Plate Bending

not so long plate

previously have shown: $M := -D \cdot \frac{d^2}{dx^2} w$. this was for single axis bending.

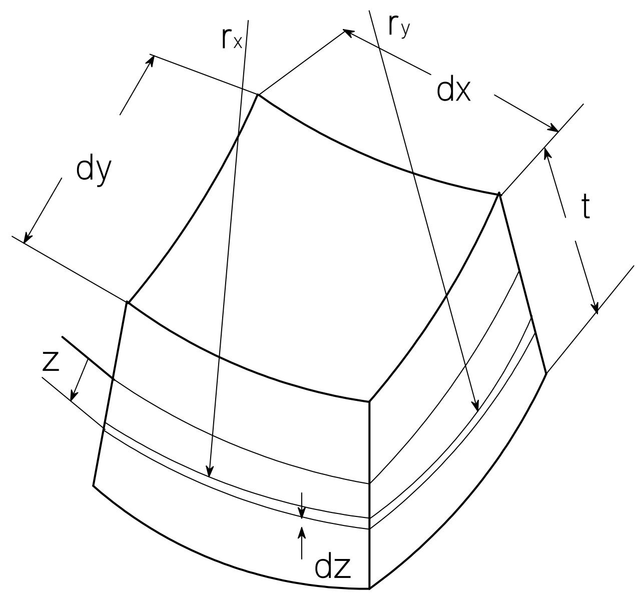
this relationship holds for the partial derivative in the respective direction fo both x and y;

assumptions:

plane cross section remains plane

small deflections $w_{max} < 3/4\ t$

stress < yield



$w = w(x, y)$

$\frac{1}{R_x} = -\frac{d^2}{dx^2} w(x,y)$ $\frac{1}{R_y} = -\frac{d^2}{dy^2} w(x,y)$

we are making no statement with respect to ε_y (or any other ε) as we did in long plate.

$\varepsilon_x := \frac{1}{R_x} \cdot z$ $\varepsilon_x := -z \cdot \frac{d^2}{dx^2} w(x,y)$ $\varepsilon_y := \frac{1}{R_y} \cdot z$ $\varepsilon_y := -z \cdot \frac{d^2}{dy^2} w(x,y)$

$\varepsilon_x := \frac{\sigma_x}{E} - \frac{\nu \cdot \sigma_y}{E}$ $\varepsilon_y := \frac{\sigma_y}{E} - \frac{\nu \cdot \sigma_x}{E}$ $\nu \cdot \varepsilon_y \rightarrow$

$\varepsilon_x + \nu \cdot \varepsilon_y$ collect, $\sigma_x, E \rightarrow$ => $\sigma_x := \frac{E}{1 - \nu^2} \cdot (\varepsilon_x + \nu \cdot \varepsilon_y)$

substituting

into

$$\varepsilon_y := -z\frac{d^2}{dy^2}w(x,y) \qquad \varepsilon_x := -z\cdot\frac{d^2}{dx^2}w(x,y) \qquad \sigma_x := \frac{E}{1-\nu^2}\cdot\left(\varepsilon_x + \nu\cdot\varepsilon_y\right)$$

$$\sigma_x \rightarrow \frac{E}{1-\nu^2}\cdot\left(-z\cdot\frac{d}{dx}\frac{d}{dx}w(x,y) - \nu\cdot z\cdot\frac{d}{dy}\frac{d}{dy}w(x,y)\right)$$

or ...

$$\sigma_x(z) := -z\frac{E}{1-\nu^2}\cdot\left(\frac{d^2}{dx^2}w(x,y) + \nu\cdot\frac{d^2}{dy^2}w(x,y)\right)$$

similarly:

$$\sigma_y := -z\frac{E}{1-\nu^2}\cdot\left(\frac{d^2}{dy^2}w(x,y) + \nu\cdot\frac{d^2}{dx^2}w(x,y)\right)$$

as in bending (applied in each of x and y direction): see figure Hughes 9.3 (below)
the lower case m denotes moment per unit length
note that the designation is changed: the subscript on the m refers to the direction of axial stress

$$m_x := \int_{\frac{-t}{2}}^{\frac{t}{2}} \sigma_x(z)\cdot z\, dz \qquad m_y := \int_{\frac{-t}{2}}^{\frac{t}{2}} \sigma_y\cdot z\, dz$$

$$m_x \rightarrow \frac{1}{12}\cdot t^3\cdot E\cdot\frac{\frac{d}{dx}\frac{d}{dx}w(x,y) + \nu\cdot\frac{d}{dy}\frac{d}{dy}w(x,y)}{-1+\nu^2} \qquad m_y \rightarrow \frac{1}{12}\cdot t^3\cdot E\cdot\frac{\frac{d}{dy}\frac{d}{dy}w(x,y) + \nu\cdot\frac{d}{dx}\frac{d}{dx}w(x,y)}{-1+\nu^2}$$

