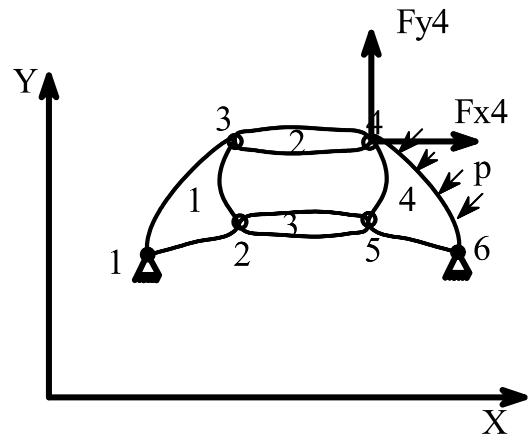
Intro to Matrix Analysis

ORIGIN := 1

consider a 2-D structure consisting of four elements, linked at pinned joints with six nodes:



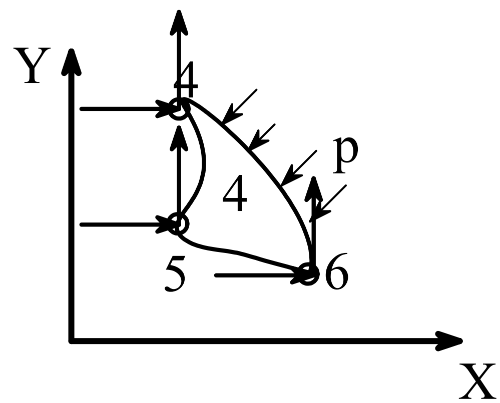
Fx and Fy are the external forces applied at node 4
p a distributed pressure on element #4

we attribute LINEAR ELASTIC behaviour to the structure and in turn to each of the elements:

consider one of the elements, say element # 4

LINEAR ELASTIC behaviour and equilibrium =>

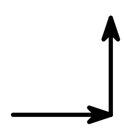
$$F = K \cdot \Delta + f_p + f_{\varepsilon 0}$$



where F = forces at nodes (any direction => two at each node in X and Y)

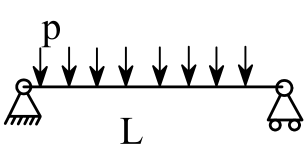
f_p is the equivalent nodal force resulting in the same REACTION to the distributed pressure

$f_{\varepsilon 0}$ is the same for initial strain/stress in the element due to fit, temperature, etc.



F = forces at each node

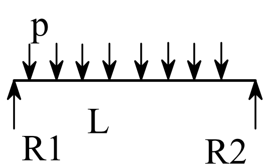
for an example of the eqivalent nodal force consider the following uniformly loaded beam:

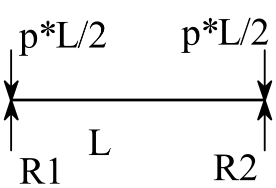


has reaction forces

$$R1 = \frac{p \cdot L}{2}$$

note: p and R have opposite directions in this example





a force of p*L/2 in the same direction as p will create the same rection force hence

$$f_p = p \cdot \frac{L}{2}$$

henceforth we will assume that the nodal forces already account for the equivalent of distributed forces and that initial stress/strain = 0 therefore: ...

$F = K \cdot \Delta$ i.e. The nodal force is linearly proportional to the displacement of the nodes

for the fourth element this is expressed as:... $nod\_el := 3$

$$Fe \rightarrow \begin{pmatrix} Fe_1 \\ Fe_2 \\ Fe_3 \end{pmatrix} \qquad Ke \rightarrow \begin{pmatrix} Ke_{1,1} & Ke_{1,2} & Ke_{1,3} \\ Ke_{2,1} & Ke_{2,2} & Ke_{2,3} \\ Ke_{3,1} & Ke_{3,2} & Ke_{3,3} \end{pmatrix} \qquad \Delta e \rightarrow \begin{pmatrix} \Delta e_1 \\ \Delta e_2 \\ \Delta e_3 \end{pmatrix}$$

Ke = element stiffness matrix which for now we will assume can be determined by experiment or analysis
similarly a matrix can be found such that:...

