Computational Ocean Acoustics

- Ray Tracing
- Wavenumber Integration
- Normal Modes
- Parabolic Equation

Normal Modes

- Modes for Range-Dependent Envir.
  - Coupled Modes (5.9)
  - One-way Coupled Modes
  - Adiabatic Modes
  - SEALAB Propagation Modeling Environment
- Modes in 3-D Environments
  - Continuously coupled modes
  - Adiabatic Approximation
  - 3-D Propagation in 2-D Environments
  - Global Propagation

3-D Modal Modeling Framework

3-D Ocean Environment

Range-Independent Sectors

[See Fig 5.19a in Jensen, Kuperman, Porter and Schmidt. *Computational Ocean Acoustics.* New York: Springer-Verlag, 2000.]

[See Jensen Fig 5.19b]

Full 3-D Mode Coupling
Strong Discontinuities

1. Pre-compute modes for all sectors
2. Each source-receiver combination
   - Horizontal ray tracing, all mode combinations
   - Local single-scattering approximation in plane geometry
   - Approximate accounting for geometric spreading $r^{-1/2}$

COMPUTATIONALLY INTENSIVE

2.5-D Modal Modeling Framework

3-D Ocean Environment

[See Jensen Fig 5.19a]

Range-Independent Sectors

[See Jensen Fig 5.19b]

In-Plane Mode Coupling
Gradual Range-Dependence

1. Pre-compute modes for all sectors
2. Each source-receiver combination
   - In-plane mode propagation between sector boundaries
   - Local single-scattering – No horizontal diffraction
   - Approximate accounting for geometric spreading $r^{-1/2}$

COMPUTATIONALLY EFFICIENT

Global Propagation

Earth is non-perfect sphere

[See Jensen, Fig 5.21]

Global Propagation

Adiabatic Mode Travel Times

$$t_m = \int_0^S \frac{\omega}{k_{rm}} \int_0^D \frac{1}{\rho(z)} \left[\frac{\Psi_m(z)}{c(z)} \right]^2 dz\, ds\ .$$

[See Jensen, Fig 5.22 and 5.23]

Computational Ocean Acoustics

- Ray Tracing
- Wavenumber Integration
- Normal Modes
- Parabolic Equation

Parabolic Equation

- Mathematical Derivation (6.2)
  - Standard Parabolic Equation (6.2.1)
  - Generalized Derivation (6.2.2)
    - Expansion of Square-root Operator
    - Rational Approximations
    - Pade' Approximations
    - Split-step Parabolic Equations
  - Phase Errors and Angular Limitations (6.2.4)

Parabolic Equations

Helmholtz Equation

$$\frac{\partial^2 p}{\partial r^2} + \frac{1}{r}\frac{\partial p}{\partial r} + \frac{\partial^2 p}{\partial z^2} + k_0^2 n^2 p = 0\,,$$

Index of Refraction

$$n(r,z) = c_0/c(r,z)$$

The Standard Parabolic Equation

Outgoing Cylindrical Wave Solution

Slowly varying depth solution (envelope)

$$p(r,z) = \psi(r,z)\, H_0^{(1)}(k_0 r),$$

Range-independent cylindrically symmetric Range-solution

Substitution into Helmholtz Equation

Use Bessel Equation

$$\frac{\partial^2 \psi}{\partial r^2} + \left(\frac{2}{H_0^{(1)}(k_0 r)}\frac{\partial H_0^{(1)}(k_0 r)}{\partial r} + \frac{1}{r}\right)\frac{\partial \psi}{\partial r} + \frac{\partial^2 \psi}{\partial z^2} + k_0^2\,(n^2-1)\,\psi = 0\,.$$

Asymptotic Hankel Function - $k_0 r \gg 1$

$$H_0^{(1)}(k_0 r) \simeq \sqrt{\frac{2}{\pi k_0 r}}\, e^{i(k_0 r - \frac{\pi}{4})}\,.$$

