Lecture 5 - 2003
Twist closed sections

As this development would be almost identical to that of the open section, some of the development is simply repeated (copied) from the open section development.
pure twist around center of rotation D => neither axial (σ) nor bending forces (Mx, My) act on section

---------------------------- from equilibrium -------------------------------

pure twist

$$\int \sigma \, dA = N_x \qquad \int \tau \cdot h_p \, dA = \int q \cdot h_p \, ds = T_p \qquad \int \sigma \, dA = 0$$

$$\int \sigma \cdot y \, dA = \text{-M.z} \qquad \int \tau \cdot \cos(\alpha) \, dA = \int q \cdot \cos(\alpha) \, ds = V_y \qquad \int \sigma \cdot y \, dA = 0$$

$$\int \sigma \cdot z \, dA = M_y \qquad \int \tau \cdot \sin(\alpha) \, dA = \int q \cdot \sin(\alpha) \, ds = V_z \qquad \int \sigma \cdot z \, dA = 0$$

a) equilibrium of wall element:

pure twist => . $\xi = \eta = 0$ =>

$$\frac{\delta v}{\delta x} = \frac{\delta \psi}{\delta x} \cdot \cos(\alpha) + \frac{\delta \eta}{\delta x} \cdot \sin(\alpha) + h_p \cdot \frac{\delta \phi}{\delta x} \qquad \text{becomes} \qquad \frac{\delta v}{\delta x} = h_D \cdot \frac{\delta \phi}{\delta x}$$

b) compatibility (shear strain)

$$\frac{d}{ds} u + \frac{d}{dx} v = \gamma$$

here is first change. we cannot set $\gamma = 0$ as we did in the open problem

=> $\frac{d}{ds} u = \gamma \cdot -\left(\frac{d}{dx} v\right)$ => $\frac{d}{ds} u = \frac{\tau}{G} \cdot -h_D \cdot \frac{\delta \phi}{\delta x}$

and integration along s =>

$$u = \int_0^s \frac{\tau}{G} \, ds - \frac{\delta \phi}{\delta x} \cdot \int h_D \, ds + u_0(x)$$

for open sections $u = -\frac{\delta \phi}{\delta x} \cdot \int h_D \, ds + u_0(x)$ as γ is small => = 0

other assumptions: section shape remains etc. same

$$\int_0^s \frac{\tau}{G}\,ds$$

See: Torsion of Thin-Walled, Noncircular Closed Shafts; Shames Section 14.5 particularly; equations 14.17, 14.18 and 14.21 (Bredt's formula)
also: Hughes 6.1.19, 6.1.21, 6.1.22 and section 6.1

$M_x = 2 \cdot q \cdot A$ $\quad q := \frac{M_x}{2 \cdot A}$ $\quad$ and ...... $M_x := G \cdot J \cdot \frac{\delta\phi}{\delta x}$ $\quad$ => $\quad q := \frac{G \cdot J}{2 \cdot A} \cdot \frac{\delta\phi}{\delta x}$

A in these relationships is the "swept area" i.e. per Shames; "total plane area vector of the area enclosed by the midline s." near 14.21

$$\int_0^s \frac{\tau}{G}\,ds = \frac{q}{G}\int_0^s \frac{1}{t}\,ds = \frac{G \cdot J \cdot \frac{\delta\phi}{\delta x}}{2 \cdot A \cdot G} \cdot \int_0^s \frac{1}{t}\,ds = \frac{J}{2 \cdot A} \cdot \int_0^s \frac{1}{t}\,ds \cdot \frac{\delta\phi}{\delta x}$$

$$J = \frac{4 \cdot A^2}{\int_0^b \frac{1}{t}\,ds}$$

integral 0 to b => circular (all way around) defining J from 14.21 (Bredt's formula)

$$u = \int_0^s \frac{\tau}{G}\,ds - \left(\int_0^s h_D\,ds\right) \cdot \frac{\delta\phi}{\delta x} + u_o(x) = \left(\frac{J}{2 \cdot A} \cdot \int_0^s \frac{1}{t}\,ds - \int_0^s h_D\,ds\right) \cdot \frac{\delta\phi}{\delta x} + u_o(x)$$

as with open sections define "sectorial" coordinate = Ω, by its derivative
Ω wrt arbitrary origin and ω wrt normalized sectorial coordinate

definition:

$$d\Omega = \left(h_D - \frac{J}{2 \cdot A} \cdot \frac{1}{t}\right) \cdot ds = d\omega \qquad \Omega = \int_0^s h_D\,ds - \frac{J}{2 \cdot A} \cdot \int_0^s \frac{1}{t}\,ds = \int_0^s h_D\,ds - 2 \cdot A \cdot \frac{\int_0^s \frac{1}{t}\,ds}{\int_0^b \frac{1}{t}\,ds}$$

the warping function then becomes (as previously): $u = -\frac{\delta\phi}{\delta x} \cdot \Omega + u_0(x) = -\phi' \cdot \Omega + u_0(x)$

the warping function Ω has a "correction" to the $\int_0^s h_D\,ds$ term of $-2 \cdot A \cdot \frac{\int_0^s \frac{1}{t}\,ds}{\int_0^b \frac{1}{t}\,ds}$

otherwise everything is identical. hD and hc still have same meaning in Ω and ω

b) warping stresses

axial stress:

as before: axial strain = du/dx => $u' = -\phi''\cdot\Omega + u'_0(x)$ and

$$\sigma = E\cdot u' = -E\cdot\phi''\cdot\Omega + E\cdot u'_0(x)$$

$\int \sigma \, dA = 0$ determines $u'_0(x)$ $\int \left(-E\cdot\phi''\cdot\Omega + E\cdot u'_0(x)'\right) dA = 0$ =>

