Computational Ocean Acoustics

- Ray Tracing
- Wavenumber Integration
- Normal Modes
- Parabolic Equation

Parabolic Equation

- Mathematical Derivation (6.2)
  - Phase Errors and Angular Limitations (6.2.4)
- Starting Fields (6.4)
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  - PE Self Starter
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Solutions by FDs and FEs

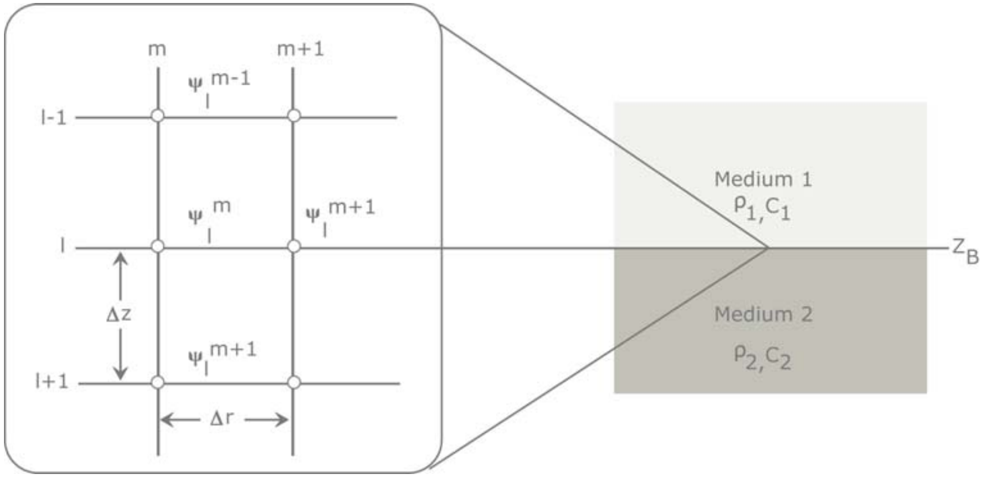
Field Equations on Horizontal Interfaces

$$\frac{\partial^2 \psi}{\partial r^2} + 2ik_0 \frac{\partial \psi}{\partial r} + k_0^2 \left(n^2 - 1\right) \psi + \frac{\partial^2 \psi}{\partial z^2} = 0 \,,$$

Boundary Conditions

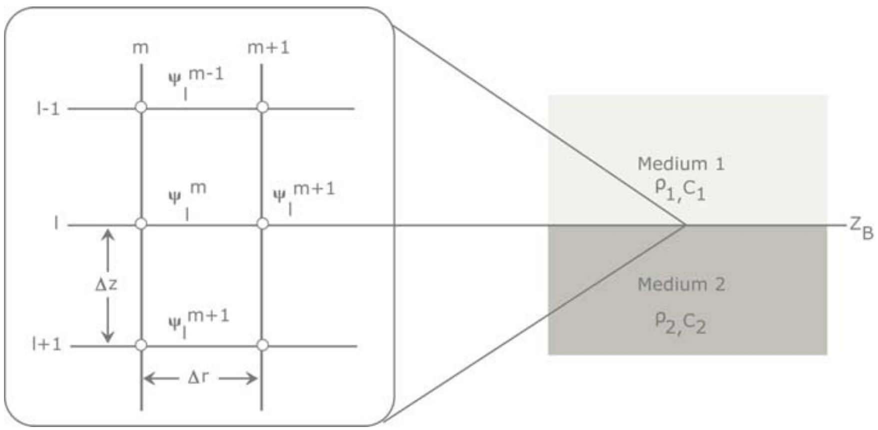
$$\psi_1(r, z_B) = \psi_2(r, z_B) \,,$$

$$\frac{1}{\rho_1} \frac{\partial \psi_1}{\partial z}\bigg|_{z_B} = \frac{1}{\rho_2} \frac{\partial \psi_2}{\partial z}\bigg|_{z_B} \,.$$



Solutions by FDs and FEs

Field Equations on Horizontal Interfaces



Medium 1:

$$\frac{\partial^2 \psi_1}{\partial r^2} + 2ik_0 \frac{\partial \psi_1}{\partial r} + k_0^2 \left(n_1^2 - 1\right) \psi_1 + \frac{\partial^2 \psi_1}{\partial z^2} = 0\,.$$