Elliptic Wave Equation

$$\frac{\partial^2 \psi}{\partial r^2} + 2ik_0\frac{\partial \psi}{\partial r} + \frac{\partial^2 \psi}{\partial z^2} + k_0^2\,(n^2-1)\,\psi = 0\,.$$

Elliptic Wave Equation

$$\frac{\partial^2 \psi}{\partial r^2} + 2ik_0 \frac{\partial \psi}{\partial r} + \frac{\partial^2 \psi}{\partial z^2} + k_0^2 \left(n^2 - 1\right) \psi = 0 \, .$$

Paraxial Approximation

Slowly varying envelope: $\partial \psi / \partial r \ll \psi / \lambda \sim ik_0 \, \psi$

$$\frac{\partial^2 \psi}{\partial r^2} \ll 2ik_0 \frac{\partial \psi}{\partial r} \, .$$

Parabolic Wave Equation

Narrow-angle approximation, valid for grazing angles less than 10-15 deg.

$$2ik_0 \frac{\partial \psi}{\partial r} + \frac{\partial^2 \psi}{\partial z^2} + k_0^2 \left(n^2 - 1\right) \psi = 0 \, ,$$

Generalized PE Derivation

PE Differential Operators

$$P = \frac{\partial}{\partial r}\,, \qquad Q = \sqrt{n^2 + \frac{1}{k_0^2}\frac{\partial^2}{\partial z^2}}\,,$$

Elliptic Wave Equation

$$\left[P^2 + 2ik_0\,P + k_0^2\,(Q^2 - 1)\right]\psi = 0\,.$$

Factorization

$$(P + ik_0 - ik_0\,Q)(P + ik_0 + ik_0\,Q)\,\psi - ik_0[P,Q]\,\psi = 0\,,$$

Operator Commutator

$$[P,Q]\,\psi = PQ\,\psi - Q\,P\,\psi\,,$$

= 0 for n=n(z), range-independent
~ 0 for n(r,z) slowly varying in r

One-way Wave Equation

$$P\psi = ik_0\,(Q-1)\,\psi\,,$$

$$\Rightarrow$$

$$\frac{\partial\psi}{\partial r} = ik_0\left(\sqrt{n^2 + \frac{1}{k_0^2}\frac{\partial^2}{\partial z^2}} - 1\right)\psi\,.$$

Ignores backscattering

Solution technique:
Approximate Pseudo-differential Operator Q

Expansion of the Square-Root Operator

Definitions

$$\varepsilon = n^2 - 1\,, \qquad \mu = \frac{1}{k_0^2}\frac{\partial^2}{\partial z^2}\,, \qquad q = \varepsilon + \mu\,,$$

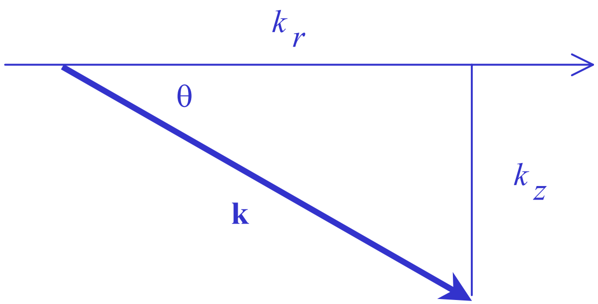
Square-root Operator

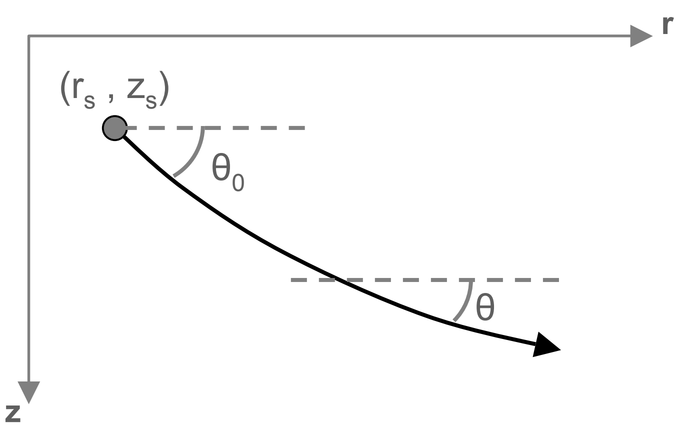
$$Q = \sqrt{1+q}\,.$$

Taylor series

$$\sqrt{1+q} = 1 + \frac{q}{2} - \frac{q^2}{8} + \frac{q^3}{16} + \cdots\,, \quad |q| < 1$$

