Ocean Acoustic Theory

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- Propagation in Layered Media
  - Reflection and Transmission
- The Ideal Waveguide
  - Image Method
  - Wavenumber Integral
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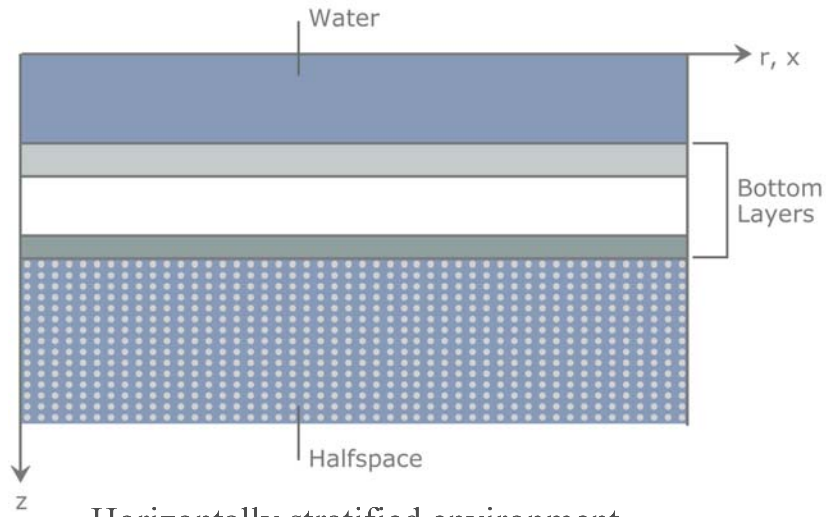
- Pekeris Waveguide



Horizontally stratified environment

Layered Media and Waveguides

Integral Transform Solution

Helmholtz Equation - Layer n

$$\left[\nabla^2 + k_n^2(z)\right] \psi(\mathbf{r}) = f(\mathbf{r}) \, ,$$

Interface Boundary Conditions

$$B\left[\psi(\mathbf{r})\right]|_{z=z_n} = 0 \, , \quad n = 1 \cdots N \, ,$$

Plane problems: Fourier Transform Solution

$$f(x,z) = \int_{-\infty}^{\infty} f(k_x,z)\, e^{ik_x x}\, dk_x \ ,$$
$$f(k_x,z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x,z)\, e^{-ik_x x}\, dx \ ,$$

Depth-Separated Wave Equation

$$\left[\frac{d^2}{dz^2} + (k^2 - k_x^2)\right]\psi(k_x,z) = S_\omega\,\frac{\delta(z-z_s)}{2\pi} \ .$$

Depth-Separated Green's Function

Superposition Principle

$$\psi(k_x,z) = -S_\omega\, G_\omega(k_x,z,z_s) = -S_\omega\,[g_\omega(k_x,z,z_s) + H_\omega(k_x,z)]$$

Source contribution: $$\left[\frac{d^2}{dz^2} + (k^2 - k_x^2)\right] g_\omega(k_x,z,z_s) = -\frac{\delta(z-z_s)}{2\pi}$$

Homogeneous Solution: $$\left[\frac{d^2}{dz^2} + (k^2 - k_x^2)\right] H_\omega(k_x,z) = 0$$

Interface Boundary Conditions

$$B\,[\psi(k_x,z_n)] = 0 \ .$$

Axisymmetric Propagation Problems: Hankel Transform Solution

$$\begin{aligned} f(r,z) &= \int_0^\infty f(k_r,z)\, J_0(k_r r)\, k_r\, dk_r \,, \\ f(k_r,z) &= \int_0^\infty f(r,z)\, J_0(k_r r)\, r\, dr \,, \end{aligned}$$

Depth-Separated Wave Equation

$$\left[\frac{d^2}{dz^2} + (k^2 - k_r^2)\right] \psi(k_r,z) = S_\omega \, \frac{\delta(z-z_s)}{2\pi} \,.$$

Superposition Principle

Depth-Separated Green's Function

$$\psi(k_r,z) = -S_\omega\, G_\omega(k_r,z,z_s) = -S_\omega\, [g_\omega(k_r,z,z_s) + H_\omega(k_r,z)]$$

Source contribution:

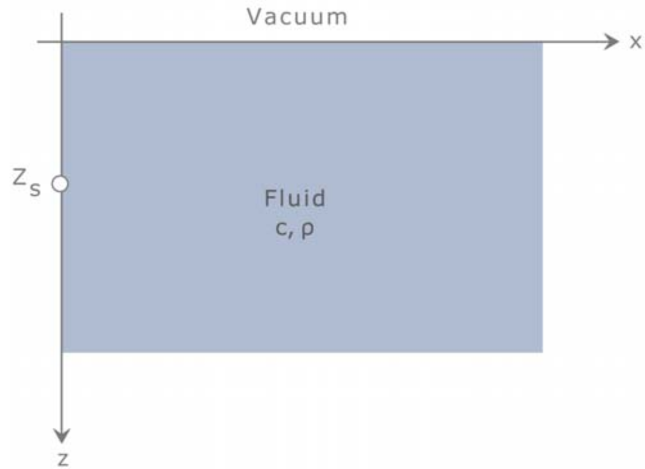
$$\left[\frac{d^2}{dz^2} + (k^2 - k_x^2)\right] g_\omega(k_x,z,z_s) \;=\; -\frac{\delta(z-z_s)}{2\pi}$$

Homogeneous Solution:

$$\left[\frac{d^2}{dz^2} + (k^2 - k_x^2)\right] H_\omega(k_x,z) \;=\; 0$$

Interface Boundary Conditions

$$B\,[\psi(k_r,z_n)] = 0 \,.$$



Point source in a fluid halfspace.

Example: Source in Fluid Halfspace

Homogeneous Solution

$$H_\omega(k_r, z) = A^+(k_r)\, e^{ik_z z} + A^-(k_r)\, e^{-ik_z z} \,,$$

Vertical Wavenumber

$$\begin{aligned} k_z &= \sqrt{k^2 - k_r^2} \\ &= \begin{cases} \sqrt{k^2 - k_r^2}\,, & k_r \le k \\ i\sqrt{k_r^2 - k^2}\,, & k_r > k\,. \end{cases} \end{aligned}$$

Radiating Waves

Evanescent Waves

Radiation Conditions

$$H_\omega(k_r, z) = \begin{cases} A^+(k_r)\, e^{ik_z z}\,, & z \to +\infty \\ A^-(k_r)\, e^{-ik_z z}\,, & z \to -\infty\,. \end{cases}$$

Source field

$$g_\omega(k_r,z,z_s) = A(k_r)\begin{cases} e^{ik_z(z-z_s)}\,, & z \ge z_s \\ e^{-ik_z(z-z_s)}\,, & z \le z_s \end{cases}$$

$$= A(k_r)\, e^{ik_z|z-z_s|}\,.$$

Integration of depth-separated wave equation over $[z_s-\epsilon, z_s+\epsilon]$:

$$\left[\frac{dg_\omega(k_r,z)}{dz}\right]_{z_s-\epsilon}^{z_s+\epsilon} + O(\epsilon) = -\frac{1}{2\pi}\,.$$

$$\Rightarrow$$

$$A(k_r) = -\frac{1}{4\pi i k_z}$$

$$\Rightarrow$$

$$g_\omega(k_r,z,z_s) = -\frac{e^{ik_z|z-z_s|}}{4\pi i k_z}\,.$$

Inverse Hankel Transform - Sommerfeld–Weyl Integral

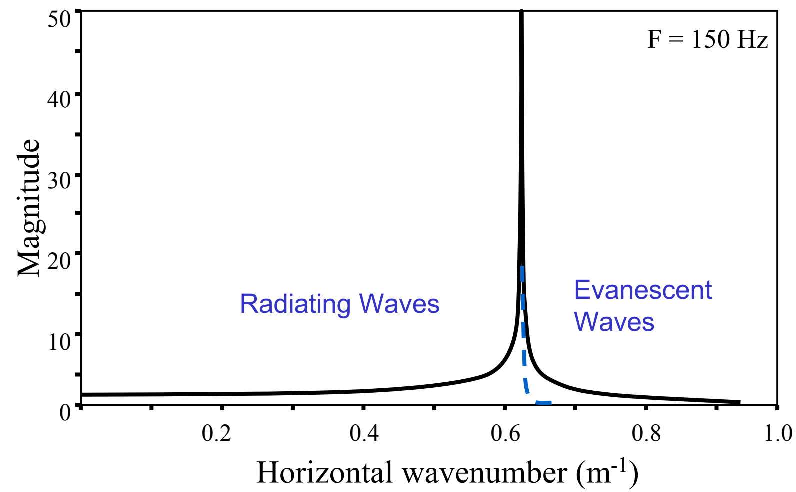
$$g_\omega(r,z,z_s) = \frac{i}{4\pi}\int_0^\infty \frac{e^{ik_z|z-z_s|}}{k_z}\, J_0(k_r r)\, k_r\, dk_r\,,$$