Taylor Series Expansion

$$\psi_{\ell-1}^m = \psi_\ell^m - \Delta z \frac{\partial \psi_\ell^m}{\partial z} + \frac{(\Delta z)^2}{2} \frac{\partial^2 \psi_\ell^m}{\partial z^2} + \cdots$$

Solve for the Second derivative of ψ

$$\frac{\partial^2 \psi_1}{\partial z^2} = -\frac{2}{(\Delta z)^2} \left(\psi_1 - \psi_{\ell-1}^m\right) + \frac{2}{\Delta z} \frac{\partial \psi_1}{\partial z}\,.$$

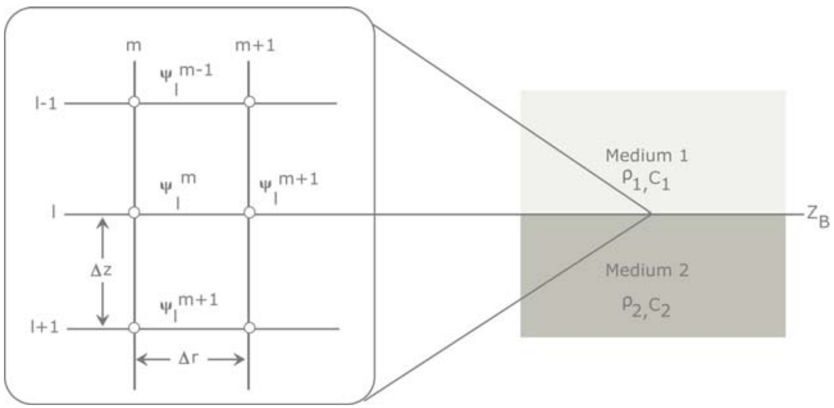
$$\frac{\partial \psi_1}{\partial z} = -\frac{\Delta z}{2} \left[\frac{\partial^2 \psi_1}{\partial r^2} + 2ik_0 \frac{\partial \psi_1}{\partial r} + k_0^2 \left(n_1^2 - 1\right) \psi_1 - \frac{2}{(\Delta z)^2} \left(\psi_1 - \psi_{\ell-1}^m\right)\right].$$

Medium 2:

$$\frac{\partial \psi_2}{\partial z} = \frac{\Delta z}{2} \left[\frac{\partial^2 \psi_2}{\partial r^2} + 2ik_0 \frac{\partial \psi_2}{\partial r} + k_0^2 \left(n_2^2 - 1\right) \psi_2 + \frac{2}{(\Delta z)^2} \left(\psi_{\ell+1}^m - \psi_2\right)\right].$$

Solutions by FDs and FEs

Field Equations on Horizontal Interfaces



Pressure Boundary Condition

$$\psi_1 = \psi_2 = \psi$$

Displacement Boundary Condition

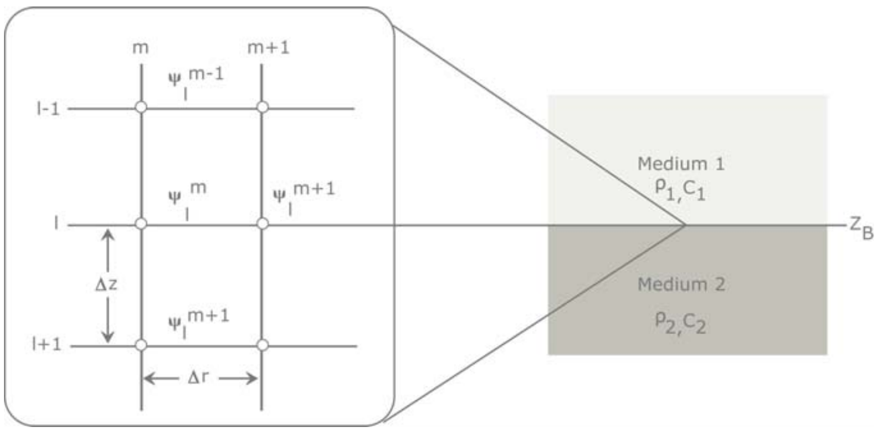
$$\frac{\partial^2 \psi}{\partial r^2} + 2ik_0 \frac{\partial \psi}{\partial r} + k_0^2 \frac{\rho_2}{\rho_1 + \rho_2} \left(n_1^2 + \frac{\rho_1}{\rho_2} n_2^2 \right) \psi - k_0^2 \psi$$
$$+ \frac{2}{(\Delta z)^2} \frac{\rho_2}{\rho_1 + \rho_2} \left(\psi_{\ell-1}^m - \frac{\rho_1 + \rho_2}{\rho_2} \psi_\ell^m + \frac{\rho_1}{\rho_2} \psi_{\ell+1}^m \right) = 0 \, .$$