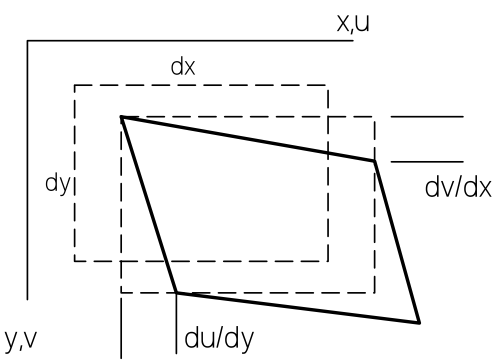
with

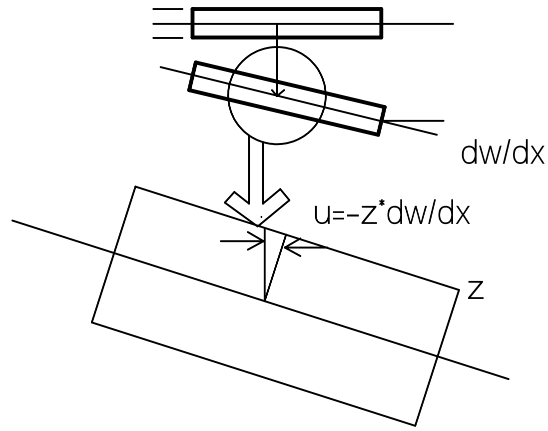
$$D := \frac{E\cdot t^3}{12\cdot\left(1-\nu^2\right)}$$

$$m_x := -D\cdot\left[\frac{d^2}{dx^2}w(x,y) + \nu\left(\frac{d^2}{dy^2}w(x,y)\right)\right] \qquad m_y := -D\cdot\left(\frac{d^2}{dy^2}w(x,y) + \nu\,\frac{d^2}{dx^2}w(x,y)\right)$$

in the general plate case there may exist a twisting moment m_{xy}
we need to derive a similar expresssion for this moment

from basic shear strain relationships: $\gamma := \frac{d}{dx}v + \frac{d}{dy}u$; $\tau := G\cdot\gamma$ and $\tau := G\cdot\frac{d}{dx}v + \frac{d}{dy}u$





from the geometry of the slope of w in each direction (x and y): $u := -z\cdot\frac{d}{dx}w(x,y)$ and $v := -z\cdot\frac{d}{dy}w(x,y)$

=> $\tau := -2\cdot G\cdot z\cdot\frac{d}{dx}\frac{d}{dy}w(x,y)$ and the twisting moment per unit length is determined by:

$$m_{xy} := -\int_{\frac{-t}{2}}^{\frac{t}{2}} \tau\cdot z\,dz \qquad m_{xy} \rightarrow \frac{1}{6}\cdot t^3\cdot G\cdot\frac{d}{dy}\frac{d}{dx}w(x,y)$$

minus sign comes from sense of mxy + and τxy(+)*z(+) see figure Hughes 9.3 below

using: $G := \frac{E}{2\cdot(1+\nu)}$ => $m_{xy} := \frac{d}{dx}\left(\frac{d}{dy}w(x,y)\right)\cdot\frac{E t^3}{12\cdot(1+\nu)}$

multiply and divide by (1-ν) => $m_{xy} := \frac{d}{dx}\left(\frac{d}{dy}w(x,y)\right)\cdot\frac{E t^3}{12\cdot(1+\nu)}\cdot\frac{(1-\nu)}{(1-\nu)}$

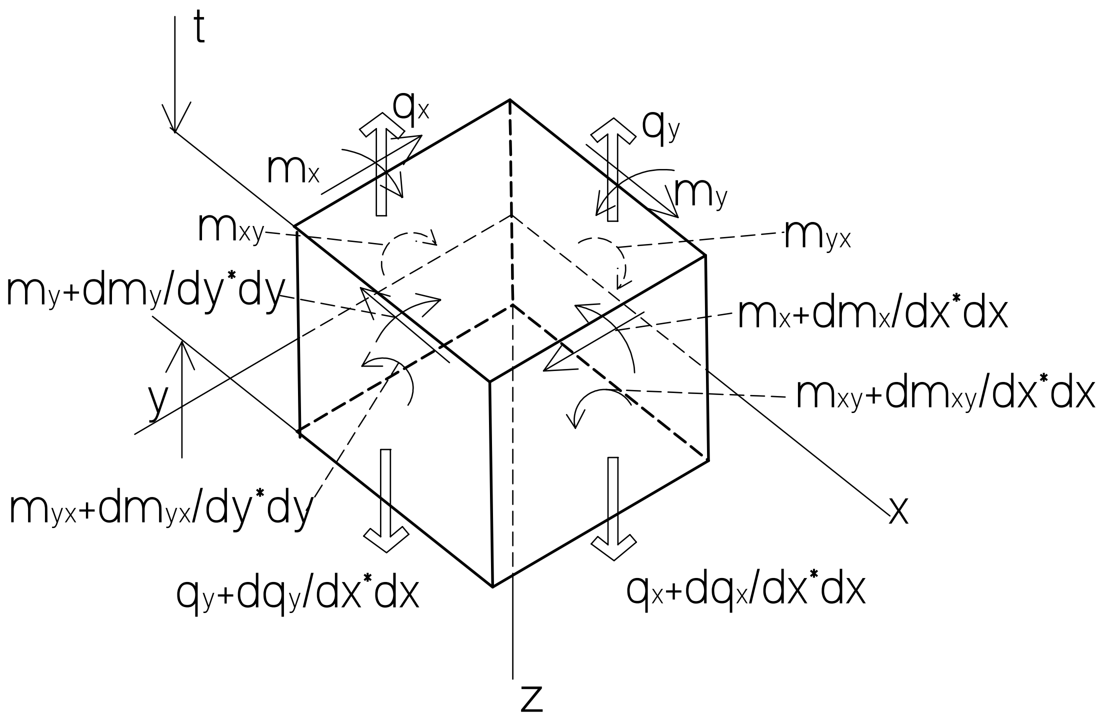
$m_{xy} := \frac{d}{dx}\left(\frac{d}{dy}w(x,y)\right)\cdot\frac{E t^3}{12\cdot\left(1-\nu^2\right)}\cdot(1-\nu)$ => $m_{xy} := D\cdot(1-\nu)\left[\frac{d}{dx}\left(\frac{d}{dy}w(x,y)\right)\right]$