Range-Independent Environment





Q = cos θ_0 relates to source angle, which – if small – justifies Taylor expansion

Plane-wave Solution

$$\psi = e^{i(k_r r \pm k_z z)} ,$$
$$k^2 = k_r^2 + k_z^2 ,$$

Grazing Angle of Propagation

$$\sin\theta = \pm\frac{k_z}{k} .$$

$$\mu = \frac{1}{k_0^2}\frac{\partial^2}{\partial z^2} = -\frac{k_z^2}{k_0^2} ,$$
$$= -n^2 \sin^2\theta .$$

Snell's Law

$$\frac{\cos\theta_0}{\cos\theta} = n ,$$
$$\Rightarrow$$
$$q = \varepsilon + \mu = (n^2 - 1) - n^2 \sin^2\theta$$
$$= -\sin^2\theta_0 .$$

Standard and Wide Angle Parabolic Equations

Square-root Operator Expansion

$$Q \simeq 1 + \frac{q}{2} = 1 + \frac{n^2 - 1}{2} + \frac{1}{2k_0^2}\frac{\partial^2}{\partial z^2}\,.$$

One-way Wave Equation

Standard PE

$$\frac{\partial \psi}{\partial r} = \frac{ik_0}{2}\left(n^2 - 1 + \frac{1}{k_0^2}\frac{\partial^2}{\partial z^2}\right)\psi\,,$$

Rational-linear Expansion

$$\sqrt{1+q} \simeq \frac{a_0 + a_1\, q}{b_0 + b_1\, q}\,,$$

Standard PE (Tappert)

$$a_0 = 1,\quad a_1 = 0.5,\quad b_0 = 1,\quad b_1 = 0$$

$$\sqrt{1+q} \simeq 1 + 0.5\, q\,, \qquad \text{Tappert}$$

Claerbout PE

$$a_0 = 1,\quad a_1 = 0.75,\quad b_0 = 1,\quad b_1 = 0.25$$

$$\sqrt{1+q} \simeq \frac{1 + 0.75\, q}{1 + 0.25\, q}\,, \qquad \text{Claerbout}$$

Greene Wide-Angle PE

$$\sqrt{1+q} \simeq \frac{0.99987 + 0.79624\, q}{1 + 0.30102\, q}\,, \qquad \text{Greene}$$

Minimizes phase errors 0-40 deg

Generalized Parabolic Equation

Square-root Operator Expansion

$$Q \simeq 1 + \frac{q}{2} = 1 + \frac{n^2 - 1}{2} + \frac{1}{2k_0^2}\frac{\partial^2}{\partial z^2} .$$

One-way Wave Equation

$$\frac{\partial \psi}{\partial r} = \frac{ik_0}{2}\left(n^2 - 1 + \frac{1}{k_0^2}\frac{\partial^2}{\partial z^2}\right)\psi ,$$

Rational-linear Expansion

$$\sqrt{1+q} \simeq \frac{a_0 + a_1\, q}{b_0 + b_1\, q} ,$$

Generalized Parabolic Equation

$$A_1 \frac{\partial \psi}{\partial r} + A_2 \frac{\partial^3 \psi}{\partial z^2 \partial r} = A_3\, \psi + A_4 \frac{\partial^2 \psi}{\partial z^2} ,$$