Interface Helmholtz Equation

$$\frac{\partial^2 \psi}{\partial r^2} + 2ik_0 \frac{\partial \psi}{\partial r} + k_0^2 \left(n^2 - 1 \right) \psi + \frac{\psi_{\ell+1}^m - 2\psi_\ell^m + \psi_{\ell-1}^m}{(\Delta z)^2} = 0 \, ,$$

Solutions by FDs and FEs

Field Equations on Horizontal Interfaces



Interface Helmholtz Equation

$$\frac{\partial^2 \psi}{\partial r^2} + 2ik_0 \frac{\partial \psi}{\partial r} + k_0^2 (n^2 - 1)\,\psi + \frac{\psi_{\ell+1}^m - 2\psi_\ell^m + \psi_{\ell-1}^m}{(\Delta z)^2} = 0\,,$$

$$\Gamma_{zz}\psi = \frac{2}{(\Delta z)^2} \frac{\rho_2}{\rho_1 + \rho_2} \left(\psi_{\ell-1}^m - \frac{\rho_1 + \rho_2}{\rho_2} \psi_\ell^m + \frac{\rho_1}{\rho_2} \psi_{\ell+1}^m \right),$$

$$\eta = \frac{\rho_2}{\rho_1 + \rho_2} \left(n_1^2 + \frac{\rho_1}{\rho_2} n_2^2 \right) - 1\,,$$

$$G = k_0^2\,\eta + \Gamma_{zz}\,,$$

$$\frac{\partial^2 \psi}{\partial r^2} + 2ik_0 \frac{\partial \psi}{\partial r} + G\,\psi = 0\,,$$

Compact Form

$$G = k_0^2\,(Q^2 - 1)\,,$$

One-Way Interface Equation

$$\begin{aligned} \frac{\partial \psi}{\partial r} &= ik_0\,(Q - 1)\,\psi \\ &= ik_0 \left(\sqrt{1+q} - 1 \right) \psi\,, \end{aligned}$$

$$q = G/k_0^2$$

Solutions by FDs and FEs

Implicit Finite Differnce Scheme (IFD)

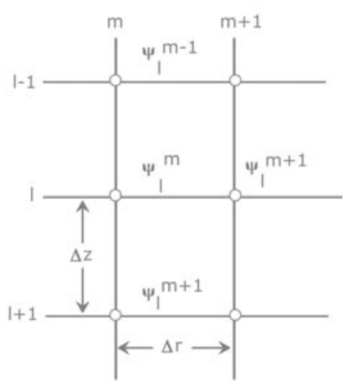
Crank–Nicolson Algorithm

$$\frac{\psi^{m+1} - \psi^{m}}{\Delta r} = i k_0 \left(\sqrt{1+q} - 1\right) \frac{\psi^{m+1} + \psi^{m}}{2},$$

$$\left[1 - \frac{i k_0 \Delta r}{2}\left(\sqrt{1+q} - 1\right)\right] \psi^{m+1} = \left[1 + \frac{i k_0 \Delta r}{2}\left(\sqrt{1+q} - 1\right)\right] \psi^{m}.$$

