Lecture 4 - 2003
Pure Twist

pure twist around center of rotation D => neither axial (σ) nor bending forces (Mx, My) act on section; as previously, D is fixed, but (for now) arbitrary point.
as before:

a) equilibrium of wall element: $\frac{d}{ds}q + \left(\frac{d}{dx}\sigma\right)\cdot t = 0$

b) compatibility (shear strain) $\frac{d}{ds}u + \frac{d}{dx}v = \gamma = 0$ small deflections

c) tangential displacement (δv) in terms of η, ζ and ϕ (geometry)

$$\frac{\delta v}{\delta x} = \frac{\delta \eta}{\delta x}\cdot\cos(\alpha) + \frac{\delta \zeta}{\delta x}\cdot\sin(\alpha) + h_p\cdot\frac{\delta \phi}{\delta x}$$

N.B. h_p =>h_D from definition of problem

further assumptions:

1) preservation of cross section shape => $\zeta = \zeta(x)$; $\eta = \eta(x)$ $\phi = \phi(x)$
2) shear though finite is small ~ 0 => $\frac{d}{ds}u = -\left(\frac{d}{dx}v\right)$
3) Hooke's law holds => $\sigma = E\cdot\frac{\delta u}{\delta x}$ axial stress

---------------------------- from equilibrium --------------------------------

		pure twist
$\int \sigma\, dA = N_x$	$\int \tau\cdot h_p\, dA = \int q\cdot h_p\, ds = T_p$	$\int \sigma\, dA = N_x = 0$
$\int \sigma\cdot y\, dA = -M.z$	$\int \tau\cdot\cos(\alpha)\, dA = \int q\cdot\cos(\alpha)\, ds = V_y$	$\int \sigma\cdot y\, dA = -M_z = 0$
$\int \sigma\cdot z\, dA = M_y$	$\int \tau\cdot\sin(\alpha)\, dA = \int q\cdot\sin(\alpha)\, ds = V_z$	$\int \sigma\cdot z\, dA = M_y = 0$

pure twist also => only ϕ is finite i.e. other displacements (and derivatives) $\zeta = \eta = 0$ =>

$\frac{\delta v}{\delta x} = \frac{\delta \eta}{\delta x}\cdot\cos(\alpha) + \frac{\delta \zeta}{\delta x}\cdot\sin(\alpha) + h_p\cdot\frac{\delta \phi}{\delta x}$ becomes $\frac{\delta v}{\delta x} = h_D\cdot\frac{\delta \phi}{\delta x}$

using negligible shear assumption $\frac{d}{ds}u = -\left(\frac{d}{dx}v\right)$ => $\frac{d}{ds}u = -h_D\cdot\frac{\delta \phi}{\delta x}$ and integration along s =>

$$u = -\frac{\delta\phi}{\delta x}\cdot\int h_D\, ds + u_0(x)$$

previously $u = -\eta'\cdot Y - \zeta'\cdot Z + u_0(x)$ which showed u linear with y and z => plane sections plane.

here - only if h_D is constant so it can come outside $h_D\left(\int 1\, ds\right)$ - is u (longitudinal displacement) linear. u is defined as warping displacement (function).

stress analysis can be made analogous for torsion and bending IF the integrand $h_D\cdot ds$ thought to be a coordinate. calculation of stresses will involve statical moments, moments of inertia and products of inertia which will be designated "sectorial" new coordinate = Ω
Ω wrt arbitrary origin and ω wrt normalized sectorial coordinate (as before like wrt center of area)

$d\Omega = h_D\cdot ds = d\omega$ the warping function then becomes:

$$u = -\frac{\delta\phi}{\delta x}\cdot\Omega + u_0(x) = -\phi'\cdot\Omega + u_0(x)$$

b) warping stresses

as before: axial strain = du/dx => $u' = -\phi''\cdot\Omega + u'_0(x)$ and

$$\sigma = E\cdot u' = -E\cdot\phi''\cdot\Omega + E\cdot u'_0(x)$$

$\int \sigma\, dA = 0$ determines $u'_0(x)$ $\int \left(-E\cdot\phi''\cdot\Omega + E\cdot u'_0(x)\right) dA = 0$ =>

$$E\cdot u'_0(x) = E\cdot\phi''\frac{\int \Omega\, dA}{A}$$ and stress becomes: $$\sigma = -E\cdot\phi''\cdot\Omega + E\cdot\phi''\cdot\frac{\int \Omega\, dA}{A} = -E\cdot\phi''\cdot\omega$$

that is: $\sigma = -E\cdot\phi''\cdot\left(\Omega - \frac{\int \Omega\, dA}{A}\right) = -E\cdot\phi''\cdot\omega$ where $\omega = \Omega - \frac{\int \Omega\, dA}{A}$

this defines the ***normalized*** coordinate in the same sense as y and Y etc.

