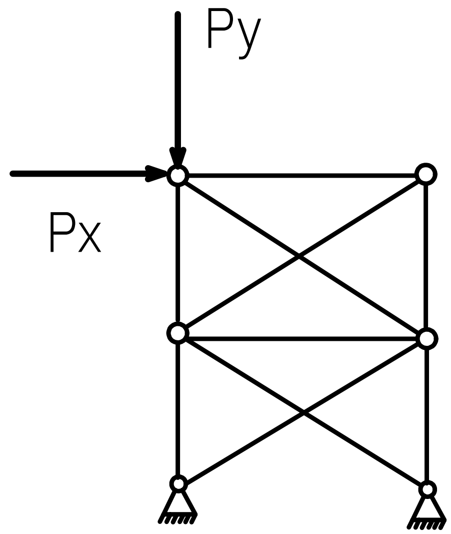
Matrix Analysis, Grillage, intro to Finite Element Modeling



suppose we were to analyze this pin-jointed structure

what are some of the analysis tools we would use?

is this statically determinant?

when we write down the model, what equations result
single
multiple?

equilibrium
compatibility of displacements
laws of material behavior

results in set of simultaneous equations in terms of structure forces and displacements
form is

$$F = K \cdot \delta$$

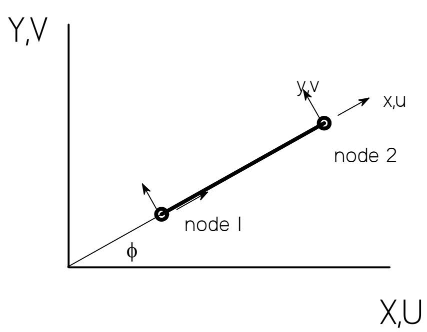
it would be nice to develop an organized approach to similar problems:

Matrix Analysis of Structures

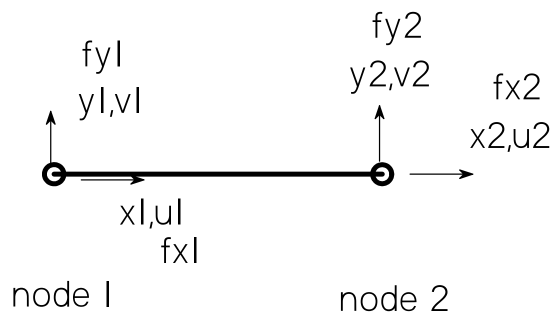
start with pin jointed frame: section 5.2

we want the law of material behavior: in this case a relation between force and displacement
we will refer to this as a stiffness matrix
and a relationship between an element and the structure it is a part of

we will address the compatibility of displacements only on a single element at this stage



defines an element and a structure coordinate system element x,y u,v



the element stiffness matrix:

$f = k_e \cdot \delta$

$$\delta = \begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{pmatrix} \qquad f = \begin{pmatrix} fx_1 \\ fy_1 \\ fx_2 \\ fy_2 \end{pmatrix}$$

note: even though v and fy = 0, will carry due to compatibility with structure

laws of material behavior (Hooke), for details including relationship of "internal" stress/force see collapsed area

$$\frac{f_1}{A} = E \cdot \frac{\Delta L}{L} = E \cdot \frac{u_1 - u_2}{L}$$

or ...

$$f_1 = \frac{A \cdot E}{L} \cdot (u_1 - u_2)$$

$$f = \begin{pmatrix} fx_1 \\ fy_1 \\ fx_2 \\ fy_2 \end{pmatrix} = \frac{A \cdot E}{L} \cdot \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{pmatrix}$$

$$k_e = \frac{A \cdot E}{L} \cdot \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

this is the in the equation **element stiffness matrix**

$f = k_e \cdot \delta$

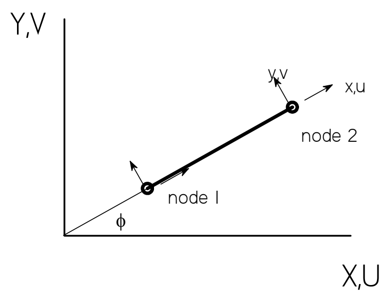
the element stress matrix is related to the internal force = -fx1 or = fx2

$$\sigma = \frac{-f_{x1}}{A} = -\frac{E}{L} \cdot (1 \;\; 0 \;\; -1 \;\; 0) \cdot \begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{pmatrix}$$

