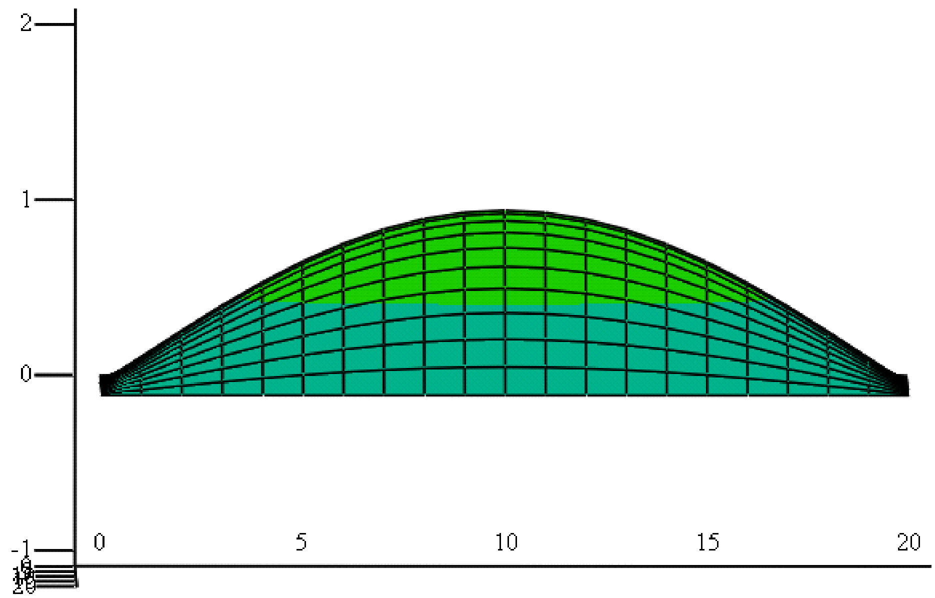
Solution of Plate Bending Equation
Uniform Load Simply Supported Free to pull in
via sinusoidal loading

loading $$p(x,y) := p_o \cdot \sin\left(\pi \cdot \frac{y}{a}\right) \cdot \sin\left(\pi \cdot \frac{x}{b}\right)$$



pxy

$w = 0$ $m_x = m_y = 0$ for $x = 0$ $y = 0$ $x = b$ $y = a$

$m_x = m_y = 0$ => $$\frac{d^2}{dx^2}w = \frac{d^2}{dy^2}w = 0$$ $x = 0$ $y = 0$ $x = b$ $y = a$

all boundary conditions satisfied if take $w(x,y) := C\cdot\sin\left(\pi\cdot\frac{y}{a}\right)\cdot\sin\left(\pi\cdot\frac{x}{b}\right)$

substitute in plate equation:

$$\frac{d^4}{dx^4}w(x,y) + 2\cdot\frac{d^2}{dx^2}\frac{d^2}{dy^2}w(x,y) + \frac{d^4}{dy^4}w(x,y) = \frac{p_o\cdot\sin\left(\pi\cdot\frac{y}{a}\right)\cdot\sin\left(\pi\cdot\frac{x}{b}\right)}{D}$$

$$\frac{d^4}{dx^4}w(x,y) + 2\cdot\frac{d^2}{dx^2}\frac{d^2}{dy^2}w(x,y) + \frac{d^4}{dy^4}w(x,y) \rightarrow 4\cdot C\cdot\sin(\pi\cdot y)\cdot\sin(\pi\cdot x)\cdot\pi^4$$

after collecting terms:

$$\frac{d^4}{dx^4}w(x,y) + 2\cdot\frac{d^2}{dx^2}\frac{d^2}{dy^2}w(x,y) + \frac{d^4}{dy^4}w(x,y) = C\cdot\sin\left(\pi\cdot\frac{y}{a}\right)\cdot\sin\left(\pi\cdot\frac{x}{b}\right)\cdot\left(\frac{\pi^4}{b^4} + 2\cdot\frac{\pi^4}{a^2\cdot b^2} + \frac{\pi^4}{a^4}\right)$$

$$= \quad C\cdot\sin\left(\pi\cdot\frac{y}{a}\right)\cdot\sin\left(\pi\cdot\frac{x}{b}\right)\cdot\left(\frac{\pi^2}{b^2} + \frac{\pi^2}{a^2}\right)^2$$

is a solution if

$$C\cdot\pi^4\cdot\sin\left(\pi\cdot\frac{y}{a}\right)\cdot\sin\left(\pi\cdot\frac{x}{b}\right)\cdot\left(\frac{1}{b^2} + \frac{1}{a^2}\right)^2 = \frac{p_o\cdot\sin\left(\pi\cdot\frac{y}{a}\right)\cdot\sin\left(\pi\cdot\frac{x}{b}\right)}{D}$$

is a solution if

$$C\cdot\pi^4\cdot\left(\frac{1}{b^2} + \frac{1}{a^2}\right)^2 = \frac{p_o}{D}$$

or ....