due to the complimentary nature of shear stress $\tau_{xy} := \tau_{yx}$

and due to the sign convention for + moment $m_{xy} := -m_{yx}$

=> $m_{yx} := -D\cdot(1-\nu)\left[\frac{d}{dx}\left(\frac{d}{dy}w(x,y)\right)\right]$

now for equilibrium of the dx, dy, dz segment:
see Hughes figure 9.3 forces and moments in a plate (below)



Hughes figure 9.3 forces and moments in a plate

q in this context is the shear force per unit length on each face:
q_x * the distance dy is the force on the x face etc...

equilibrium of vertical forces => p is the lateral (+z direction) load per unit area - not shown on figure

$$\left[q_x + \left(\frac{d}{dx}q_x\right)\cdot dx\right]\cdot dy - q_x\cdot dy + \left[q_y + \left(\frac{d}{dy}q_y\right)\cdot dy\right]\cdot dx - q_y\cdot dx + p\cdot dx\cdot dy = 0 \qquad \Rightarrow \qquad \frac{d}{dx}q_x + \frac{d}{dy}q_y + p = 0$$

moments wrt the x axis =>

$$\left[m_{xy} + \left(\frac{d}{dx}m_{xy}\right)\cdot dx\right]\cdot dy - m_{xy}\cdot dy + m_y\cdot dx - \left[m_y + \left(\frac{d}{dy}m_y\right)\cdot dy\right]\cdot dx + q_y\cdot dx\cdot dy = 0 \qquad \Rightarrow \frac{d}{dx}m_{xy} - \frac{d}{dy}m_y + q_y = 0$$

and similarly: taking moments wrt the y-axis =>

$$\frac{d}{dy}m_{yx} + \frac{d}{dx}m_x - q_x = 0$$

or ... since $m_{xy} := -m_{yx}$ $\quad -\left(\frac{d}{dy}m_{xy}\right) + \frac{d}{dx}m_x - q_x = 0$

substituting the moment relations

$q_x := -\left(\frac{d}{dy}m_{xy}\right) + \frac{d}{dx}m_x$ and $q_y := -\left(\frac{d}{dx}m_{xy}\right) + \frac{d}{dy}m_y$

into the shear equation => $\frac{d}{dx}\left[-\left(\frac{d}{dy}m_{xy}\right) + \frac{d}{dx}m_x\right] + \frac{d}{dy}\left[-\left(\frac{d}{dx}m_{xy}\right) + \frac{d}{dy}m_y\right] + p = 0$

$$\frac{d^2}{dx^2}m_x - 2\cdot\frac{d}{dx}\frac{d}{dy}m_{yx} + \frac{d^2}{dy^2}m_y + p = 0$$

substituting the relations fro m_x, m_y and m_{xy} above =>

$$\frac{d^2}{dx^2}\left[-D\cdot\left[\frac{d^2}{dx^2}w(x,y) + \nu\left(\frac{d^2}{dy^2}w(x,y)\right)\right]\right] - 2\cdot\frac{d}{dx}\left[\frac{d}{dy}D\cdot(1-\nu)\left[\frac{d}{dx}\left(\frac{d}{dy}w(x,y)\right)\right]\right] + \frac{d^2}{dy^2}\left[-D\cdot\left(\frac{d^2}{dy^2}w(x,y) + \nu\cdot\frac{d^2}{dx^2}w(x,y)\right)\right]$$
$$= -p$$

the terms with ν cancel and the result is: $\frac{d^4}{dx^4}w(x,y) + 2\cdot\frac{d^2}{dx^2}\frac{d^2}{dy^2}w(x,y) + \frac{d^4}{dy^4}w(x,y) = \frac{p}{D}$