Ocean Acoustic Theory

- Acoustic Wave Equation
- Integral Transforms
- Helmholtz Equation
- Source in Unbounded and Bounded Media
- Reflection and Transmission
- The Ideal Waveguide
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  - Wavenumber Integral
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- Pekeris Waveguide

The Wave Equation

Conservation of Mass

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot \rho \mathbf{v}$$

Euler's Equation (Equation of motion)

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = -\frac{1}{\rho} \nabla p(\rho)$$

Constitutive Equation

$$p = p_0 + \rho' \left[\frac{\partial p}{\partial \rho}\right]_S + \frac{1}{2} (\rho')^2 \left[\frac{\partial^2 p}{\partial \rho^2}\right]_S + \cdots$$

Speed of Sound

$$c^2 \equiv \left[\frac{\partial p}{\partial \rho}\right]_S \quad \text{(Sound speed)}$$

The Linear Wave Equation

$$\begin{aligned}
\frac{\partial \rho'}{\partial t} &= -\rho_0 \nabla \cdot \mathbf{v} \, , \\
\frac{\partial \mathbf{v}}{\partial t} &= -\frac{1}{\rho_0} \nabla p'(\rho) \, , \\
p' &= \rho' c^2 \, .
\end{aligned}$$

Wave Equation for Pressure

$$\rho \nabla \cdot \left(\frac{1}{\rho} \nabla p\right) - \frac{1}{c^2} \frac{\partial^2 p}{\partial t^2} = 0 \, ,$$

The Wave Equation

Wave Equation for Particle Velocity

$$\frac{1}{\rho}\nabla\left(\rho c^2\,\nabla\cdot\mathbf{v}\right)-\frac{\partial^2\mathbf{v}}{\partial t^2}=\mathbf{0}\,.$$

Wavefield Potentials

Wave Equation for Velocity Potential

Constant density ρ:

$$\mathbf{v}=\nabla\phi\,.$$

$$\nabla\left(c^2\,\nabla^2\phi-\frac{\partial^2\phi}{\partial t^2}\right)=\mathbf{0}\,.$$

$$\nabla^2\phi-\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2}=0\,,$$

Wave Equation for Displacement Potential

$$\mathbf{u}=\nabla\psi\,,$$

$$\nabla^2\psi-\frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2}=0\,.$$

$$p=-K\,\nabla^2\psi\,,$$

$$K=\rho c^2\,.$$

$$p=-\rho\,\frac{\partial^2\psi}{\partial t^2}\,.$$

Solution of the Wave Equation

- 4-D Partial Differential Equation
- Analytical solutions only for few canonical problems
- Direct Numerical Solution (FDM, FEM)
  - Computationally intensive ($\Delta x \ll \lambda$, $\Delta t \ll T$).
- Dimension Reduction for PDE
  - Geometrical symmetries (Plane or axisymmetric problems)
  - Integral transforms
  - Analytical or numerical solution of ODE or low dimensional PDE.
  - Evaluation of inverse transforms (analytical or numerical)

The Helmholtz Equation

Frequency–time Fourier transform pair

$$
\begin{aligned}
f(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\omega)\, e^{-i\omega t} d\omega \,, \\
f(\omega) &= \int_{-\infty}^{\infty} f(t)\, e^{i\omega t} dt \,,
\end{aligned}
$$

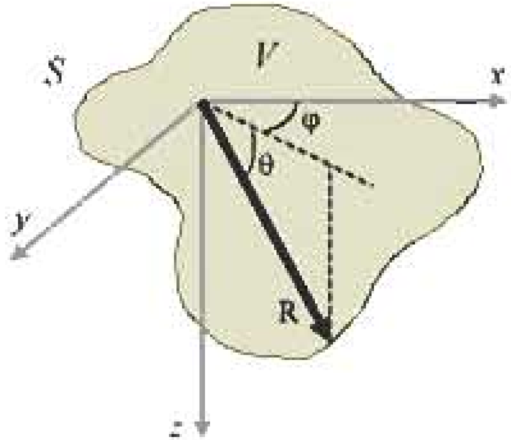
Helmholtz Equation

$$
\left[\nabla^2 + k^2(\mathbf{r})\right] \psi(\mathbf{r}, \omega) = 0 \,,
$$

$$
k(\mathbf{r}) = \frac{\omega}{c(\mathbf{r})} \,.
$$

Solution of Hemholtz Equation

- Dimensionality of the problem.
- Medium wavenumber variation $k(\mathbf{r})$, i.e., the sound speed variation $c(\mathbf{r})$.
- Boundary conditions.
- Source–receiver geometry.
- Frequency and bandwidth.



