} \mathbf{R}_1^j & \mathbf{R}_2^j \\ \mathbf{R}_3^j & \mathbf{R}_4^j \end{bmatrix} \begin{bmatrix} \mathbf{a}^j \\ \mathbf{b}^j \end{bmatrix},$$

$$\mathbf{R}_1^j = \frac{1}{2}\left(\widetilde{\mathbf{C}}^j + \widehat{\mathbf{C}}^j\right)\mathbf{H}_1^j,$$

$$\mathbf{R}_2^j = \frac{1}{2}\left(\widetilde{\mathbf{C}}^j - \widehat{\mathbf{C}}^j\right),$$

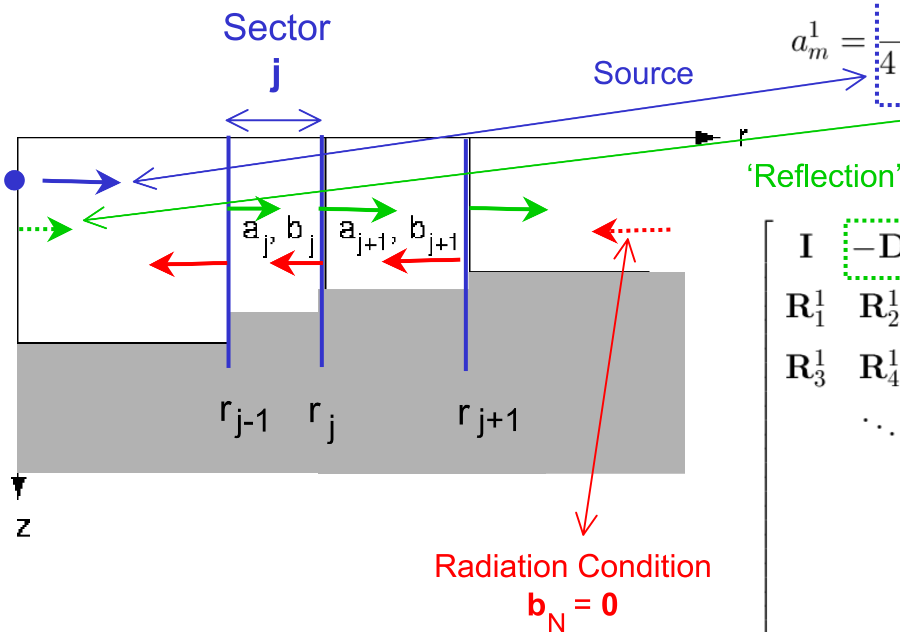
$$\mathbf{R}_3^j = \frac{1}{2}\left(\widetilde{\mathbf{C}}^j - \widehat{\mathbf{C}}^j\right)\left(\mathbf{H}_2^{j+1}\right)^{-1}\mathbf{H}_1^j,$$

$$\mathbf{R}_4^j = \frac{1}{2}\left(\widetilde{\mathbf{C}}^j + \widehat{\mathbf{C}}^j\right)\left(\mathbf{H}_2^{j+1}\right)^{-1}.$$

Initial Condition at Origin

'Rigid' Condition

$$a_m^1 = \frac{i}{4\,\rho(z_s)}\,\Psi_m(z_s)\,H_0^{(1)}(k_{rm}^1 r_1) + b_m^1\,\frac{H_0^{(1)}(k_{rm}^1 r_1)}{H_0^{(2)}(k_{rm}^1 r_1)}, \qquad m = 1,\ldots M\,.$$



Block-Diagonal Matrix Problem

$$\begin{bmatrix}
\mathbf{I} & -\mathbf{D} & 0 & & & & & & \\
\mathbf{R}_1^1 & \mathbf{R}_2^1 & \mathbf{I} & 0 & & & & & \\
\mathbf{R}_3^1 & \mathbf{R}_4^1 & 0 & \mathbf{I} & & & & & \\
 & \ddots & \ddots & \ddots & \ddots & & & & \\
 & & & \mathbf{R}_1^{N-2} & \mathbf{R}_2^{N-2} & \mathbf{I} & 0 & & \\
 & & & \mathbf{R}_3^{N-2} & \mathbf{R}_4^{N-2} & 0 & \mathbf{I} & & \\
 & & & & & \mathbf{R}_1^{N-1} & \mathbf{R}_2^{N-1} & \mathbf{I} \\
 & & & & & \mathbf{R}_3^{N-1} & \mathbf{R}_4^{N-1} & 0
\end{bmatrix}
\begin{bmatrix} \mathbf{a}^1 \\ \mathbf{b}^1 \\ \mathbf{a}^2 \\ \mathbf{b}^2 \\ \vdots \\ \mathbf{a}^{N-1} \\ \mathbf{b}^{N-1} \\ \mathbf{a}^N \end{bmatrix}
=
\begin{bmatrix} \mathbf{s} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix},$$