Square-root Approximation

$$\sqrt{1+q} \simeq \frac{a_0 + a_1 q}{b_0 + b_1 q},$$



$$\left[1 - \frac{i k_0 \Delta r}{2}\left(\frac{a_0 + a_1\left(\eta + \frac{\Gamma_{zz}}{k_0^2}\right)}{b_0 + b_1\left(\eta + \frac{\Gamma_{zz}}{k_0^2}\right)} - 1\right)\right] \psi^{m+1}$$

$$= \left[1 + \frac{i k_0 \Delta r}{2}\left(\frac{a_0 + a_1\left(\eta + \frac{\Gamma_{zz}}{k_0^2}\right)}{b_0 + b_1\left(\eta + \frac{\Gamma_{zz}}{k_0^2}\right)} - 1\right)\right] \psi^{m}.$$

Implicit Finite Difference Scheme

$$\left[b_0 + b_1\eta - \frac{ik_0\,\Delta r}{2}\left[(a_0-b_0)+(a_1-b_1)\eta\right]\right]\psi^{m+1}$$
$$+\frac{1}{k_0^2}\left[b_1 - \frac{ik_0\Delta r}{2}(a_1-b_1)\right]\Gamma_{zz}\,\psi^{m+1}$$
$$=\left[b_0 + b_1\eta + \frac{ik_0\,\Delta r}{2}\left[(a_0-b_0)+(a_1-b_1)\eta\right]\right]\psi^{m}$$
$$+\frac{1}{k_0^2}\left[b_1 + \frac{ik_0\,\Delta r}{2}(a_1-b_1)\right]\Gamma_{zz}\,\psi^{m}\,.$$

$$w_1 = b_0 + \frac{ik_0\,\Delta r}{2}(a_0-b_0)\,,$$
$$w_1^* = b_0 - \frac{ik_0\,\Delta r}{2}(a_0-b_0)\,,$$
$$w_2 = b_1 + \frac{ik_0\,\Delta r}{2}(a_1-b_1)\,,$$
$$w_2^* = b_1 - \frac{ik_0\,\Delta r}{2}(a_1-b_1)\,.$$

$$\left(\frac{w_1^*}{w_2^*}+\eta\right)\psi_\ell^{m+1} + \frac{1}{k_0^2}\left[\frac{2}{(\Delta z)^2}\,\frac{\rho_2}{\rho_1+\rho_2}\right]$$
$$\times\left(\psi_{\ell-1}^{m+1} - \frac{\rho_1+\rho_2}{\rho_2}\,\psi_\ell^{m+1} + \frac{\rho_1}{\rho_2}\,\psi_{\ell+1}^{m+1}\right)$$
$$=\left(\frac{w_1+w_2\,\eta}{w_2^*}\right)\psi_\ell^m + \frac{1}{k_0^2}\left(\frac{w_2}{w_2^*}\right)\left[\frac{2}{(\Delta z)^2}\,\frac{\rho_2}{\rho_1+\rho_2}\right]$$
$$\times\left(\psi_{\ell-1}^{m} - \frac{\rho_1+\rho_2}{\rho_2}\,\psi_\ell^{m} + \frac{\rho_1}{\rho_2}\,\psi_{\ell+1}^{m}\right).$$

Vector Form

$$[1, u, v]\begin{bmatrix}\psi_{\ell-1}^{m+1}\\ \psi_\ell^{m+1}\\ \psi_{\ell+1}^{m+1}\end{bmatrix} = \frac{w_2}{w_2^*}\,[1,\hat{u},v]\begin{bmatrix}\psi_{\ell-1}^{m}\\ \psi_\ell^{m}\\ \psi_{\ell+1}^{m}\end{bmatrix},$$

$$u = \frac{\rho_1+\rho_2}{\rho_2}\left[\frac{k_0^2\,(\Delta z)^2}{2}\left(\frac{w_1^*}{w_2^*}\right)-1\right]$$
$$+\frac{k_0^2\,(\Delta z)^2}{2}\left[(n_1^2-1)+\frac{\rho_1}{\rho_2}\,(n_2^2-1)\right],$$

$$v = \frac{\rho_1}{\rho_2},$$

$$\hat{u} = \frac{\rho_1+\rho_2}{\rho_2}\left[\frac{k_0^2\,(\Delta z)^2}{2}\left(\frac{w_1}{w_2}\right)-1\right]$$
$$+\frac{k_0^2\,(\Delta z)^2}{2}\left[(n_1^2-1)+\frac{\rho_1}{\rho_2}\,(n_2^2-1)\right].$$