$$A_1 = b_0 + b_1\left(n^2 - 1\right) ,$$

$$A_2 = b_1 / k_0^2 ,$$

$$A_3 = ik_0\left[(a_0 - b_0) + (a_1 - b_1)(n^2 - 1)\right] ,$$

$$A_4 = i(a_1 - b_1)/k_0 .$$

Padé Approximation

$$\sqrt{1+q} = 1 + \sum_{j=1}^{m} \frac{a_{j,m}\, q}{1 + b_{j,m}\, q} + O(q^{2m+1}) \,,$$

$$a_{j,m} = \frac{2}{2m+1} \sin^2\left(\frac{j\pi}{2m+1}\right) \,,$$

$$b_{j,m} = \cos^2\left(\frac{j\pi}{2m+1}\right) \,.$$

First-order Padé Approximation

$$\sqrt{1+q} \simeq 1 + \frac{0.50\, q}{1 + 0.25\, q} = \frac{1 + 0.75 q}{1 + 0.25 q} \,,$$

Second-order Padé Approximation

$$\sqrt{1+q} \simeq 1 + \frac{0.13820\, q}{1 + 0.65451\, q} + \frac{0.36180\, q}{1 + 0.09549\, q} \,,$$

Very-Wide-Angle Padé Parabolic Equation (Collins)

$$\frac{\partial \psi}{\partial r} = i k_0 \left[\sum_{j=1}^{m} \frac{a_{j,m}\left(n^2 - 1 + \frac{1}{k_0^2}\frac{\partial^2}{\partial z^2}\right)}{1 + b_{j,m}\left(n^2 - 1 + \frac{1}{k_0^2}\frac{\partial^2}{\partial z^2}\right)} \right] \psi \,,$$

Split-Step PEs

Square-root operator, Feit–Fleck splitting

$$Q = \sqrt{1+\varepsilon+\mu} \simeq \sqrt{1+\mu} + \sqrt{1+\varepsilon} - 1\,,$$

Standard PE – $\mu \simeq 0$

$$\frac{\partial \psi}{\partial r} = \frac{ik_0}{2}\left(n^2 - 1 + \frac{1}{k_0^2}\frac{\partial^2}{\partial z^2}\right)\psi\,,$$

Thomson–Chapman PE

$$\frac{\partial \psi}{\partial r} = ik_0\left(n - 2 + \sqrt{1 + \frac{1}{k_0^2}\frac{\partial^2}{\partial z^2}}\right)\psi\,.$$

LOGPE

$$\frac{\partial \psi}{\partial r} = ik_0\left\{\ln n + \frac{1}{2}\ln\left[\cos^2\left(-\frac{i}{k_0}\frac{\partial}{\partial z}\right)\right]\right\}\psi\,,$$

Phase Errors and Angular Limitations

Claerbout's wide-angle PE

$$\frac{\partial \psi}{\partial r} = ik_0 \left(\frac{1 + 0.75\, q}{1 + 0.25\, q} - 1 \right) \psi \, ,$$

Range-Independent Environment

$$\left(k^2(z) + 3k_0^2 + \frac{\partial^2}{\partial z^2} \right) \frac{\partial \psi}{\partial r} = 2ik_0 \left(k^2(z) - k_0^2 + \frac{\partial^2}{\partial z^2} \right) \psi \, .$$

Separation of Variables.

$$\psi = \Phi(r)\, \Psi(z) \, ,$$

$= k_{rm}$

$$\left[\frac{d^2 \Psi}{dz^2} + k^2(z)\, \Psi \right] \left(\frac{d\Phi}{dr} - 2ik_0\, \Phi \right) + \left[3k_0^2 \frac{d\Phi}{dr} + 2ik_0^3\, \Phi \right] \Psi = 0 \, ,$$

Phase Errors and Angular Limitations

Vertical 'Modal' Equation

$$\frac{d^2\Psi}{dz^2} + \left[k^2(z) - k_{rm}^2\right]\Psi = 0\,,$$

Horizontal Parabolic Equation

$$\frac{d\Phi}{dr} - ik_0\,\frac{2k_{rm}^2 - 2k_0^2}{3k_0^2 + k_{rm}^2}\,\Phi = 0\,.$$