=>

$$Se = \frac{E}{L} \cdot (-1 \;\; 0 \;\; 1 \;\; 0)$$

now let's connect to the structure coordinate system:



structure forces

$$F = \begin{pmatrix} Fx_1 \\ Fy_1 \\ Fx_2 \\ Fy_2 \end{pmatrix}$$

$$fx_1 = Fx_1 \cdot \cos(\phi) + Fy_1 \cdot \sin(\phi)$$

$$fy_1 = -Fx_1 \cdot \sin(\phi) + Fy_1 \cdot \cos(\phi)$$

same for node 2

N.B. ϕ is the angle measured CCW from the structure X to the element x coordinate direction

if substitute $\lambda = \cos(\phi)$ $\mu = \sin(\phi)$

$$f = \begin{pmatrix} fx_1 \\ fy_1 \\ fx_2 \\ fy_2 \end{pmatrix} = \begin{pmatrix} \lambda & \mu & 0 & 0 \\ -\mu & \lambda & 0 & 0 \\ 0 & 0 & \lambda & \mu \\ 0 & 0 & -\mu & \lambda \end{pmatrix} \cdot \begin{pmatrix} Fx_1 \\ Fy_1 \\ Fx_2 \\ Fy_2 \end{pmatrix}$$

define T $f = T \cdot F$

$$T := \begin{pmatrix} \lambda & \mu & 0 & 0 \\ -\mu & \lambda & 0 & 0 \\ 0 & 0 & \lambda & \mu \\ 0 & 0 & -\mu & \lambda \end{pmatrix}$$

$$T^T \rightarrow \begin{pmatrix} \lambda & -\mu & 0 & 0 \\ \mu & \lambda & 0 & 0 \\ 0 & 0 & \lambda & -\mu \\ 0 & 0 & \mu & \lambda \end{pmatrix}$$

$$T^{-1} \text{ simplify } \rightarrow \begin{pmatrix} \frac{\lambda}{\lambda^2+\mu^2} & \frac{-\mu}{\lambda^2+\mu^2} & 0 & 0 \\ \frac{\mu}{\lambda^2+\mu^2} & \frac{\lambda}{\lambda^2+\mu^2} & 0 & 0 \\ 0 & 0 & \frac{\lambda}{\lambda^2+\mu^2} & \frac{-\mu}{\lambda^2+\mu^2} \\ 0 & 0 & \frac{\mu}{\lambda^2+\mu^2} & \frac{\lambda}{\lambda^2+\mu^2} \end{pmatrix}$$

but ... recall ... $\lambda := \cos(\phi)$ $\mu := \sin(\phi)$

$$\lambda^2 + \mu^2 \text{ simplify } \rightarrow 1$$

restating ...

$$T := \begin{pmatrix} \lambda & \mu & 0 & 0 \\ -\mu & \lambda & 0 & 0 \\ 0 & 0 & \lambda & \mu \\ 0 & 0 & -\mu & \lambda \end{pmatrix}$$

$$T^T \rightarrow \begin{pmatrix} \cos(\phi) & -\sin(\phi) & 0 & 0 \\ \sin(\phi) & \cos(\phi) & 0 & 0 \\ 0 & 0 & \cos(\phi) & -\sin(\phi) \\ 0 & 0 & \sin(\phi) & \cos(\phi) \end{pmatrix}$$

$$T^{-1} \text{ simplify } \rightarrow \begin{pmatrix} \cos(\phi) & -\sin(\phi) & 0 & 0 \\ \sin(\phi) & \cos(\phi) & 0 & 0 \\ 0 & 0 & \cos(\phi) & -\sin(\phi) \\ 0 & 0 & \sin(\phi) & \cos(\phi) \end{pmatrix}$$

this matrix has the special property that the inverse is = to the transpose the same transformation relation applies to displacement

structure element

$$\Delta = \begin{pmatrix} U_1 \\ V_1 \\ U_2 \\ V_2 \end{pmatrix} \qquad \delta = \begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{pmatrix} \qquad \delta = T \cdot \Delta$$

now we have all that is required to determine the stiffness matrix in structure coordinates:
taking our element stiffness equation and substituting the two transformation equations we just developed:

$$f = k_e \cdot \delta \qquad f = T \cdot F \qquad \delta = T \cdot \Delta$$

$$f = T \cdot F = k_e \cdot \delta = k_e \cdot T \cdot \Delta \quad \Rightarrow \quad T \cdot F = k_e \cdot T \cdot \Delta$$

pre-multiply by T inverse (= T transform) $T^T \cdot T \cdot F = F = T^T \cdot k_e \cdot T \cdot \Delta$

so our structure stiffness matrix = $K_e = T^T \cdot k_e \cdot T$

this is the **element stiffness in structure coordinates**

or

$$k_e := \frac{A \cdot E}{L} \cdot \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad K_e := T^T \cdot k_e \cdot T$$

$$K_e \rightarrow \begin{pmatrix} \cos(\phi)^2 \cdot A \cdot \frac{E}{L} & \cos(\phi) \cdot A \cdot \frac{E}{L} \cdot \sin(\phi) & -\cos(\phi)^2 \cdot A \cdot \frac{E}{L} & -\cos(\phi) \cdot A \cdot \frac{E}{L} \cdot \sin(\phi) \\ \cos(\phi) \cdot A \cdot \frac{E}{L} \cdot \sin(\phi) & \sin(\phi)^2 \cdot A \cdot \frac{E}{L} & -\cos(\phi) \cdot A \cdot \frac{E}{L} \cdot \sin(\phi) & -\sin(\phi)^2 \cdot A \cdot \frac{E}{L} \\ -\cos(\phi)^2 \cdot A \cdot \frac{E}{L} & -\cos(\phi) \cdot A \cdot \frac{E}{L} \cdot \sin(\phi) & \cos(\phi)^2 \cdot A \cdot \frac{E}{L} & \cos(\phi) \cdot A \cdot \frac{E}{L} \cdot \sin(\phi) \\ -\cos(\phi) \cdot A \cdot \frac{E}{L} \cdot \sin(\phi) & -\sin(\phi)^2 \cdot A \cdot \frac{E}{L} & \cos(\phi) \cdot A \cdot \frac{E}{L} \cdot \sin(\phi) & \sin(\phi)^2 \cdot A \cdot \frac{E}{L} \end{pmatrix}$$

$$\lambda \rightarrow \cos(\phi) \qquad \mu \rightarrow \sin(\phi)$$

$$\frac{K_e}{\left(\frac{A \cdot E}{L}\right)} \rightarrow \begin{pmatrix} \cos(\phi)^2 & \cos(\phi) \cdot \sin(\phi) & -\cos(\phi)^2 & -\cos(\phi) \cdot \sin(\phi) \\ \cos(\phi) \cdot \sin(\phi) & \sin(\phi)^2 & -\cos(\phi) \cdot \sin(\phi) & -\sin(\phi)^2 \\ -\cos(\phi)^2 & -\cos(\phi) \cdot \sin(\phi) & \cos(\phi)^2 & \cos(\phi) \cdot \sin(\phi) \\ -\cos(\phi) \cdot \sin(\phi) & -\sin(\phi)^2 & \cos(\phi) \cdot \sin(\phi) & \sin(\phi)^2 \end{pmatrix}$$

eqn. 5.2.9 in terms of cos and sin

we will now address multiple elements in a system demonstrating the process referred to as

assembly

to address this we will first approach it using the result from previous

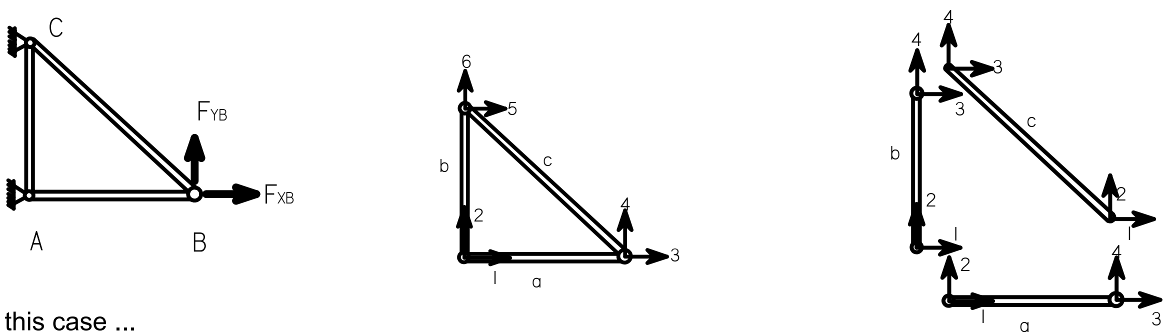
ORIGIN := 1

$$K_{i,j} = \sum_{ie=1}^{n\_elements} \left(Ke_{ie}\right)_{i,j}$$

i = 1 ..... number of nodes (forces, n per node)
j = 1 ..... number of nodes (displacements, n per node)