Implicit Finite Difference Scheme

Global Matrix Equation

$$
\begin{bmatrix}
u_1 & v_1 & & & & & \\
1 & u_2 & v_2 & & & & \\
 & 1 & u_3 & v_3 & & & \\
 & & \ddots & \ddots & \ddots & & \\
 & & & 1 & u_{N-2} & v_{N-2} & \\
 & & & & 1 & u_{N-1} & v_{N-1} \\
 & & & & & 1 & u_N \\
h & & & & & &
\end{bmatrix}
\begin{bmatrix}
\psi_1 \\ \psi_2 \\ \psi_3 \\ \vdots \\ \psi_{N-2} \\ \psi_{N-1} \\ \psi_N
\end{bmatrix}^{m+1}
$$

$$
= \left(\frac{w_2}{w_2^*}\right)
\begin{bmatrix}
\hat{u}_1 & v_1 & & & & & \\
1 & \hat{u}_2 & v_2 & & & & \\
 & 1 & \hat{u}_3 & v_3 & & & \\
 & & \ddots & \ddots & \ddots & & \\
 & & & 1 & \hat{u}_{N-2} & v_{N-2} & \\
 & & & & 1 & \hat{u}_{N-1} & v_{N-1} \\
 & & & & & 1 & \hat{u}_N
\end{bmatrix}
\begin{bmatrix}
\psi_1 \\ \psi_2 \\ \psi_3 \\ \vdots \\ \psi_{N-2} \\ \psi_{N-1} \\ \psi_N
\end{bmatrix}^{m} .
$$

$$
[1, u, v] \begin{bmatrix} \psi_{\ell-1}^{m+1} \\ \psi_{\ell}^{m+1} \\ \psi_{\ell+1}^{m+1} \end{bmatrix} = \frac{w_2}{w_2^*} [1, \hat{u}, v] \begin{bmatrix} \psi_{\ell-1}^{m} \\ \psi_{\ell}^{m} \\ \psi_{\ell+1}^{m} \end{bmatrix} ,
$$

$$
\begin{aligned}
u &= \frac{\rho_1+\rho_2}{\rho_2}\left[\frac{k_0^2\,(\Delta z)^2}{2}\left(\frac{w_1^*}{w_2^*}\right) - 1\right] \\
&\quad + \frac{k_0^2\,(\Delta z)^2}{2}\left[(n_1^2-1) + \frac{\rho_1}{\rho_2}(n_2^2-1)\right], \\
v &= \frac{\rho_1}{\rho_2}, \\
\hat{u} &= \frac{\rho_1+\rho_2}{\rho_2}\left[\frac{k_0^2\,(\Delta z)^2}{2}\left(\frac{w_1}{w_2}\right) - 1\right] \\
&\quad + \frac{k_0^2\,(\Delta z)^2}{2}\left[(n_1^2-1) + \frac{\rho_1}{\rho_2}(n_2^2-1)\right].
\end{aligned}
$$

$$
\begin{aligned}
w_1 &= b_0 + \frac{ik_0\,\Delta r}{2}(a_0 - b_0)\,, \\
w_1^* &= b_0 - \frac{ik_0\,\Delta r}{2}(a_0 - b_0)\,, \\
w_2 &= b_1 + \frac{ik_0\,\Delta r}{2}(a_1 - b_1)\,, \\
w_2^* &= b_1 - \frac{ik_0\,\Delta r}{2}(a_1 - b_1)\,.
\end{aligned}
$$

Error Analysis

Standard PE - Compact Form

$$\frac{\partial \psi}{\partial r} = (A + B)\,\psi = U(r,z)\,\psi\,,$$

$$A = \frac{ik_0}{2}\left[n^2(r,z) - 1\right], \qquad B = \frac{i}{2k_0}\frac{\partial^2}{\partial z^2}\,.$$

Power-series Marching Solution

$$\psi_{j+1} = \left[1 + U\Delta r + (U' + U^2)\frac{(\Delta r)^2}{2} + (U'' + 2UU' + U'U + U^3)\frac{(\Delta r)^3}{\boxed{6}}\right]_j \psi_j\,.$$