$\sigma e = Se \cdot \Delta e$ Se = element stress matrix

remember that F and Δ in this two dimensional example each have two components X & Y

$$Fe := \begin{pmatrix} FeX_1 \\ FeY_1 \\ FeX_2 \\ FeY_2 \\ FeX_3 \\ FeY_3 \end{pmatrix} \qquad \Delta e := \begin{pmatrix} U_1 \\ V_1 \\ U_2 \\ V_2 \\ U_3 \\ V_3 \end{pmatrix}$$

U is X displacement
V is Y displacement

and Ke is a 6 x 6 matrix of coefficients.
we will express the relationship as above until later

this examaple is 2-D pinned and involves only X and Y forces and displacements
were this to include clamped joints, there would be a moment and resulting rotation θ for 3 components at each node Fx, Fy, M and U, V, θ
if this were 3_D, there would be three forces and three moments at each node

later we will also see the concepts of "force" and "displacement" to be generalized and include imposed moments and resulting rotation θ and termed "degrees of freedom"

the solution to these problems involves three concepts:
- equilibrium (of "generalized" forces)
- compatibility (of displacements or "degrees of freedom")
- material behavior

and we will operate in three coordinate systems:
- global or overall structure
- element in structure system and ...
- an element coordinate system

above we have expressed the linear elastic behavior of the element in the structure coordinate system and could do the same for each element. We might have to "pad" some matrices (add some 0 to get the same number of rows and columns for operations below.

$$Fe := \begin{pmatrix} Fe_1 \\ Fe_2 \\ Fe_3 \end{pmatrix} \qquad Ke := \begin{pmatrix} Ke_{1,1} & Ke_{1,2} & Ke_{1,3} \\ Ke_{2,1} & Ke_{2,2} & Ke_{2,3} \\ Ke_{3,1} & Ke_{3,2} & Ke_{3,3} \end{pmatrix} \qquad \Delta e := \begin{pmatrix} \Delta_1 \\ \Delta_2 \\ \Delta_3 \end{pmatrix}$$

let's now operate in the structure coordinate system and develop some information about the system K (stiffness matrix) in the relation:

$F = K \cdot \Delta$ where ....

$$F \rightarrow \begin{pmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \end{pmatrix} \qquad K \rightarrow \begin{pmatrix} K_{1,1} & K_{1,2} & K_{1,3} & K_{1,4} & K_{1,5} & K_{1,6} \\ K_{2,1} & K_{2,2} & K_{2,3} & K_{2,4} & K_{2,5} & K_{2,6} \\ K_{3,1} & K_{3,2} & K_{3,3} & K_{3,4} & K_{3,5} & K_{3,6} \\ K_{4,1} & K_{4,2} & K_{4,3} & K_{4,4} & K_{4,5} & K_{4,6} \\ K_{5,1} & K_{5,2} & K_{5,3} & K_{5,4} & K_{5,5} & K_{5,6} \\ K_{6,1} & K_{6,2} & K_{6,3} & K_{6,4} & K_{6,5} & K_{6,6} \end{pmatrix} \qquad \Delta \rightarrow \begin{pmatrix} \Delta_1 \\ \Delta_2 \\ \Delta_3 \\ \Delta_4 \\ \Delta_5 \\ \Delta_6 \end{pmatrix}$$

with each node having two or more degrees of freedom i.e.
F is an n x 1 vector
K is a n x n matrix
Δ is an n x 1 vector
where n is the number of degrees of freedom at each node

now to address the structure let's "pad" the element and express the components of nodal force and displacement in strucuture coordinates: the element node 1 corresponds to structure node 4 etc. so we could first say: ....