Grazing Angle Representation

$$k_x = k\cos\theta\,,$$

$$k_z = k\sin\theta\,,$$

$$\frac{dk_x}{d\theta} = -k_z\,.$$



Magnitude of the depth-dependent Green's function for point source in an infinite medium. Solid curve: $z - z_s = \lambda/10$; dashed curve: $z - z_s = 2\,\lambda$.

$$\Rightarrow$$

$$g_\omega(\mathbf{r},\mathbf{r}') \simeq \frac{i}{4\pi}\int_{-k}^{k} \frac{e^{ik_z|z-z_s|}}{k_z}\, e^{ik_x x}\, dk_x$$

$$= \frac{i}{4\pi}\int_0^\pi e^{ik\,|z-z_s|\sin\theta + ikx\cos\theta}\, d\theta\,.$$

Halfspace Problem: Surface and Radiation Conditions

$$\psi(k_r,0) \quad \equiv 0$$

$$\psi(k_r,z) \;\; \text{radiating for} \;\; z \to \infty$$

$$\Rightarrow$$

$$\begin{aligned}\psi(k_r,0) &= -S_\omega\left[g_\omega(k_r,0,z_s) + H_\omega(k_r,0)\right] \\ &= S_\omega\left[\frac{e^{ik_z z_s}}{4\pi i k_z} - A^+(k_r)\right] = 0\,,\end{aligned}$$

Total field

$$\psi(k_r,z) = S_\omega\left[\frac{e^{ik_z|z-z_s|}}{4\pi i k_z} - \frac{e^{ik_z(z+z_s)}}{4\pi i k_z}\right].$$

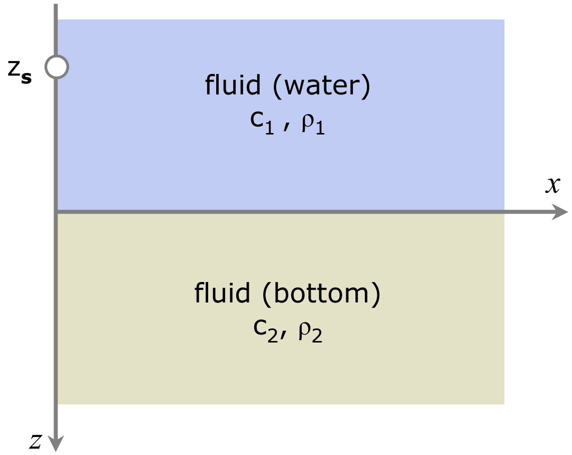
Loyd-Mirror Minima and Maxima

$$\begin{aligned}\sin\theta_{\max} &= \frac{(2m-1)\,\pi}{2kz_s}\,, \\ \sin\theta_{\min} &= \frac{(m-1)\,\pi}{kz_s}\,.\end{aligned}$$

Free Surface Reflection Coefficient

$$R = -1\,.$$

[See Fig 2.7 in Jensen, Kuperman, Porter and Schmidt. *Computational Ocean Acoustics.* New York: Springer-Verlag, 2000.]



Two fluid halfspaces

Reflection and Transmission

Homogeneous Solutions

$$H_{\omega,1}(k_r,z) \;=\; A_1^-(k_r)\, e^{-ik_{z,1}z}$$
$$H_{\omega,2}(k_r,z) \;=\; A_2^+(k_r)\, e^{ik_{z,2}z}$$

Source Field

$$g_{\omega,1}(k_r,z,z_s) \;=\; -\frac{e^{ik_z|z-z_s|}}{4\pi i k_z}\,.$$

Boundary Conditions

Continuity of vertical displacement

$$\frac{\partial \psi_1(k_r,z)}{\partial z} \;=\; \frac{\partial \psi_2(k_r,z)}{\partial z}\,, \qquad z=0$$

$$\Leftrightarrow$$

$$k_{z,2}\, A_2^+(k_r) + k_{z,1}\, A_1^-(k_r) \;=\; k_{z,1}\, g_{\omega,1}(k_r,0,z_s)\,.$$