Radial Solution

$$\Phi(r) = \Phi(r_0)\;\exp\left[ik_0\left(\frac{2k_{rm}^2 - 2k_0^2}{3k_0^2 + k_{rm}^2}\right)(r - r_0)\right].$$

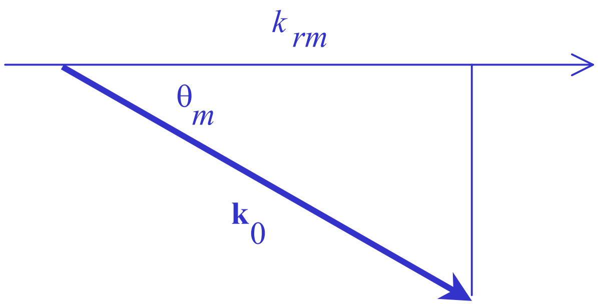
Acoustic Pressure

$$p(r,z) = p(r_0,z)\,\sqrt{\frac{r_0}{r}}\;\exp\left[ik_0\left(\frac{k_0^2 + 3k_{rm}^2}{3k_0^2 + k_{rm}^2}\right)(r - r_0)\right].$$

Phase Errors and Angular Limitations

Exact Modal Phase

$$\exp[ik_{rm}(r - r_0)]$$



$$k_{rm} = k_0 \cos\theta_m = k_0 \varphi$$

$$\varphi = cos(\theta_m) = \sqrt{1 - \sin^2\theta}, \qquad \text{Helmholtz}$$

Clairbout Modal Phase

$$\varphi = \frac{1 + 3\cos^2\theta_m}{3 + \cos^2\theta_m} = \frac{1 - 0.75\,\sin^2\theta_m}{1 - 0.25\,\sin^2\theta_m}. \qquad \text{Claerbout}$$

PE Modal Phases

$$Q = \sqrt{1 - \sin^2\theta_m}, \qquad \text{Helmholtz}$$

$$Q_1 = 1 - \frac{\sin^2\theta_m}{2}, \qquad \text{Tappert}$$

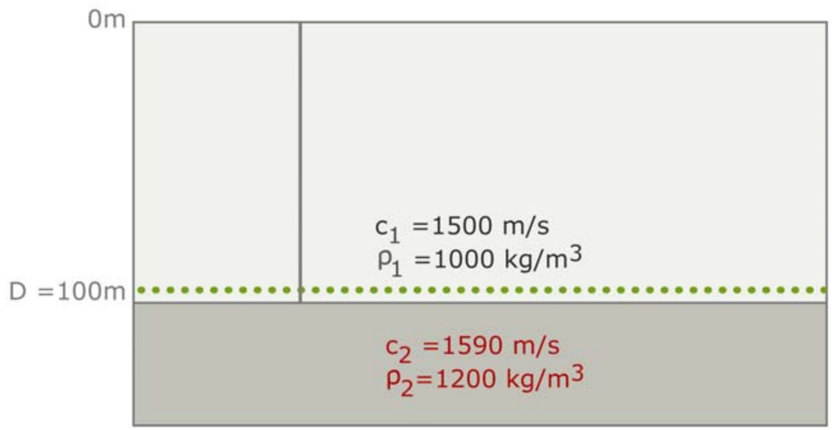
$$Q_2 = \frac{1 - 0.75\,\sin^2\theta_m}{1 - 0.25\,\sin^2\theta_m}, \qquad \text{Claerbout, Padé (1)}$$

$$Q_3 = \frac{0.99987 - 0.79624\,\sin^2\theta_m}{1 - 0.30102\,\sin^2\theta_m}, \qquad \text{Greene}$$

$$Q_4 = 1 - \frac{0.13820\,\sin^2\theta_m}{1 - 0.65451\,\sin^2\theta_m} - \frac{0.36180\,\sin^2\theta_m}{1 - 0.09549\,\sin^2\theta_m}. \qquad \text{Padé (2)}$$

[See Jensen Fig 6.1]

PE Workshop Case 3B



[See Jensen Fig 6.2]