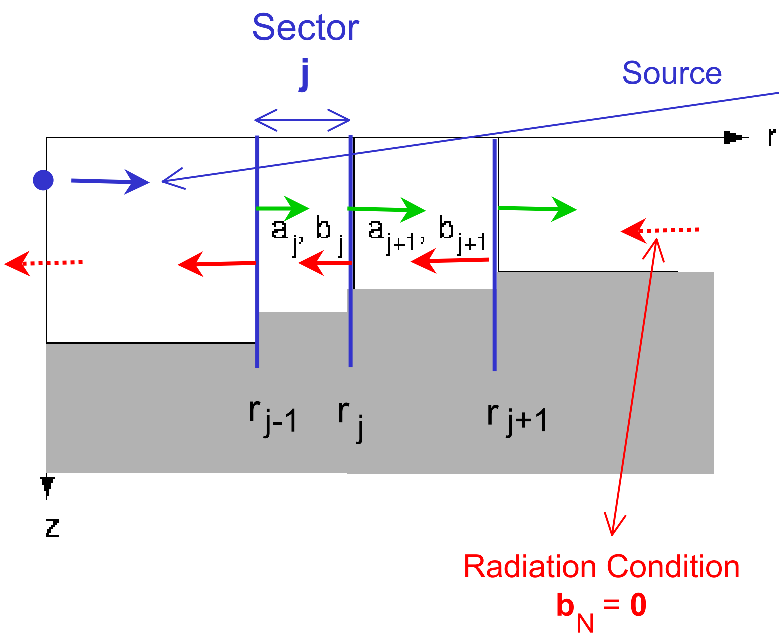
$$d_{ii} = \frac{H_0^{(1)}(k_{ri}^1 r_1)}{H_0^{(2)}(k_{ri}^1 r_1)},$$

$$s_m = \frac{i}{4\,\rho(z_s)}\,\Psi_m(z_s)\,H_0^{(1)}(k_{rm}^1 r_1)\,.$$

Initial Condition at Origin

'Transparent' Condition

$$a_m^1 = \frac{i}{4\,\rho(z_s)}\,\Psi_m(z_s)\,H_0^{(1)}(k_{rm}^1 r_1) \qquad m = 1,\ldots M\,.$$



Block-Diagonal Matrix Problem

Reflection'