$$C = \frac{p_o}{D\cdot\pi^4}\cdot\frac{1}{\left(\frac{1}{b^2} + \frac{1}{a^2}\right)^2}$$

$$w(x,y) := \frac{p_o}{D\cdot\pi^4}\cdot\frac{1}{\left(\frac{1}{b^2} + \frac{1}{a^2}\right)^2}\cdot\sin\left(\pi\cdot\frac{y}{a}\right)\cdot\sin\left(\pi\cdot\frac{x}{b}\right)$$

is the displacement for a sinusoidal loading in x and y
moments and stresses are determined from:

$$m_x := -D\cdot\left[\frac{d^2}{dx^2}w(x,y) + \nu\left(\frac{d^2}{dy^2}w(x,y)\right)\right] \qquad m_y := -D\cdot\left(\frac{d^2}{dy^2}w(x,y) + \nu\cdot\frac{d^2}{dx^2}w(x,y)\right)$$

and $\sigma_x := \frac{m_x}{I} \cdot z_{max}$ $I := \frac{t^3}{12}$ $\sigma_x := \frac{m_x}{I} \cdot \frac{t}{2}$ $\sigma_x := m_x \cdot \frac{6}{t^2}$

this result can first be generalized to:

$$w(x,y) := \frac{p_o}{D \cdot \pi^4} \cdot \frac{1}{\left(\frac{n^2}{b^2} + \frac{m^2}{a^2}\right)^2} \cdot \sin\left(m\pi \cdot \frac{y}{a}\right) \cdot \sin\left(n\pi \cdot \frac{x}{b}\right)$$

for a loading of a higher order sinusoidal loading

$$p(x,y) := p_o \cdot \sin\left(m\pi \cdot \frac{y}{a}\right) \cdot \sin\left(n\pi \cdot \frac{x}{b}\right)$$