Crank–Nicolson Algorithm

$$\frac{\psi_{j+1} - \psi_j}{\Delta r} = \frac{U_{j+1}\psi_{j+1} + U_j\psi_j}{2}\,,$$

$$\psi_{j+1} = \frac{1 + U_j\frac{\Delta r}{2}}{1 - U_{j+1}\frac{\Delta r}{2}}\,\psi_j\,.$$

Series Expansion

$$\psi_{j+1} = \left[\left(1 + U_j\frac{\Delta r}{2}\right)\left(1 + U_{j+1}\frac{\Delta r}{2} + U_{j+1}^2\frac{(\Delta r)^2}{4} + U_{j+1}^3\frac{(\Delta r)^3}{8}\right)\right]\psi_j\,.$$

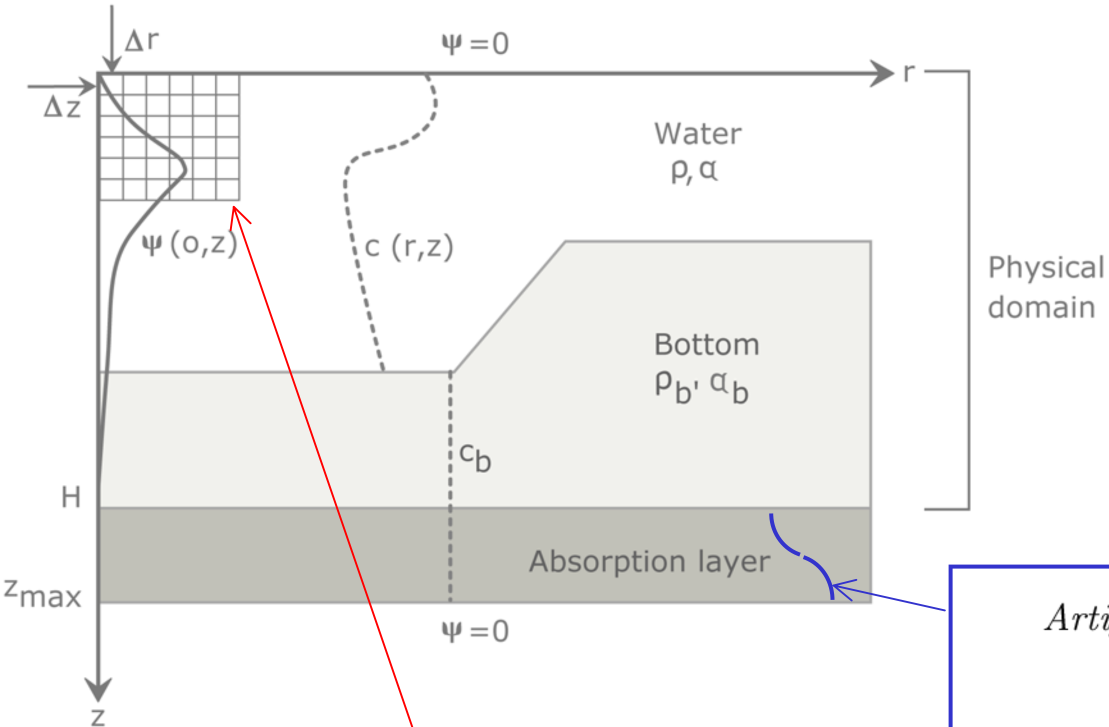
$$U_{j+1} = U_j + U_j'\,\Delta r + U_j''\frac{(\Delta r)^2}{2} + \cdots$$

$$\psi_{j+1} = \left[1 + U\Delta r + (U' + U^2)\frac{(\Delta r)^2}{2} + (U'' + 2UU' + U'U + U^3)\frac{(\Delta r)^3}{\boxed{4}}\right]_j \psi_j\,.$$

Truncation Error

$$E_{\rm CN} = -\frac{(\Delta r)^3}{12}\left(U'' + 2UU' + U'U + U^3\right)_j \psi_j\,.$$

Numerical Implementation



Artificial Absorbtion Layer

$$n^2 = n_b^2 + i\alpha \exp\left[-\left(\frac{z - z_{\max}}{D}\right)^2\right],$$

Range-Depth Discretization

$$\Delta z \leq \lambda/4$$
$$\Delta r \simeq 2 - 5\,\Delta z$$

Convergence Analysis !!