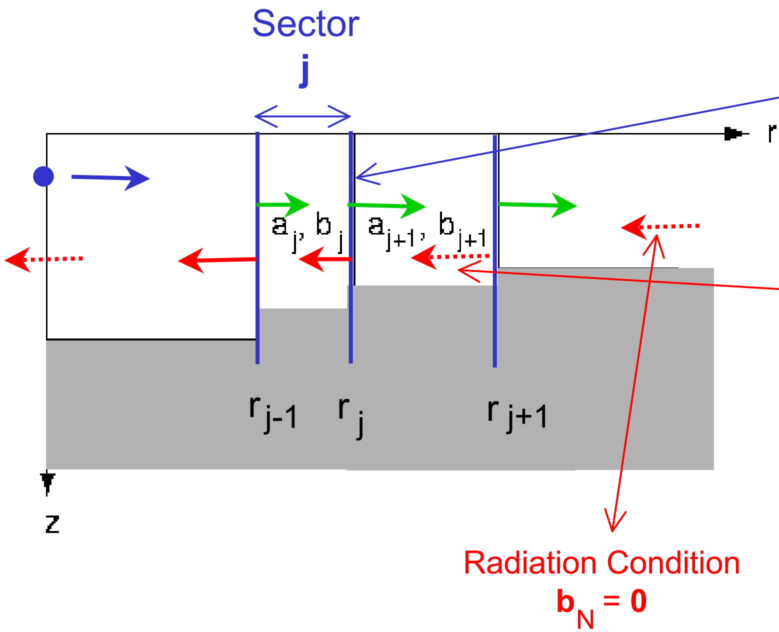
$$\begin{bmatrix} \mathbf{I} & 0 & 0 & & & & \\ \mathbf{R}_1^1 & \mathbf{R}_2^1 & \mathbf{I} & 0 & & & \\ \mathbf{R}_3^1 & \mathbf{R}_4^1 & 0 & \mathbf{I} & & & \\ & \ddots & \ddots & \ddots & \ddots & & \\ & & \mathbf{R}_1^{N-2} & \mathbf{R}_2^{N-2} & \mathbf{I} & 0 & \\ & & \mathbf{R}_3^{N-2} & \mathbf{R}_4^{N-2} & 0 & \mathbf{I} & \\ & & & & \mathbf{R}_1^{N-1} & \mathbf{R}_2^{N-1} & \mathbf{I} \\ & & & & \mathbf{R}_3^{N-1} & \mathbf{R}_4^{N-1} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{a}^1 \\ \mathbf{b}^1 \\ \mathbf{a}^2 \\ \mathbf{b}^2 \\ \vdots \\ \mathbf{a}^{N-1} \\ \mathbf{b}^{N-1} \\ \mathbf{a}^N \end{bmatrix} = \begin{bmatrix} \mathbf{s} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix},$$

$$s_m = \frac{i}{4\,\rho(z_s)}\,\Psi_m(z_s)\,H_0^{(1)}(k_{rm}^1 r_1)\,.$$

One-Way Coupled Modes

Coupling Equations Interface j

$$\begin{bmatrix} \mathbf{a}^{j+1} \\ \mathbf{b}^{j+1} \end{bmatrix} = \begin{bmatrix} \mathbf{R}_1 & \mathbf{R}_2 \\ \mathbf{R}_3 & \mathbf{R}_4 \end{bmatrix} \begin{bmatrix} \mathbf{a}^{j} \\ \mathbf{b}^{j} \end{bmatrix} .$$



Ignore Backscatter from next interface:

$$\mathbf{b}^{j+1} = \mathbf{0}$$

Back-Scattered Amplitudes

$$\mathbf{b}^{j} = -\mathbf{R}_4^{-1} \mathbf{R}_3 \, \mathbf{a}^{j} .$$

Forward-Scattered Amplitudes

$$\mathbf{a}^{j+1} = \left(\mathbf{R}_1 - \mathbf{R}_2 \, \mathbf{R}_4^{-1} \mathbf{R}_3 \right) \mathbf{a}^{j} ,$$

Approximate Single-Scatter Solution

$$\mathbf{a}^{j+1} = \mathbf{R}_1 \, \mathbf{a}^{j} .$$

$$\mathbf{R}_1^j = \frac{1}{2} \left(\widetilde{\mathbf{C}}^j + \widehat{\mathbf{C}}^j \right) \mathbf{H}_1^j ,$$

Mean of pressure and velocity coupling

Other: $p/(\rho c)^{1/2}$ continuous

Continuous Mode Coupling

Helmholtz Equation

$$\frac{\rho}{r}\frac{\partial}{\partial r}\left(\frac{r}{\rho}\frac{\partial p}{\partial r}\right) + \rho\frac{\partial}{\partial z}\left(\frac{1}{\rho}\frac{\partial p}{\partial z}\right) + \frac{\omega^2}{c^2(r,z)}\,p = -\frac{\delta(r)\,\delta(z-z_s)}{2\pi r}\,.$$

Range Factorization

$$p(r,z) = \sum_m \Phi_m(r)\,\Psi_m(r,z)\,,$$

Local Modes $\Psi_m(r,z)$

$$\rho(r,z)\,\frac{\partial}{\partial z}\left[\frac{1}{\rho(r,z)}\frac{\partial \Psi_m(r,z)}{\partial z}\right] + \left[\frac{\omega^2}{c^2(r,z)} - k_{rm}^2(r)\right]\Psi_m(r,z) = 0\,.$$