Continuity of pressure

$$\rho_1\, \psi_1(k_r,z) \;=\; \rho_2\, \psi_2(k_r,z)\,, \qquad z=0$$

$$\Leftrightarrow$$

$$\rho_2\, A_2^+ - \rho_1\, A_1^- \;=\; \rho_1\, g_{\omega,1}(k_r,0,z_s)$$

Reflected and Transmitted Waves

$$A_1^- \;=\; \frac{\rho_2 k_{z,1} - \rho_1 k_{z,2}}{\rho_2 k_{z,1} + \rho_1 k_{z,2}}\, g_{\omega,1}(k_r,0,z_s)\,,$$

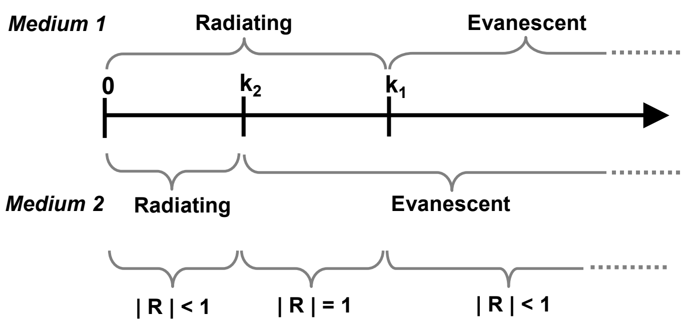
$$A_2^+ \;=\; \frac{2\rho_1 k_{z,1}}{\rho_2 k_{z,1} + \rho_1 k_{z,2}}\, g_{\omega,1}(k_r,0,z_s)\,.$$

Reflection Coefficient for Fluid–Fluid interface

$$R \;=\; \frac{\rho_2 k_{z,1} - \rho_1 k_{z,2}}{\rho_2 k_{z,1} + \rho_1 k_{z,2}}\,,$$

$$T \;=\; \frac{2\rho_1 k_{z,1}}{\rho_2 k_{z,1} + \rho_1 k_{z,2}}\,.$$

Example: Hard Bottom – $c_2 > c_1$



Spectral domains for a hard bottom, $k_2 < k_1$.

1. $k_r < k_2$: Waves are *propagating* vertically in both media and energy will be transmitted into the bottom: $|R| < 1$.
2. $k_2 < k_r < k_1$: Waves are *propagating* in the upper halfspace (water) but are *evanescent* in the lower halfspace (bottom): $|R| = 1$.
3. $k_1 < k_r$: Waves are *evanescent* in depth in both media: $|R| < 1$.

[see Jensen, Fig 2.10]

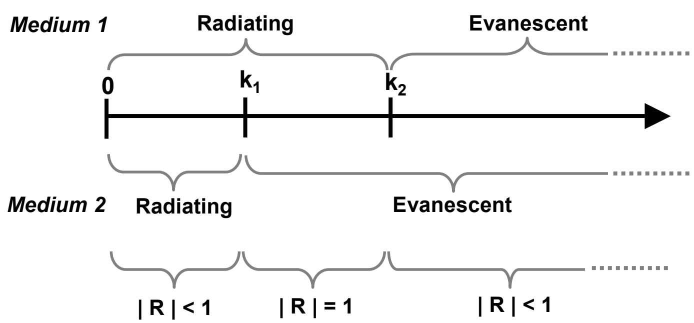
Magnitude and Phase

$$R(\theta) = |R(\theta)|\, e^{-i\phi(\theta)} \,,$$

Critical Angle

$$\theta_c = \arccos\left(k_2/k_1\right)$$

Example: Soft Bottom – $c_2 < c_1$



[see Jensen, Fig 2.12]

Spectral domains for a soft bottom, $k_1 < k_2$.

1. $k_r < k_1$: Waves are *propagating* vertically in both media and energy will be transmitted into the bottom: $|R| < 1$.
2. $k_1 < k_r < k_2$: Waves are *evanescent* in the upper halfspace (water) but *propagating* in the lower halfspace (bottom): $|R| = 1$.
3. $k_2 < k_r$: Waves are *evanescent* in depth in both media: $|R| < 1$.