$$E\cdot u'_0(x) = E\cdot\phi''\frac{\int \Omega \, dA}{A}$$

and stress becomes: $\sigma = -E\cdot\phi''\cdot\Omega + E\cdot\phi''\cdot\frac{\int \Omega \, dA}{A} = -E\cdot\phi''\cdot\omega$

that is: $\sigma = -E\cdot\phi''\cdot\left(\Omega - \frac{\int \Omega \, dA}{A}\right) = -E\cdot\phi''\cdot\omega$ where $\omega = \Omega - \frac{\int \Omega \, dA}{A}$

axial stress: $\sigma = -E\cdot\phi''\cdot\omega$

shear stress

shear flow follows from integration of $\frac{d}{ds}q + \left(\frac{d}{dx}\sigma\right)\cdot t = 0$ along s and leads to :

$$\frac{d}{ds}q = -\left(\frac{d}{dx}\sigma\right) \quad => \quad q(s,x) = -\int \frac{d}{dx}\sigma \, ds + q_1(x)$$

using the expression for axial stress $\sigma = E\cdot u' = -E\cdot\phi''\cdot\omega$

$$q(s,x) = q_1(x) - \int_0^s \left(\frac{d}{dx}\sigma\right)\cdot t \, ds = q_1(x) - \int_0^s -E\cdot\phi'''\cdot\omega\cdot t \, ds = q_1(x) + E\cdot\phi'''\left(\int_0^s \omega\cdot t \, ds\right)$$

where $q_1(x)$ is f(x) unlike open section we cannot set it = 0 $q_1(x) \neq 0$

we can superpose an open and closed problem setting the "slip" i.e. γ at an arbitrary cut = 0
this is equivalent to collecting all the s variation into the open solution and the x variation into the constant

$$q_{open}(s,x) = \tau_{open} \cdot t = \frac{-T_{\omega}}{I_{\omega\omega}} \cdot Q_{\omega} \qquad Q_{\omega} = \int \omega \, dA = \int_0^s \omega \cdot t \, ds$$

the ω derived above is the value with the constant of integration set to zero, i.e starting from open end.

$$q(s,x) = q_1(x) + q_{open}(s,x) = q_1(x) - \frac{T_{\omega}}{I_{\omega\omega}} \cdot Q_{\omega}$$

no slip => $\displaystyle \int \gamma \, ds = 0 = \int \frac{\tau}{G} \, ds = \int \frac{q}{t \cdot G} \, ds = 0$

N.B. these integrals are circular i.e. no slip results are for complete way around the closed section

$$\Rightarrow \quad 0 = \int \frac{q}{t} \, ds = \int \frac{q_1(x) - \frac{T_{\omega}}{I_{\omega\omega}} \cdot Q_{\omega}}{t} \, ds = q_1(x) \cdot \int \frac{1}{t} \, ds - \frac{T_{\omega}}{I_{\omega\omega}} \cdot \int Q_{\omega} \, ds$$

$$\Rightarrow \quad q_1(x) = \frac{\frac{T_{\omega}}{I_{\omega\omega}} \cdot \int Q_{\omega} \, ds}{\int \frac{1}{t} \, ds}$$

so we can say:

$$q(s,x) = -\left[\frac{T_{\omega}}{I_{\omega\omega}} \cdot \left(Q_{\omega} - \frac{\int Q_{\omega} \, ds}{\int \frac{1}{t} \, ds} \right) \right]$$

thus a "correction" $\dfrac{\int Q_{\omega} \, ds}{\int \frac{1}{t} \, ds}$ is applied to Q_{ω} for the closed section. the ω is for the closed section (with it's correction applied) *c) Center of twist*

as for an open section, the second and third equilibrium condition above requires:

$$\int \sigma \cdot y \, dA = 0 \qquad \int \sigma \cdot z \, dA = 0 \qquad \text{for pure twist}$$

using $\sigma = -E \cdot \phi'' \cdot \omega$ this requires $\int \omega \cdot y \, dA = 0$ and $\int \omega \cdot z \, dA = 0$ as $E \neq 0$ and $\phi'' \neq 0$

as shown above this relationship is identical with the new "corrrected" ω so the shear center and center of twist can be calculated the same way.

$$y_D = \frac{\left(I_{y\omega c} \cdot I_z - I_{yz} \cdot I_{z\omega c}\right)}{\left(I_y \cdot I_z - I_{yz}^{\ 2}\right)} \qquad \text{and ...} \qquad z_D = \frac{\left(-I_{z\omega c} \cdot I_y + I_{yz} \cdot I_{y\omega c}\right)}{\left(I_y \cdot I_z - I_{yz}^{\ 2}\right)}$$

and for principal axes $I_{yz} = 0$ $\qquad I_{yz} := 0$

$$y_D := \frac{\left(I_{y\omega c} \cdot I_z - I_{yz} \cdot I_{z\omega c}\right)}{\left(I_y \cdot I_z - I_{yz}^{\ 2}\right)} \qquad z_D := \frac{\left(-I_{z\omega c} \cdot I_y + I_{yz} \cdot I_{y\omega c}\right)}{\left(I_y \cdot I_z - I_{yz}^{\ 2}\right)}$$

$$y_D \rightarrow \frac{I_{y\omega c}}{I_y} \qquad \text{and ...} \qquad z_D \rightarrow \frac{-I_{z\omega c}}{I_z}$$