Substitution into Helmholtz Equation

$$\sum_m \frac{\rho}{r}\frac{\partial}{\partial r}\left(\frac{r}{\rho}\frac{\partial(\Phi_m\Psi_m)}{\partial r}\right) + \sum_m k_{rm}^2(r)\Phi_m\Psi_m = -\frac{\delta(r)\,\delta(z-z_s)}{2\pi r}\,,$$

Rearranging Terms

$$\sum_m \left[\frac{\rho}{r}\frac{\partial}{\partial r}\left(\frac{r}{\rho}\frac{\partial \Phi_m}{\partial r}\right)\Psi_m + 2\,\frac{\partial \Phi_m}{\partial r}\,\frac{\partial \Psi_m}{\partial r} + \frac{\rho}{r}\frac{\partial}{\partial r}\left(\frac{r}{\rho}\frac{\partial \Psi_m}{\partial r}\right)\Phi_m\right]$$
$$+\sum_m k_{rm}^2(r)\Phi_m\Psi_m = -\frac{\delta(r)\,\delta(z-z_s)}{2\pi r}\,.$$

Orthogonality Operator

$$\rho = \rho(z) : \quad \int (\cdot) \, \frac{\Psi_n(r,z)}{\rho(z)} \, dz \, ,$$

$$\frac{1}{r}\frac{d}{dr}\left(r\,\frac{d\Phi_n}{dr}\right) + \sum_m 2B_{mn}\frac{d\Phi_m}{dr} + \sum_m A_{mn}\Phi_m + k_{rn}^2(r)\,\Phi_n = -\frac{\delta(r)\,\Psi_n(z_s)}{2\pi r},$$

$$A_{mn} = \int \frac{1}{r}\frac{\partial}{\partial r}\left(r\,\frac{\partial \Psi_m}{\partial r}\right)\frac{\Psi_n}{\rho}\,dz \, ,$$

$$B_{mn} = \int \frac{\partial \Psi_m}{\partial r}\,\frac{\Psi_n}{\rho}\,dz \, .$$

ODE for Continuously Coupled Modes

Solved e.g. by FD

$$B_{mn} = -B_{nm}$$

Orthogonality

$$\int \frac{\Psi_m(z)\,\Psi_n(z)}{\rho(z)}\,dz = \delta_{mn} \, ,$$

$$\int \frac{\partial \Psi_m(z)}{\partial r}\,\frac{\Psi_n(z)}{\rho(z)}\,dz + \int \frac{\Psi_m(z)}{\rho(z)}\,\frac{\partial \Psi_n(z)}{\partial r}\,dz = 0 \, .$$

Adiabatic Approximation

Decoupled Equations

$$\frac{1}{r}\frac{d}{dr}\left(r\,\frac{d\Phi_n}{dr}\right) + k_{rn}^2(r)\,\Phi_n = -\frac{\delta(r)\,\Psi_n(z_s)}{2\pi r}\,,$$

WKB Approximation

$$\Phi_n(r) \simeq A\,\frac{e^{i\int_0^r k_{rn}(r')\,dr'}}{\sqrt{k_{rn}(r)}}\,.$$

Range-independent Source Condition

$$A = \frac{i}{\rho(z_s)\sqrt{8\pi r}}\,e^{-i\pi/4}\,\Psi_n(z_s)\,.$$

Adiabatic Mode Approximation

$$p(r,z) \simeq \frac{i}{\rho(z_s)\sqrt{8\pi r}}\,e^{-i\pi/4}\sum_{m=1}^{\infty}\Psi_m(z_s)\,\Psi_m(r,z)\,\frac{e^{i\int_0^r k_{rm}(r')\,dr'}}{\sqrt{k_{rm}(r)}}\,.$$

Reciprocal Adiabatic Approximation (ad hoc)

$$p(r,z) \simeq \frac{i}{\rho(z_s)\sqrt{8\pi}}\,e^{-i\pi/4}\sum_{m=1}^{\infty}\Psi_m(z_s)\,\Psi_m(r,z)\,\frac{e^{i\int_0^r k_{rm}(r')\,dr'}}{\sqrt{\int_0^r k_{rm}(r')\,dr'}}\,.$$

Warm-Core Eddy Propagation

[Examples: See Jensen Figs 5.17, 5.18, 5.19a]