....

as an aside: $u' = -\phi''\cdot\omega$ => $u = -\phi'\cdot\omega + \text{constant}$ $d\omega = h_D\cdot ds$

in this sense, ω is defined as the unit warping function
displacement per unit change in rotation
dependent only on s within a constant

....

shear flow follows from integration of $\frac{d}{ds}q + \left(\frac{d}{dx}\sigma\right)\cdot t = 0$ along s as above and leads to :

$$\frac{d}{ds}q = -\left(\frac{d}{dx}\sigma\right)\cdot t \qquad => \qquad q(s,x) = -\int \frac{d}{dx}\sigma\cdot t\, ds + q_1(x)$$

using the expression for axial stress $\sigma = E\cdot u' = -E\cdot\phi''\cdot\omega$

$$q(s,x) = q_1(x) - \int_0^s \left(\frac{d}{dx}\sigma\right)\cdot t\, ds = q_1(x) - \int_0^s -E\cdot\phi'''\cdot\omega\cdot t\, ds = q_1(x) + E\cdot\phi'''\left(\int_0^s \omega\cdot t\, ds\right)$$

where $q_1(x)$ is f(z) and represents the shear flow at the start of the region. it is 0 at a stress free boundary which is convenient for an open section: $q_1(x) = 0$

as before if we designate the integrals which are the static moments of the cross section area: e.g. Qy and Qz:

$$Q_\omega = \int \omega\, dA = \int_0^s \omega\cdot t\, ds$$

= "sectorial statical moment of the cut-off portion of the cross section"

therefore: $q(s,x) = E\cdot\phi'''\cdot Q_\omega$

designate torsional moment wrt D by Tω $\int \tau\cdot h_D\, dA = \int q\cdot h_D\, ds = T_\omega$

now, since $d\Omega = h_D\cdot ds = d\omega$ => $\int q\cdot h_D\, ds = \int q\, d\omega$ and using integration by parts

parts: $\int u\, dv = (u\cdot v)(b) - (u\, v)(0) - \int v\, du$ $\qquad u = q \quad v = \omega \quad du = dq \quad dv = d\omega$

integration along s and as dq = δq/δs*ds

$$\int q\, d\omega = q\cdot\omega(s = b) - q\cdot\omega(s = 0) - \int \omega\, dq = q\cdot\omega(s = b) - q\cdot\omega(s = 0) - \int \omega\cdot\frac{\delta q}{\delta s}\, ds$$

$q\cdot\omega(s = b) = 0$ and $q\cdot\omega(s = 0) = 0$ as q(s=b) and q(s=0) = 0 (stress free ends)

now using equilibrium: $\frac{d}{ds}q + \left(\frac{d}{dx}\sigma\right)\cdot t = 0$

$$\int q\,d\omega = 0 - \int \omega\cdot\frac{\delta q}{\delta s}\,ds = \int \omega\cdot\frac{d}{dx}\sigma\cdot t\,ds$$ substituting $\sigma = -E\cdot\phi''\cdot\omega$ from above $\frac{d}{dx}\sigma = -E\cdot\phi'''\cdot\omega$ =>

$$\int q\,d\omega = \int \omega\cdot\frac{d}{dx}\sigma\cdot t\,ds = -E\cdot\phi'''\cdot\int \omega\cdot\omega\cdot t\,ds = -E\cdot\phi'''\cdot I_{\omega\omega}$$

where
similar to Iz $I_{\omega\omega} = \int \omega\cdot\omega\,dA = \int \omega\cdot\omega\cdot t\,ds$ $\quad I_z = \int y\cdot y\,dA = \int y\cdot y\cdot t\,ds$ N.B. sometimes this is represented by Iyy