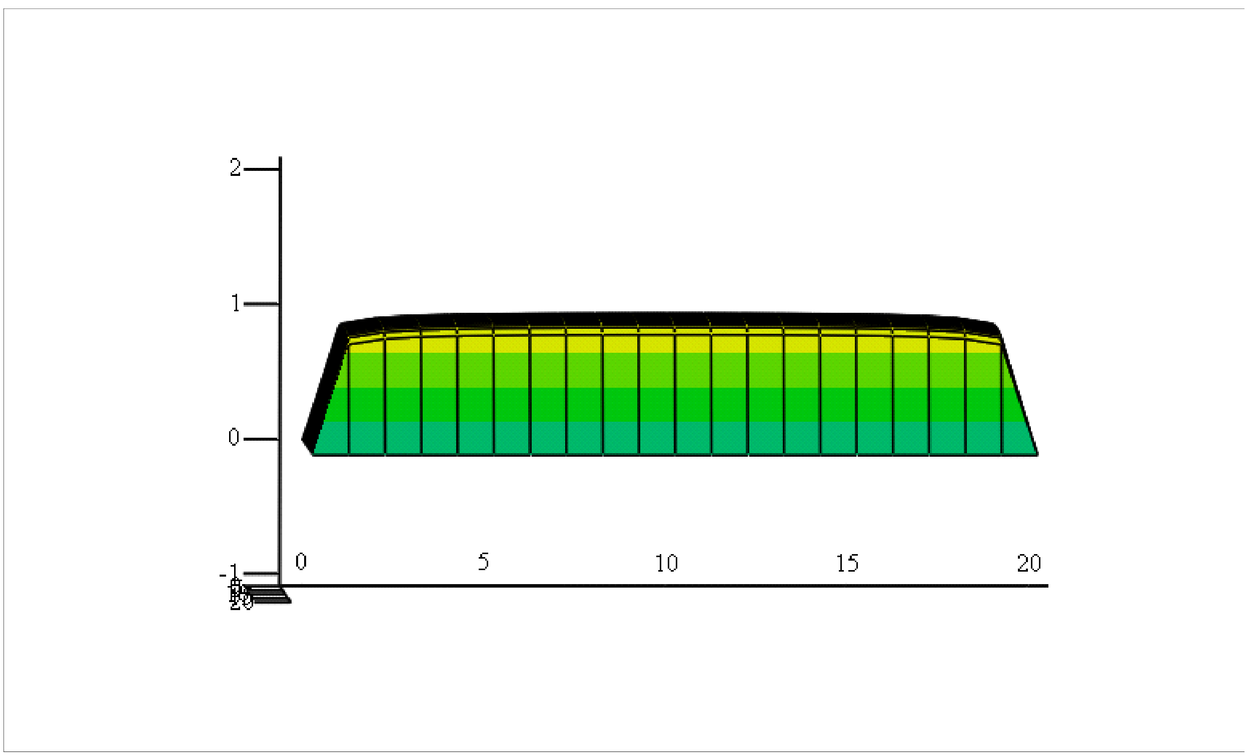
now consider a uniform load: p_o

represent p_o in a double fourier series:

$$p = f(x,y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{mn} \cdot \sin\left(\frac{m \cdot \pi \cdot y}{a}\right) \cdot \sin\left(\frac{n \cdot \pi \cdot x}{b}\right)$$

coefficients amn can be determined and are: $a_{mn} = \frac{16 \cdot p_o}{\pi^2 \cdot m \cdot n}$ odd coefficients even = 0

for example on a square plate with infinity = 20 i.e. 20 terms in the series: $N \equiv 20$ $M \equiv 20$



eacl pxy

$a_{mn} \cdot \sin\left(\frac{\quad}{a}\right) \cdot \sin\left(\frac{\quad}{b}\right)$

note here that we are using m, and n odd the displacement for each loading element

is ...

$$w(x,y) := \frac{1}{D\cdot\pi^4}\cdot\frac{a_{mn}}{\left(\frac{n^2}{b^2}+\frac{m^2}{a^2}\right)^2}\cdot\sin\left(\frac{m\cdot\pi\cdot y}{a}\right)\cdot\sin\left(\frac{n\cdot\pi\cdot x}{b}\right)$$

from above

this result can first be generalized to:

for a loading of a higher order sinusoidal loading

$$w(x,y) := \frac{p_o}{D\cdot\pi^4}\cdot\frac{1}{\left(\frac{n^2}{b^2}+\frac{m^2}{a^2}\right)^2}\cdot\sin\left(m\pi\cdot\frac{y}{a}\right)\cdot\sin\left(n\pi\cdot\frac{x}{b}\right)$$

$$p(x,y) := p_o\cdot\sin\left(m\pi\cdot\frac{y}{a}\right)\cdot\sin\left(n\pi\cdot\frac{x}{b}\right)$$

so by superposition of lots of the components of the fourier expansion of p_o is ...

$$w(x,y) := \frac{1}{\pi^4\cdot D}\cdot\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_{mn}}{\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2}\cdot\sin\left(\frac{m\cdot\pi\cdot y}{a}\right)\cdot\sin\left(\frac{n\cdot\pi\cdot x}{b}\right)$$

$$a_{mn} := \frac{16\cdot p_o}{\pi^2\cdot m\cdot n}$$

substituting for amn

$$w(x,y) := \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{16\cdot p_o}{\pi^6\cdot D\cdot m\cdot n\cdot\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2}\cdot\sin\left(m\cdot\pi\cdot\frac{y}{a}\right)\cdot\sin\left(n\cdot\pi\cdot\frac{x}{b}\right)$$

m, n, odd => substitute 2*m-1 and 2*n-1 for m, n

$$w(x,y) := \sum^{\infty}\sum^{\infty}\frac{16\cdot p_o}{\qquad\qquad 2}\cdot\sin\left[(2\cdot m-1)\cdot\pi\cdot\frac{y}{a}\right]\cdot\sin\left[(2\cdot n-1)\cdot\pi\cdot\frac{x}{b}\right]$$

$$\sum_{m=1}\ \sum_{n=1}\ \frac{\quad}{\pi^6\cdot D\cdot(2\cdot m-1)\cdot(2\cdot n-1)\cdot\left[\frac{(2\cdot m-1)^2}{a^2}+\frac{(2\cdot n-1)^2}{b^2}\right]^2}\quad \lfloor \qquad a\rfloor \quad \lfloor \qquad b\rfloor$$

if we want to look at maximum deflection: x = b/2 y = a/2 expand here

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from previous lecture:

$$m_x := -D\cdot\left[\frac{d^2}{dx^2}w(x,y)+\nu\left(\frac{d^2}{dy^2}w(x,y)\right)\right] \qquad m_y := -D\cdot\left(\frac{d^2}{dy^2}w(x,y)+\nu\cdot\frac{d^2}{dx^2}w(x,y)\right)$$

and we can solve just as above for the single half waves:

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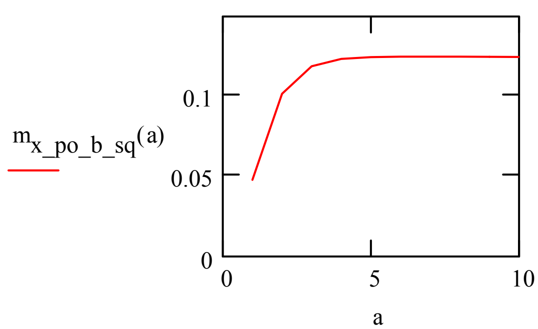
plot as a function of a/b i.e. a with b = 1

$a := 1..10 \qquad b := 1 \qquad \nu := 0.3$

$M := 10 \qquad N := 10$

$$m_{x\_po\_b\_sq}(a) := \sum_{m=1}^{M}\ \sum_{n=1}^{N} \frac{16}{\pi^4\cdot(2\cdot m-1)\cdot(2\cdot n-1)\cdot\left[\frac{(2\cdot m-1)^2}{\left(\frac{a}{b}\right)^2}+(2\cdot n-1)^2\right]^2}\cdot\left[(2\cdot n-1)^2+\nu\cdot\frac{(2\cdot m-1)^2}{\left(\frac{a}{b}\right)^2}\right]\cdot(-1)^{(m}$$



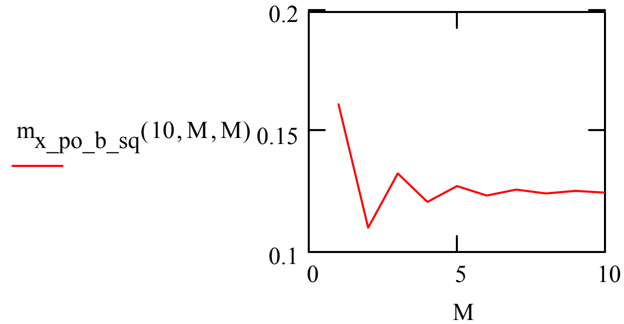
$m_{x\_po\_b\_sq}(10) = 0.125$

what length (ratio) is needed to declare a plate long???

see how many terms we need to obtain convergence:

$$m_{x\_po\_b\_sq}(a,M,N) := \sum_{m=1}^{M}\ \sum_{n=1}^{N} \frac{16}{\pi^4\cdot(2\cdot m-1)\cdot(2\cdot n-1)\cdot\left[\frac{(2\cdot m-1)^2}{\left(\frac{a}{b}\right)^2}+(2\cdot n-1)^2\right]^2}\cdot\left[(2\cdot n-1)^2+\nu\cdot\frac{(2\cdot m-1)^2}{\left(\frac{a}{b}\right)^2}\right]\cdot($$

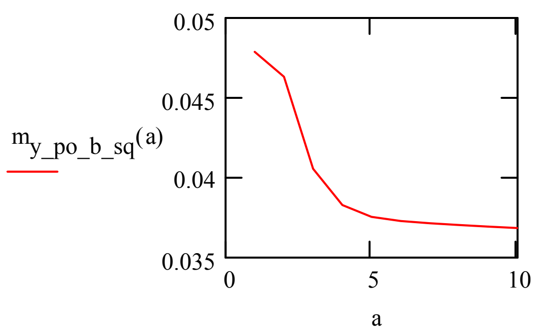
$M := 1..10$



the corresponding y direction moment is: ν is associated with the m term vs. the n term see above my)

$M := 10$ $N := 10$ $b := 1$ $a := 1..10$ $\nu := 0.3$ max at a/2 b/2

$$m_{y\_po\_b\_sq}(a) := \sum_{m=1}^{M}\sum_{n=1}^{N} \frac{16}{\pi^4\cdot(2\cdot m-1)\cdot(2\cdot n-1)\cdot\left[\frac{(2\cdot m-1)^2}{\left(\frac{a}{b}\right)^2}+(2\cdot n-1)^2\right]^2}\cdot\left[\left[\nu\cdot\left[(2\cdot n-1)^2\right]\right]+\frac{(2\cdot m-1)^2}{\left(\frac{a}{b}\right)^2}\right]\cdot(-$$