$\left(Ke_{ie}\right)_{i,j}$ = n x n matrix linear elastically connecting force at element node i to displacement node j where
n = number of dof per node

Matrix Analysis Example Hughes figure 5.12   page 191 ff



in this case ...

$n\_elements := 3$ $n\_nodes := 3$

$n\_free := 2$ number of degrees of freedom per node

$n\_dof := n\_nodes \cdot n\_free$ total number of degrees of freedom in structure

$nod\_el := 2$ nodes per element

let's define the structure in a matrix listing the nodes associated with each element as follows

$$elem := \begin{pmatrix} 1 & 2 \\ 1 & 3 \\ 2 & 3 \end{pmatrix}$$

next, let's expand this matrix to the degrees of freedom sometimes referred to as the topology matrix or location matrix

$ie := 1 .. n\_elements$ $j := 1 .. n\_free$ $k := 0 .. n\_free - 1$

odd dof k = 1
even dof k = 0

$$top_{ie,\, n\_free \cdot j - k} := n\_free\, elem_{ie,j} - k$$

$$top = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 5 & 6 \\ 3 & 4 & 5 & 6 \end{pmatrix}$$

this says for example, the second degree of freedom in element #2 lines up with the second dof in the structure ...

$ie := 2$ $jj := 2$ $top_{ie,jj} = 2$

$ie := 3$ $jj := 1$ $top_{ie,jj} = 3$

while the first dof of element 3 lines up with the third dof in the structure now represent each of the three stiffness matrices as follows:

$$Ke_1 := \begin{pmatrix} a11 & a12 & a13 & a14 \\ a21 & a22 & a23 & a24 \\ a31 & a32 & a33 & a34 \\ a41 & a42 & a43 & a44 \end{pmatrix} \qquad Ke_2 := \begin{pmatrix} b11 & b12 & b13 & b14 \\ b21 & b22 & b23 & b24 \\ b31 & b32 & b33 & b34 \\ b41 & b42 & b43 & b44 \end{pmatrix} \qquad Ke_3 := \begin{pmatrix} c11 & c12 & c13 & c14 \\ c21 & c22 & c23 & c24 \\ c31 & c32 & c33 & c34 \\ c41 & c42 & c43 & c44 \end{pmatrix}$$

we could develop each expanded stiffness matrix ...

$i := 1 .. nod\_eln\_free \qquad j := 1 .. nod\_eln\_free$

$Ke1_{6,6} := 0$

$ie := 1$

$Ke1_{top_{ie,i}, top_{ie,j}} := (Ke_{ie})_{i,j} \qquad Ke_1 := Ke1$

$$Ke_1 \rightarrow \begin{pmatrix} a11 & a12 & a13 & a14 & 0 & 0 \\ a21 & a22 & a23 & a24 & 0 & 0 \\ a31 & a32 & a33 & a34 & 0 & 0 \\ a41 & a42 & a43 & a44 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$Ke2_{6,6} := 0$

$ie := 2$

$(Ke2)_{top_{ie,i}, top_{ie,j}} := (Ke_{ie})_{i,j} \qquad Ke_2 := Ke2$

$$Ke_2 \rightarrow \begin{pmatrix} b11 & b12 & 0 & 0 & b13 & b14 \\ b21 & b22 & 0 & 0 & b23 & b24 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ b31 & b32 & 0 & 0 & b33 & b34 \\ b41 & b42 & 0 & 0 & b43 & b44 \end{pmatrix}$$