$$Fe \rightarrow \begin{pmatrix} Fe_4 \\ Fe_5 \\ Fe_6 \end{pmatrix} \qquad Ke \rightarrow \begin{pmatrix} Ke_{1,1} & Ke_{1,2} & Ke_{1,3} \\ Ke_{2,1} & Ke_{2,2} & Ke_{2,3} \\ Ke_{3,1} & Ke_{3,2} & Ke_{3,3} \end{pmatrix} \qquad \Delta \rightarrow \begin{pmatrix} \Delta_4 \\ \Delta_5 \\ \Delta_6 \end{pmatrix} \qquad Ke \cdot \Delta \rightarrow \begin{pmatrix} Ke_{1,1} \cdot \Delta_4 + Ke_{1,2} \cdot \Delta_5 + Ke_{1,3} \cdot \Delta_6 \\ Ke_{2,1} \cdot \Delta_4 + Ke_{2,2} \cdot \Delta_5 + Ke_{2,3} \cdot \Delta_6 \\ Ke_{3,1} \cdot \Delta_4 + Ke_{3,2} \cdot \Delta_5 + Ke_{3,3} \cdot \Delta_6 \end{pmatrix}$$

or ... with no loss in accuracy padding the nodes not related to the fourth element ...

$$Fe \rightarrow \begin{pmatrix} Fe_1 \\ Fe_2 \\ Fe_3 \\ Fe_4 \\ Fe_5 \\ Fe_6 \end{pmatrix} \qquad Ke \rightarrow \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & Ke_{1,1} & Ke_{1,2} & Ke_{1,3} \\ 0 & 0 & 0 & Ke_{2,1} & Ke_{2,2} & Ke_{2,3} \\ 0 & 0 & 0 & Ke_{3,1} & Ke_{3,2} & Ke_{3,3} \end{pmatrix} \qquad \Delta \rightarrow \begin{pmatrix} \Delta_1 \\ \Delta_2 \\ \Delta_3 \\ \Delta_4 \\ \Delta_5 \\ \Delta_6 \end{pmatrix}$$

note that in this expression, we have expanded ("padded") the F and Δ vectors to include the unrelated nodes with no loss in accuracy as ...

$$Ke \cdot \Delta \rightarrow \begin{pmatrix} 0 \\ 0 \\ 0 \\ Ke_{1,1} \cdot \Delta_4 + Ke_{1,2} \cdot \Delta_5 + Ke_{1,3} \cdot \Delta_6 \\ Ke_{2,1} \cdot \Delta_4 + Ke_{2,2} \cdot \Delta_5 + Ke_{2,3} \cdot \Delta_6 \\ Ke_{3,1} \cdot \Delta_4 + Ke_{3,2} \cdot \Delta_5 + Ke_{3,3} \cdot \Delta_6 \end{pmatrix}$$

which compares with the values above

we're now going to change nomenclature to allow including an additional element say element #3

first so we can keep track we'll rename the previous sttiffness matrix Ke4

$$Fe4 \rightarrow \begin{pmatrix} Fe4_1 \\ Fe4_2 \\ Fe4_3 \\ Fe4_4 \\ Fe4_5 \\ Fe4_6 \end{pmatrix} \qquad Ke4 \rightarrow \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & Ke4_{1,1} & Ke4_{1,2} & Ke4_{1,3} \\ 0 & 0 & 0 & Ke4_{2,1} & Ke4_{2,2} & Ke4_{2,3} \\ 0 & 0 & 0 & Ke4_{3,1} & Ke4_{3,2} & Ke4_{3,3} \end{pmatrix} \qquad Ke4 \cdot \Delta \rightarrow \begin{pmatrix} 0 \\ 0 \\ 0 \\ Ke4_{1,1} \cdot \Delta_4 + Ke4_{1,2} \cdot \Delta_5 + Ke4_{1,3} \cdot \Delta_6 \\ Ke4_{2,1} \cdot \Delta_4 + Ke4_{2,2} \cdot \Delta_5 + Ke4_{2,3} \cdot \Delta_6 \\ Ke4_{3,1} \cdot \Delta_4 + Ke4_{3,2} \cdot \Delta_5 + Ke4_{3,3} \cdot \Delta_6 \end{pmatrix}$$