Homogeneous medium occupying the volume V bounded by the surface S.

Helmholtz Equation for Homogeneous Media

Cartesian Coordinates

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2},$$

$$\psi(x, y, z) = \begin{cases} A\, e^{i\mathbf{k}\cdot\mathbf{r}} \\ B\, e^{-i\mathbf{k}\cdot\mathbf{r}}, \end{cases}$$

Wavefronts: $\mathbf{k}\cdot\mathbf{r} = \text{const}$

1-D propagation: $k_y, k_z = 0$:

$$\psi(x) = \begin{cases} A\, e^{ikx} & \text{Forward propagating} \\ B\, e^{-ikx} & \text{Backward propagating} \end{cases}$$

Cylindrical Cordinates

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2} + \frac{\partial^2}{\partial z^2} \; .$$

Axial Symmetry

$$\left[\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} + k^2\right] \psi(r) = 0 \; ,$$

bessel Functions

$$\psi(r) = \begin{cases} A\, J_0(kr) \\ B\, Y_0(kr) \; , \end{cases}$$

Hankel Functions

$$\psi(r) = \begin{cases} C H_0^{(1)}(kr) &= C\,[J_0(kr) + iY_0(kr)] \\ D H_0^{(2)}(kr) &= D\,[J_0(kr) - iY_0(kr)] \; . \end{cases}$$

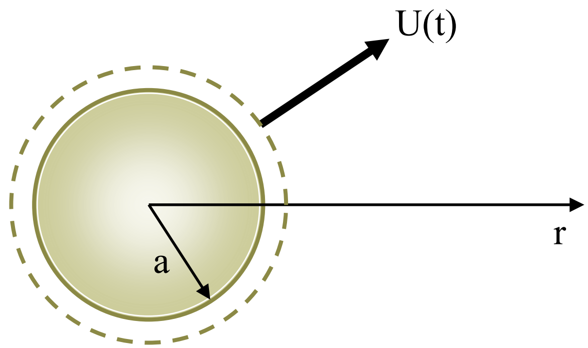
$$H_0^{(1)}(kr) \simeq \sqrt{\frac{2}{\pi kr}}\, e^{i(kr-\pi/4)} \text{ Diverging waves}$$

$$H_0^{(2)}(kr) \simeq \sqrt{\frac{2}{\pi kr}}\, e^{-i(kr-\pi/4)} \text{ Converging waves}$$

Spherical Cordinates

$$\left[\frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} + k^2\right] \psi(r) = 0 \, ,$$

$$\psi(r) = \begin{cases} (A/r)\, e^{ikr} & \text{Diverging waves} \\ (B/r)\, e^{-ikr} & \text{Converging waves} \end{cases}$$



Vibrating sphere in an infinite fluid medium.

Source in Unbounded Medium

Frequency Domain

$$u_r(a) = U(\omega)\,.$$

Spherical geometry solution

$$\begin{aligned}\psi(r) &= A\,\frac{e^{ikr}}{r}\,,\\ u_r(r) &= \frac{\partial\psi(r)}{\partial r} = A\,e^{ikr}\left(\frac{ik}{r}-\frac{1}{r^2}\right).\end{aligned}$$

Simple point source: $ka \ll 1$

$$\begin{aligned}u_r(a) &= A\,e^{ika}\,\frac{ika-1}{a^2} \simeq -\frac{A}{a^2}\,,\\ A &= -a^2\,U(\omega)\,.\\ &\Rightarrow\\ \psi(r) &= -S_\omega\,\frac{e^{ikr}}{4\pi r}\,.\\ S_\omega &= 4\pi a^2\,U(\omega)\end{aligned}$$

Green's function

,

$$g_\omega(r,0) = \frac{e^{ikr}}{4\pi r} ,$$

Source at r_0

$$g_\omega(\mathbf{r},\mathbf{r}_0) = \frac{e^{ikR}}{4\pi R} , \quad R = |\mathbf{r} - \mathbf{r}_0| .$$

Helmholtz Equation for Green's function

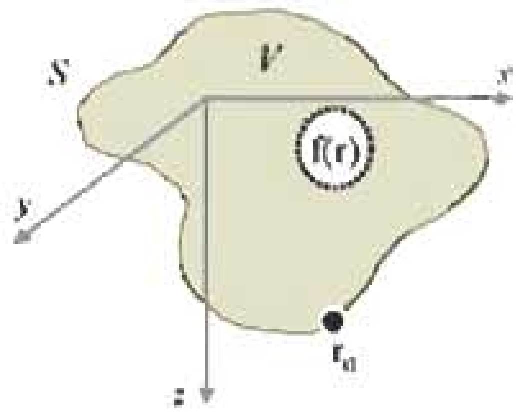
$$\left[\nabla^2 + k^2\right] g_\omega(\mathbf{r},\mathbf{r}_0) = -\delta(\mathbf{r} - \mathbf{r}_0) ,$$