Variable Density

$$\rho \nabla \cdot \left(\frac{1}{\rho} \nabla p \right) + k_0^2 n^2 p = 0 \,,$$

$$\tilde{p} = \frac{p}{\sqrt{\rho}} \,,$$

$$\nabla^2 \tilde{p} + k_0^2 \tilde{n}^2 \tilde{p} = 0 \,,$$

Effective Index of Refraction

$$\tilde{n}^2 = n^2 + \frac{1}{2k_0^2} \left[\frac{1}{\rho} \nabla^2 \rho - \frac{3}{2\rho^2} (\nabla \rho)^2 \right] .$$

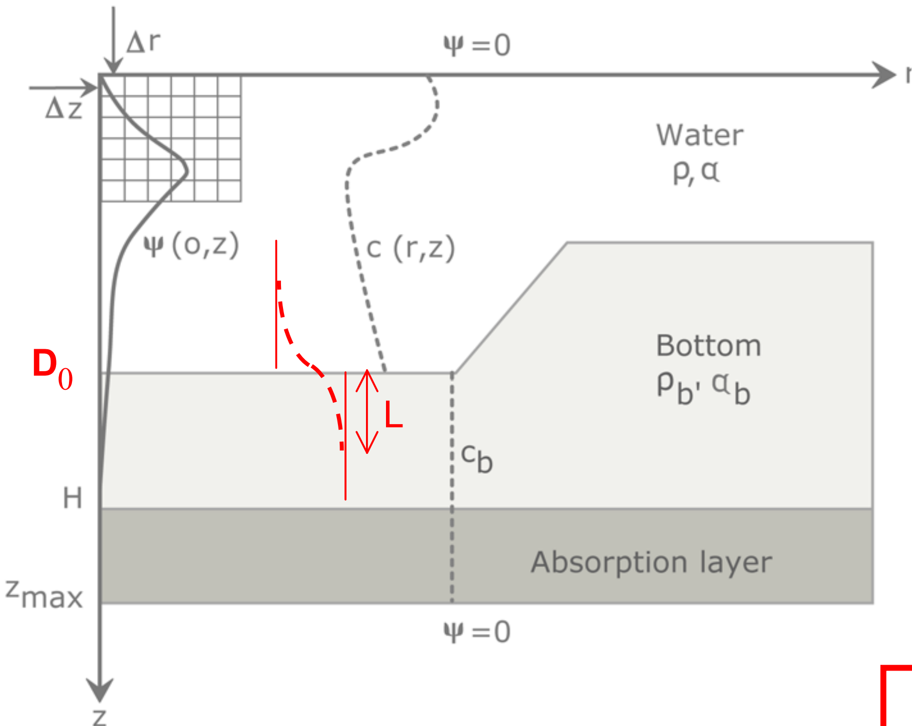
Depth Derivatives

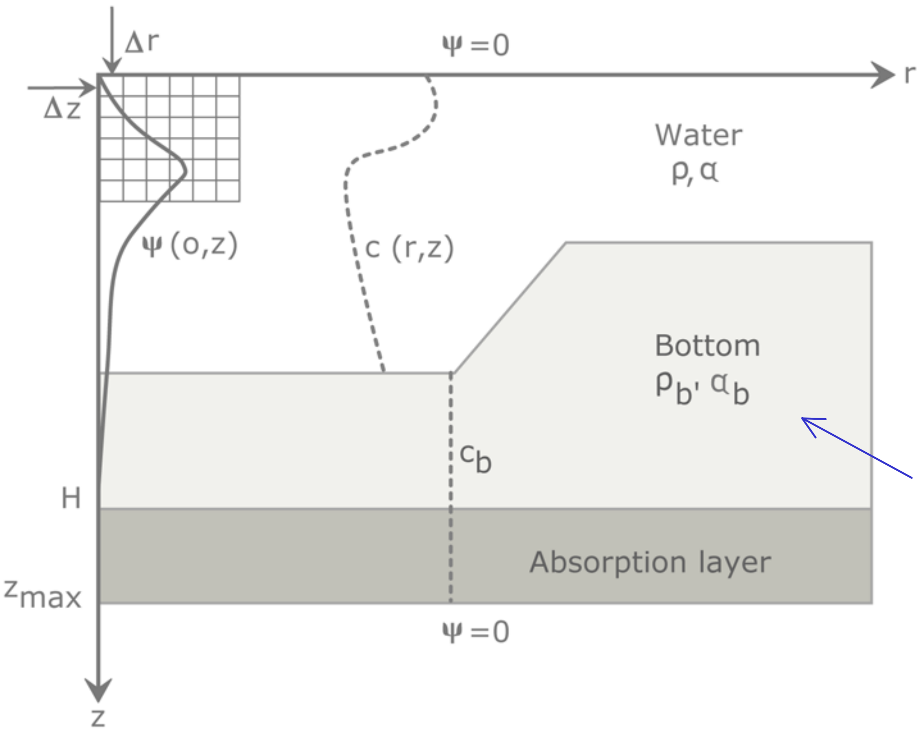
$$\tilde{n}^2 = n^2 + \frac{1}{2k_0^2} \left[\frac{1}{\rho} \frac{\partial^2 \rho}{\partial z^2} - \frac{3}{2\rho^2} \left(\frac{\partial \rho}{\partial z} \right)^2 \right] .$$

Smoothing

$$\rho(z) = \frac{1}{2}(\rho_2 + \rho_1) + \frac{1}{2}(\rho_2 - \rho_1) \tanh \left(\frac{z - D_0}{L} \right) ,$$

$$k_0 L \simeq 2 \,.$$





Attenuation