now suppose another element (#3) has nodes 2 and 5 so node 5 is a common node

$$Fe3 \rightarrow \begin{pmatrix} Fe3_1 \\ Fe3_2 \\ Fe3_3 \\ Fe3_4 \\ Fe3_5 \\ Fe3_6 \end{pmatrix} \quad Ke3 \rightarrow \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & Ke3_{1,1} & 0 & 0 & Ke3_{1,2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & Ke3_{2,1} & 0 & 0 & Ke3_{2,2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad Ke3\cdot\Delta \rightarrow \begin{pmatrix} 0 \\ Ke3_{1,1}\cdot\Delta_2 + Ke3_{1,2}\cdot\Delta_5 \\ 0 \\ 0 \\ Ke3_{2,1}\cdot\Delta_2 + Ke3_{2,2}\cdot\Delta_5 \\ 0 \end{pmatrix}$$

N.B. only components for F2 and F5 and .. only two nodes for this element

now we can use equilibrium for forces at the nodes as follows from these two elements: ...
obviously complete equilibrium requires all nodes ...

$F := Fe3 + Fe4$

this states that the external force at each node is in equilibrium with the components of that force for each element

note .. elements with no connection contribute nothing ...

let's look at node 5 ...

$Fe3_5 \rightarrow Fe3_5 \qquad Fe4_5 \rightarrow Fe4_5 \qquad F_5 \rightarrow Fe3_5 + Fe4_5$

$Fe3 := Ke3\cdot\Delta \qquad Fe4 := Ke4\cdot\Delta$

Δ is common (compatibility)

$F := (Ke3 + Ke4)\cdot\Delta \qquad F_5 \rightarrow Ke3_{2,1}\cdot\Delta_2 + Ke4_{2,1}\cdot\Delta_4 + \left(Ke3_{2,2} + Ke4_{2,2}\right)\cdot\Delta_5 + Ke4_{2,3}\cdot\Delta_6$

and if we sum Ke3 and Ke 4 to get K (for these two elements)

$K := Ke3 + Ke4$

$$K \rightarrow \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & Ke3_{1,1} & 0 & 0 & Ke3_{1,2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & Ke4_{1,1} & Ke4_{1,2} & Ke4_{1,3} \\ 0 & Ke3_{2,1} & 0 & Ke4_{2,1} & Ke3_{2,2} + Ke4_{2,2} & Ke4_{2,3} \\ 0 & 0 & 0 & Ke4_{3,1} & Ke4_{3,2} & Ke4_{3,3} \end{pmatrix}$$

$F := K\cdot\Delta \qquad F_5 \rightarrow Ke3_{2,1}\cdot\Delta_2 + Ke4_{2,1}\cdot\Delta_4 + \left(Ke3_{2,2} + Ke4_{2,2}\right)\cdot\Delta_5 + Ke4_{2,3}\cdot\Delta_6$

same as above... CONCLUSION

K - the structure stiffness matrix is determined by the sum of element stiffness matrices in structure coordinates (expanded to include all nodes)

i.e. ...

$$K_{i,j} = \sum_{ie=1}^{n\_elements} \left(Ke_{ie}\right)_{i,j}$$

i = 1 ..... number of nodes (forces, n per node)
j = 1 ..... number of nodes (displacements, n per node)

$\left(Ke_{ie}\right)_{i,j}$ = n x n matrix linear elastically connecting force at element node i to displacement node j where
n = number of dof per node

reference Zienkiewicz expresses the importance of this relationship ...

" .... ***general assembly process can be found to be the common and fundamental feature of ALL finite element calculations and should be understood*** ..."