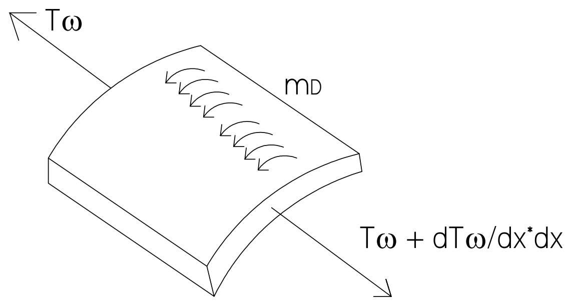
going back to the relationship for torsional moment, where we have derived relationships for $\int q\cdot h_D\,ds$

=>

$$T_\omega = \int q\cdot h_D\,ds = \int q\,d\omega = -E\cdot\phi'''\cdot I_{\omega\omega}$$ therefore: $\phi''' = \frac{-T_\omega}{E\cdot I_{\omega\omega}}$

if we think of a distributed torsional load (moment/unit length) m_D;

equilibrium over element dz =>



$$-T_\omega + m_D\cdot dx + \left[T_\omega + \left(\frac{d}{dx}T_\omega\right)\cdot dx\right] = 0$$ => $T'_\omega = -m_D$

and just as $M'_y = V_z$ the warping moment M'_ω may be defined as $M'_\omega = T_\omega$

thus: $\phi'' = \frac{-M_\omega}{E\cdot I_{\omega\omega}}$ and the stresses are as follows:

$\sigma = -E\cdot\phi''\cdot\omega = \frac{M_\omega}{I_{\omega\omega}}\cdot\omega$ and ... from $q(s,x) = E\cdot\phi'''\cdot Q_\omega$ $\quad q(s,x) = \tau\cdot t = \frac{-T_\omega}{I_{\omega\omega}}\cdot Q_\omega$

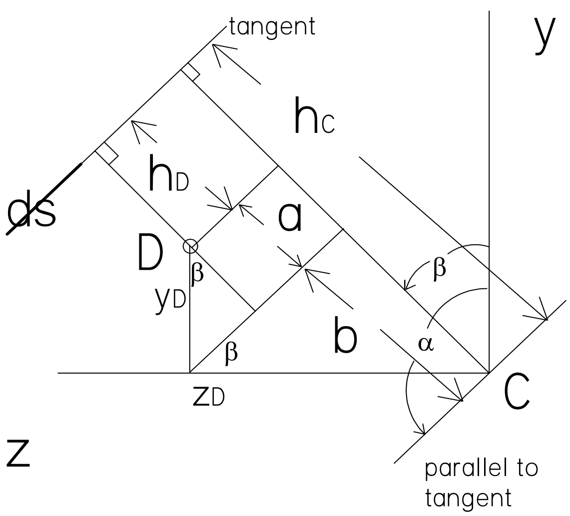
c) Center of twist

calculation of the sectorial quantities assumes center of twist y_D and z_D are known. the second and third equilibrium conditions above require:

$$\int \sigma \cdot y \, dA = 0 \qquad \int \sigma \cdot z \, dA = 0 \qquad \text{for pure twist}$$

using $\sigma = -E \cdot \phi'' \cdot \omega$ this requires $\int \omega \cdot y \, dA = 0$ and $\int \omega \cdot z \, dA = 0$ as $E \neq 0$ and $\phi'' \neq 0$

now for some geometry: determine distance from tangent to wall from h_D in terms of the coordinates of the center, the angle (α) that the y axis would have to rotate to line up with the positive direction of the tangent and the perpendicular distance from the origin of the centroidal coordinates h_C



$$h_D = h_C - a - b$$

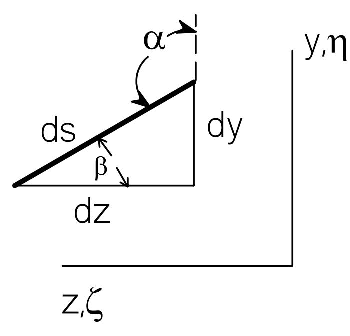
$$a = y_D \cdot \cos(\beta) \qquad b = z_D \cdot \sin(\beta)$$