The Point Source Field

Wavenumber Integration Kernel

$$\psi(k_r, z) = \begin{cases} -S_\omega \left[g_{\omega,1}(k_r, z, z_s) + H_{\omega,1}(k_r, z) \right] , & z < 0 \\ -S_\omega \, H_{\omega,2}(k_r, z) , & z > 0 , \end{cases}$$

Reflected Field

[see Jensen, Fig 2.14 and 2.15]

$$\begin{aligned} \psi_R(r, z) &= \int_0^\infty A_1^-(k_r) \, e^{-ik_{z,1}z} \, J_0(k_r r) \, k_r \, dk_r \\ &= \frac{1}{2} \int_{-\infty}^\infty A_1^-(k_r) \, e^{-ik_{z,1}z} \, H_0^{(1)}(k_r r) \, k_r \, dk_r \, . \end{aligned}$$

Farfield Approximation - $k_r r \gg 1$

$$\psi_R(r, z) = \frac{S_\omega \, e^{-i\pi/4}}{4\pi\sqrt{2\pi r}} \int_{-\infty}^\infty |R(k_r)| \, \frac{\sqrt{k_r}}{ik_{z,1}} \, e^{-i[\phi(k_r) + k_{z,1}(z+z_s) - k_r r]} \, dk_r \, .$$

The Point Source Field

Wavenumber Integration Kernel

$$\psi(k_r, z) = \begin{cases} -S_\omega\, [g_{\omega,1}(k_r, z, z_s) + H_{\omega,1}(k_r, z)]\,, & z < 0 \\ -S_\omega\, H_{\omega,2}(k_r, z)\,, & z > 0\,, \end{cases}$$

Reflected Field

$$\begin{aligned} \psi_R(r, z) &= \int_0^\infty A_1^-(k_r)\, e^{-ik_{z,1}z}\, J_0(k_r r)\, k_r\, dk_r \\ &= \frac{1}{2}\int_{-\infty}^\infty A_1^-(k_r)\, e^{-ik_{z,1}z}\, H_0^{(1)}(k_r r)\, k_r\, dk_r\,. \end{aligned}$$

Farfield Approximation - $k_r r \gg 1$

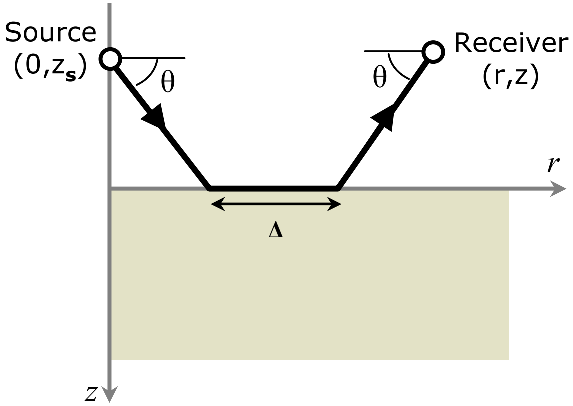
$$\psi_R(r, z) = \frac{S_\omega\, e^{-i\pi/4}}{4\pi\sqrt{2\pi r}} \int_{-\infty}^\infty |R(k_r)|\, \frac{\sqrt{k_r}}{ik_{z,1}}\, e^{-i[\phi(k_r)+k_{z,1}(z+z_s)-k_r r]}\, dk_r\,.$$

Stationary Phase Approximation - $k_{z,1}(z+z_s) \gg 1$,

$$\begin{aligned} \frac{\partial}{\partial k_r}\left[\phi(k_r) + k_{z,1}(z+z_s) - k_r r\right] &= 0 \\ &\Leftrightarrow \\ \frac{\partial \phi(k_r)}{\partial k_r} - \frac{k_r\,(z+z_s)}{k_{z,1}} - r &= 0 \end{aligned}$$

$$\Leftrightarrow$$

$$r = \frac{\partial \phi(k_r)}{\partial k_r} - (z+z_s)\cot\theta\,.$$



1. $\theta < \theta_c$: $\Delta = \partial\phi/\partial k_r > 0$,
2. $\theta > \theta_c$: $\Delta = \partial\phi/\partial k_r = 0$.

Critical Range

$$r_c = -(z+z_s)\cot\theta_c$$

Spherical wave incident on half-space: *direct, reflected, transmitted,* and *head/lateral/conical* waves

$$S(t) = \begin{cases} \sin(\omega_c t) - \frac{1}{2}\sin(2\omega_c t) & \text{for} \quad 0 < t < 1/f_c \\ 0 & \text{else} \end{cases} .$$

[See Fig 8.2 in Jensen, Kuperman, Porter and Schmidt. *Computational Ocean Acoustics.* New York: Springer-Verlag, 2000.]

Spherical wave incident on a halfspace