Integrate over spherical volume V of radius $\epsilon \to 0$:

$$\begin{aligned}
\int_V -\delta(\mathbf{r} - \mathbf{r}_0) dV &= -1 \\
\int_V k^2 g_\omega(\mathbf{r},\mathbf{r}_0) dV &\to_{\epsilon\to 0} 0 \\
\int_V \nabla^2 g_\omega(\mathbf{r},\mathbf{r}_0) dV &= \int_S \frac{\partial}{\partial R} g_\omega(\mathbf{r},\mathbf{r}_0) dS \\
&= \int_S \left[-\frac{e^{ik\epsilon}}{4\pi\epsilon^2} + \frac{ike^{ik\epsilon}}{4\pi\epsilon}\right] dS \\
&= 4\pi\epsilon^2 \left[-\frac{e^{ik\epsilon}}{4\pi\epsilon^2} + \frac{ike^{ik\epsilon}}{4\pi\epsilon}\right] \to_{\epsilon\to 0} -1
\end{aligned}$$

Reciprocity

$$g_\omega(\mathbf{r},\mathbf{r}_0) = g_\omega(\mathbf{r}_0,\mathbf{r}) ,$$



Sources in a volume V bounded by the surface S.

Source in Bounded Medium

Inhomogeneous Helmholtz Equation

$$\left[\nabla^2 + k^2\right] \psi(\mathbf{r}) = f(\mathbf{r}) \,.$$

General Green's Function

$$G_\omega(\mathbf{r}, \mathbf{r}_0) = g_\omega(\mathbf{r}, \mathbf{r}_0) + H_\omega(\mathbf{r}) \,,$$

$$\left[\nabla^2 + k^2\right] H_\omega(\mathbf{r}) = 0 \,.$$

$$\Rightarrow$$

$$\left[\nabla^2 + k^2\right] G_\omega(\mathbf{r}, \mathbf{r}_0) \;=\; -\delta(\mathbf{r} - \mathbf{r}_0) \,.$$

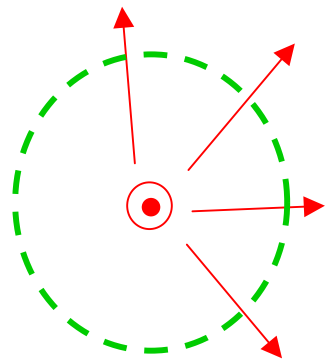
Green's Theorem

$$\psi(\mathbf{r}) = \int_S \left[G_\omega(\mathbf{r}, \mathbf{r}_0) \frac{\partial \psi(\mathbf{r}_0)}{\partial \mathbf{n}_0} - \psi(\mathbf{r}_0) \frac{\partial G_\omega(\mathbf{r}, \mathbf{r}_0)}{\partial \mathbf{n}_0} \right] dS_0 - \int_V f(\mathbf{r}_0)\, G_\omega(\mathbf{r}, \mathbf{r}_0)\, dV_0 ,$$

Source in infinite medium

$$\psi(\mathbf{r}) = -\int_V f(\mathbf{r}_0)\, g_\omega(\mathbf{r},\mathbf{r}_0)\, dV_0 \,.$$

For any imaginary surface enclosing the sources:



$$\int_S \left[g_\omega(\mathbf{r},\mathbf{r}_0)\, \frac{\partial \psi(\mathbf{r}_0)}{\partial \mathbf{n}_0} - \psi(\mathbf{r}_0)\, \frac{\partial g_\omega(\mathbf{r},\mathbf{r}_0)}{\partial \mathbf{n}_0} \right] dS_0 = 0 \,.$$

$$\Rightarrow \quad \int_S \frac{e^{ikR}}{4\pi R} \left[\frac{\partial \psi(\mathbf{r}_0)}{\partial R} - ik\, \psi(\mathbf{r}_0) \right] dS_0 = 0 \,.$$

Radiation condition

$$R \left[\frac{\partial}{\partial R} - ik \right] \psi(\mathbf{r}_0) \to 0 \,, \quad R = |\mathbf{r} - \mathbf{r}_0| \to \infty \,.$$

$$\Rightarrow \quad \int_S \frac{e^{ikR}}{4\pi R} \left[\frac{\partial \psi(\mathbf{r}_0)}{\partial R} - ik\, \psi(\mathbf{r}_0) \right] dS_0 = 0 \,.$$