$$\beta = \alpha - \frac{\pi}{2} \qquad \cos(\beta) = \cos\left(\alpha - \frac{\pi}{2}\right) = \sin(\alpha)$$

$$\sin(\beta) = \sin\left(\alpha - \frac{\pi}{2}\right) = -\cos(\alpha)$$

$$h_D = h_C - a - b = h_C - y_D \cdot \cos(\beta) - z_D \cdot \sin(\beta) = h_C - y_D \cdot \sin(\alpha) + z_D \cdot \cos(\alpha)$$

multiply by ds and apply geometry:



$$h_D \cdot ds = h_C \cdot ds - y_D \cdot \sin(\alpha) \cdot ds + z_D \cdot \cos(\alpha) \cdot ds$$

$$ds \cdot \cos(\beta) = dz = ds \cdot \sin(\alpha)$$

$$ds \cdot \sin(\beta) = -dy = ds \cdot (-\cos(\alpha)) \qquad dy = ds \cdot \cos(\alpha)$$

$$h_D \cdot ds = h_C \cdot ds - y_D \cdot \sin(\alpha) \cdot ds + z_D \cdot \cos(\alpha) \cdot ds = h_C \cdot ds - y_D \cdot dz + z_D \cdot dy$$

sectorial coordinate ω = hds => $h_D \cdot ds = d\omega_D = h_C \cdot ds - y_D \cdot dz + z_D \cdot dy = d\omega_C - y_D \cdot dz + z_D \cdot dy$

which is now integrated: $\omega_D = \omega_C - y_D \cdot z + z_D \cdot y$

and introduced into the equilibrium equations above where ω is ω_D:

$$\int \omega \cdot y \, dA = 0 \text{ and } \int \omega \cdot z \, dA = 0 \implies$$

$$\int \omega_D \cdot y \, dA = 0 = \int (\omega_C - y_D \cdot z + z_D \cdot y) \cdot y \, dA \quad \text{and ....} \quad \int \omega_D \cdot z \, dA = 0 = \int (\omega_C - y_D \cdot z + z_D \cdot y) \cdot z \, dA$$

now using second moment nomenclature (including treating ω as a coordinate) =>

$$\int \omega_C \cdot y \, dA - y_D \int y \cdot z \, dA + z_D \cdot \int y \cdot y \, dA = 0$$

becomes

$$I_{z\omega c} - y_D \cdot I_{yz} - z_D \cdot I_z = 0$$

recall that $I_{y\omega c}$ is referred to C for ω

and ....

$$\int \omega_C \cdot z \, dA - y_D \int z \cdot z \, dA + z_D \cdot \int y \cdot z \, dA = 0$$

becomes

$$I_{y\omega c} - y_D \cdot I_y + z_D \cdot I_{yz} = 0$$

which provides two equations in two unknowns y_D and z_D

Given

$$I_{y\omega c} - y_D \cdot I_y + z_D \cdot I_{yz} = 0 \qquad I_{z\omega c} - y_D \cdot I_{yz} + z_D \cdot I_z = 0 \qquad \begin{pmatrix} y_D \\ z_D \end{pmatrix} := \text{Find}(y_D, z_D)$$

$$y_D \rightarrow \frac{I_{y\omega c} \cdot I_z - I_{yz} \cdot I_{z\omega c}}{I_y \cdot I_z - I_{yz}^2} \quad \text{and ...} \quad z_D \rightarrow \frac{-I_{z\omega c} \cdot I_y + I_{yz} \cdot I_{y\omega c}}{I_y \cdot I_z - I_{yz}^2}$$

and for principal axes $I_{yz} = 0$

$$I_{yz} := 0 \qquad y_D := \frac{(I_{y\omega c} \cdot I_z - I_{yz} \cdot I_{z\omega c})}{(I_y \cdot I_z - I_{yz}^2)} \qquad z_D := \frac{(-I_{z\omega c} \cdot I_y + I_{yz} \cdot I_{y\omega c})}{(I_y \cdot I_z - I_{yz}^2)}$$

$$y_D \rightarrow \frac{I_{y\omega c}}{I_y} \quad \text{and ...} \quad z_D \rightarrow \frac{-I_{z\omega c}}{I_z}$$