Complex Wavenumber

$$k = \frac{\omega}{c} + i\alpha\,, \qquad \alpha > 0\,.$$

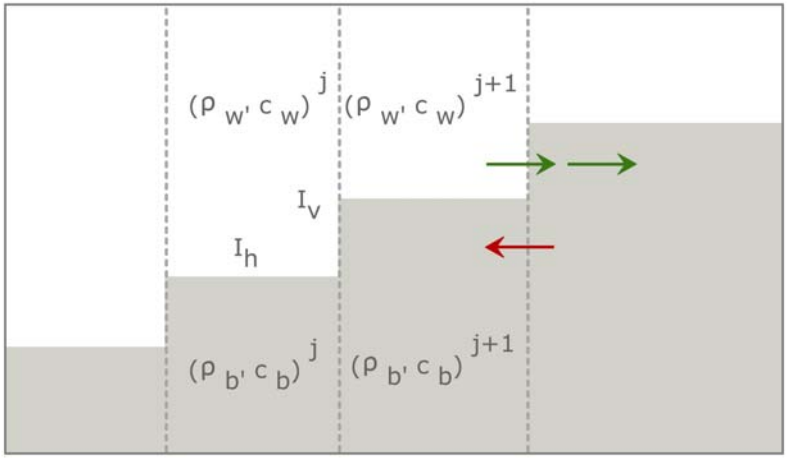
$$\alpha^{(\lambda)} = -20\log\left(\frac{e^{-\alpha(r+\lambda)}}{e^{-\alpha r}}\right) = \alpha\lambda\; 20\log e\,,$$

Complex Index of Refraction

$$n^2 = \left(\frac{k}{k_0}\right)^2 \simeq \left(\frac{c_0}{c}\right)^2\left[1 + i\,\frac{2\alpha c}{\omega}\right].$$

$$\simeq \left(\frac{c_0}{c}\right)^2\left[1 + i\,\frac{\alpha^{(\lambda)}}{27.29}\right].$$

Energy Conservation in PEs



[See Fig 6.8, 6.10-6.12 in Jensen, Kuperman, Porter and Schmidt. *Computational Ocean Acoustics.* New York: Springer-Verlag, 2000.]

Upslope: Energy loss

Downslope: Energy Gain

$p/\sqrt{\rho}$ Accounts for density variation

$p/\sqrt{\rho c}$ Energy Conserving

Student Demos
Wavenumber Integration Models