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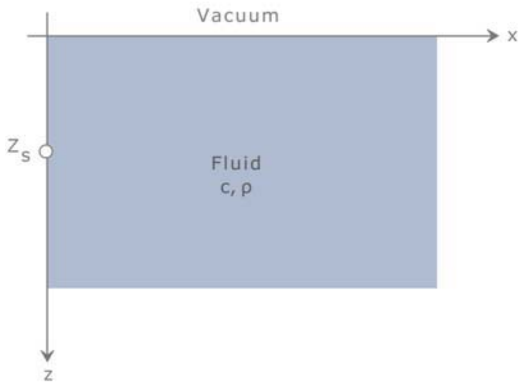

Point Source in Fluid Halfspace

Acoustic Pressure

$$p(\mathbf{r}) = \rho\omega^2\,\psi(\mathbf{r})\;,$$

Pressure-release boundary condition

$$\psi(\mathbf{r}_0) \equiv 0\,, \quad \mathbf{r}_0 = (x,y,0)\;.$$

Green's theorem

$$\psi(\mathbf{r}) = \int_S G_\omega(\mathbf{r},\mathbf{r}_0)\,\frac{\partial\psi(\mathbf{r}_0)}{\partial\mathbf{n}_0}\,dS_0 - \int_V f(\mathbf{r}_0)\,G_\omega(\mathbf{r},\mathbf{r}_0)\,dV_0\;.$$

Simple point source

$$f(\mathbf{r}_0) = S_\omega\,\delta(\mathbf{r}_0 - \mathbf{r}_s)\;.$$

Green's Function

$$\begin{aligned}
\text{Choose}\quad G_\omega(\mathbf{r},\mathbf{r}_0) &\equiv 0 \quad \text{for} \quad \mathbf{r}_0 = (x,y,0) \\
G_\omega(\mathbf{r},\mathbf{r}_0) &= g_\omega(\mathbf{r},\mathbf{r}_0) + H_\omega(\mathbf{r}) \\
&= \frac{e^{ikR}}{4\pi R} - \frac{e^{ikR'}}{4\pi R'} \\
&\Rightarrow \\
\psi(\mathbf{r}) &= -S_\omega\,G_\omega(\mathbf{r},\mathbf{r}_s)\;.
\end{aligned}$$

with

$$\begin{aligned}
R &= \sqrt{(x-x_s)^2 + (y-y_s)^2 + (z-z_s)^2}\;, \\
R' &= \sqrt{(x-x_s)^2 + (y-y_s)^2 + (z+z_s)^2}\;.
\end{aligned}$$

Acoustic Pressure

$$p(\mathbf{r}) = \rho\omega^2\psi(\mathbf{r}) = -\rho\omega^2 S_\omega\left[\frac{e^{ikR}}{4\pi R} - \frac{e^{ikR'}}{4\pi R'}\right]\;,$$

Transmission Loss

$$\text{TL}(\mathbf{r}, \mathbf{r}_s) = -20 \log_{10} \left| \frac{p(\mathbf{r}, \mathbf{r}_s)}{p(R = 1m)} \right| \; ,$$

$$\begin{aligned} p(R=1) &= \rho\omega^2 \psi(\omega, R=1) \\ &= -\rho\omega^2 S_\omega \frac{e^{ik}}{4\pi} = 1 \\ &\Rightarrow \\ S_\omega &= -\frac{4\pi}{\rho\omega^2} \end{aligned}$$

Transmission Loss Pressure

$$P(\mathbf{r}, \mathbf{r}_s) = \frac{p(\mathbf{r}, \mathbf{r}_s)}{p(R = 1m)} \; ,$$

where

$$\left[\nabla^2 + k^2\right] \Psi(\mathbf{r}, \mathbf{r}_s) = -\frac{4\pi}{\rho\omega^2} \, \delta(\mathbf{r} - \mathbf{r}_s) \; .$$

Transmission Loss Helmholtz Equation

$$\left[\nabla^2 + k^2\right] P(\mathbf{r}, \mathbf{r}_s) = -4\pi \, \delta(\mathbf{r} - \mathbf{r}_s) \; .$$

Density Variations

$$\rho \nabla \cdot \left[\rho^{-1} \nabla P(\mathbf{r}, \mathbf{r}_s)\right] + k^2 P(\mathbf{r}, \mathbf{r}_s) = -4\pi \, \delta(\mathbf{r} - \mathbf{r}_s) \; .$$