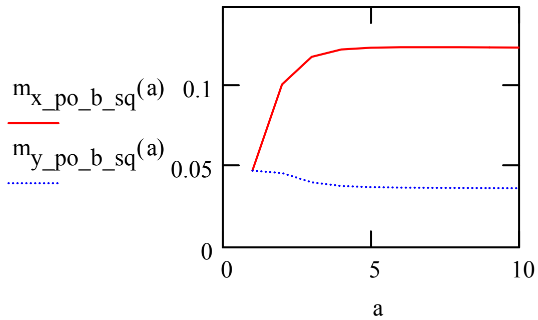
$m_{y\_po\_b\_sq}(10) = 0.037$

$m_{x\_po\_b\_sq}(10,10,10)\cdot\nu = 0.037$

restating original form of mx/po*b^2

$$m_{x\_po\_b\_sq}(a) := \sum_{m=1}^{M}\sum_{n=1}^{N} \frac{16}{\pi^4\cdot(2\cdot m-1)\cdot(2\cdot n-1)\cdot\left[\frac{(2\cdot m-1)^2}{2}+(2\cdot n-1)^2\right]^2}\cdot\left[(2\cdot n-1)^2+\nu\cdot\frac{(2\cdot m-1)^2}{\left(\frac{a}{b}\right)^2}\right]\cdot(-1)^{(m}$$

$$\left[\left(\frac{a}{b}\right)^2 \right] \quad \left[\left(b \right) \right]$$



part of figure 9.5 in Hughes

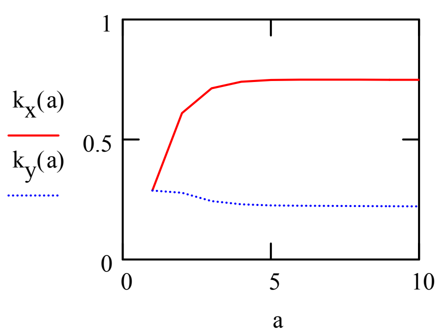
stress is related to the moment as before

$$\sigma_x := \frac{m_x}{I} \cdot z_{max} \qquad I := \frac{t^3}{12} \qquad \sigma_x := \frac{m_x}{I} \cdot \frac{t}{2} \qquad \sigma_x := m_x \cdot \frac{6}{t^2}$$

at maximum mid point: considering:

$$\sigma_x = k \cdot p_o \cdot \left(\frac{b}{t}\right)^2 \qquad \frac{\sigma_x}{\left[p_o \cdot \left(\frac{b}{t}\right)^2\right]} = k \qquad k = \frac{m_x \cdot \frac{6}{t^2}}{\left[p_o \cdot \left(\frac{b}{t}\right)^2\right]} = 6 \cdot m_{x\_po\_b\_sq}$$

therefore: $k_x(a) := 6 \cdot m_{x\_po\_b\_sq}(a)$ similarly $k_y(a) := 6 \cdot m_{y\_po\_b\_sq}(a)$

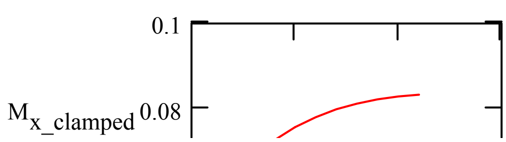


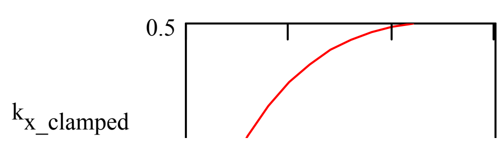
$k_x(10) = 0.748$

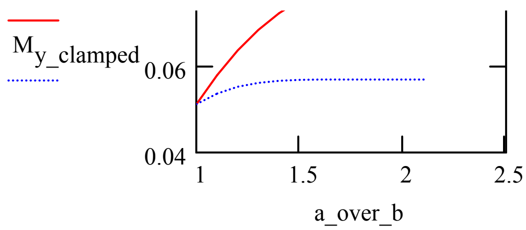
$k_y(10) = 0.221$

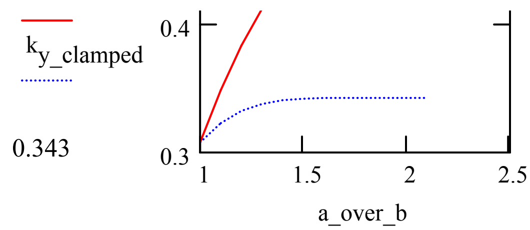
$k_x(10) \cdot \nu = 0.224$

the clamped situation is considerably more complicated
the results are developed in Timoshenko
results are shown in the plot:





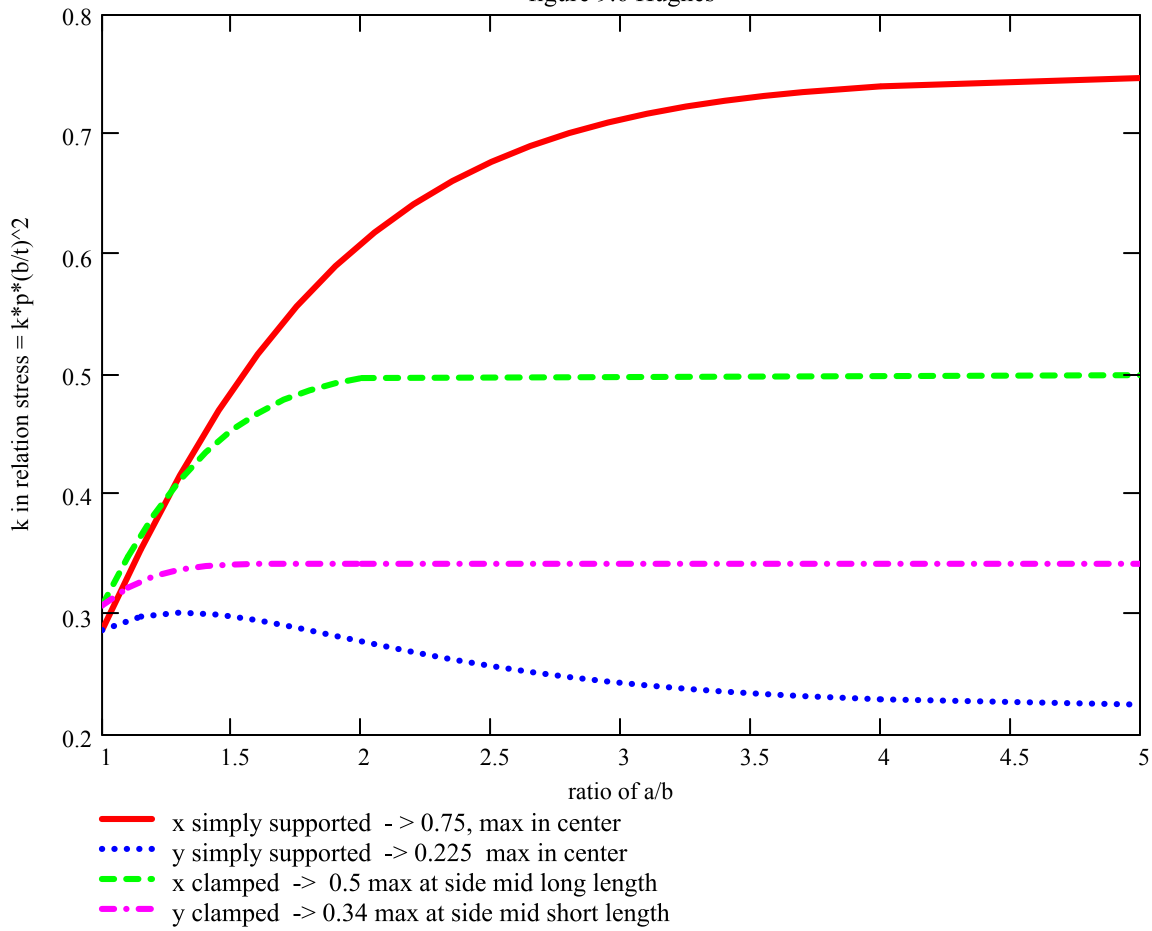




N.B. this NOT the same My and ky as for the simply supported case.
In the clamped case, there is an axial stress in the y (long direction) even for a long plate.
The y axis stress is the maximum at the midpoint of the short side (x = b/2 at the edge y = 0 and y = a)
the x axis stress is maximum at the midpoint of the long side (y = a/2 at the edge x = 0 and x = b)

the bottom line plot for simply supported and clamped/clamped plates under uniform pressure is

figure 9.6 Hughes



Mx is max (k = 0.5) at x = a/2, i.e. at ends of short side, middle of long side
My is max (k = 0.34) at y = b/2, i.e. at ends of long side middle of short side
(think of situation in square => both are max (k = ~0.3) and equal on sides in middle)