$Ke3_{6,6} := 0$

$ie := 3$

$Ke3_{top_{ie,i}, top_{ie,j}} := (Ke_{ie})_{i,j} \qquad Ke_3 := Ke3$

$$Ke_3 \rightarrow \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & c11 & c12 & c13 & c14 \\ 0 & 0 & c21 & c22 & c23 & c24 \\ 0 & 0 & c31 & c32 & c33 & c34 \\ 0 & 0 & c41 & c42 & c43 & c44 \end{pmatrix}$$

and then add $\quad K := Ke_1 + Ke_2 + Ke_3$

$$K \rightarrow \begin{pmatrix} a11 + b11 & a12 + b12 & a13 & a14 & b13 & b14 \\ a21 + b21 & a22 + b22 & a23 & a24 & b23 & b24 \\ a31 & a32 & a33 + c11 & a34 + c12 & c13 & c14 \\ a41 & a42 & a43 + c21 & a44 + c22 & c23 & c24 \\ b31 & b32 & c31 & c32 & b33 + c33 & b34 + c34 \\ b41 & b42 & c41 & c42 & b43 + c43 & b44 + c44 \end{pmatrix}$$

eqn. at top of page 193 in text

or ... we could just add in the appropriate term according to the topology matrix ...
redefining the element stiffness matrices ...

$$Ke_1 := \begin{pmatrix} a11 & a12 & a13 & a14 \\ a21 & a22 & a23 & a24 \\ a31 & a32 & a33 & a34 \\ a41 & a42 & a43 & a44 \end{pmatrix} \quad Ke_2 := \begin{pmatrix} b11 & b12 & b13 & b14 \\ b21 & b22 & b23 & b24 \\ b31 & b32 & b33 & b34 \\ b41 & b42 & b43 & b44 \end{pmatrix} \quad Ke_3 := \begin{pmatrix} c11 & c12 & c13 & c14 \\ c21 & c22 & c23 & c24 \\ c31 & c32 & c33 & c34 \\ c41 & c42 & c43 & c44 \end{pmatrix}$$

iniitialize K ... $K_{n\_dof, n\_dof} := 0$

$ie := 1 .. n\_elements$ $\quad K_{top_{ie,i}, top_{ie,j}} := K_{top_{ie,i}, top_{ie,j}} + \left(Ke_{ie}\right)_{i,j}$

$$K \rightarrow \begin{pmatrix} a11 + b11 & a12 + b12 & a13 & a14 & b13 & b14 \\ a21 + b21 & a22 + b22 & a23 & a24 & b23 & b24 \\ a31 & a32 & a33 + c11 & a34 + c12 & c13 & c14 \\ a41 & a42 & a43 + c21 & a44 + c22 & c23 & c24 \\ b31 & b32 & c31 & c32 & b33 + c33 & b34 + c34 \\ b41 & b42 & c41 & c42 & b43 + c43 & b44 + c44 \end{pmatrix}$$

eqn. at top of page 193 in text

$ie := 3$

this algorithm takes the i,j element in the ie th stiffness matrix (in structure coordinates) and adds it to the row and column determined by the ie'th row and i = j 'th column in the global stiffness matrix.

so now we have the Stiffness matrix for the structure (it's singular)
next we apply the boundary conditions ...
this step removes the rows and columns of the constraints from the equations
it is the equivalent of writing the compatibility of displacements equations for the free node in this example

$ii := 1 .. n\_dof$

$F_{ii} := F_{ii}$ $\quad \Delta_{ii} := \Delta_{ii}$

and only degrees of freedom 3 and 4 are unconstrained

therefore the reduced equations become

$$F \rightarrow \begin{pmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \end{pmatrix} \quad \Delta \rightarrow \begin{pmatrix} \Delta_1 \\ \Delta_2 \\ \Delta_3 \\ \Delta_4 \\ \Delta_5 \\ \Delta_6 \end{pmatrix}$$

$F_{red} := \mathrm{submatrix}(F, 3, 4, 1, 1)$ $\quad F_{red} \rightarrow \begin{pmatrix} F_3 \\ F_4 \end{pmatrix}$

$K_{red} := \mathrm{submatrix}(K, 3, 4, 3, 4)$

$$K_{red} \rightarrow \begin{pmatrix} a33 + c11 & a34 + c12 \\ a43 + c21 & a44 + c22 \end{pmatrix}$$

and we can solve for Δ3 and Δ4

$$\begin{pmatrix}\Delta_3\\ \Delta_4\end{pmatrix} := K_{red}^{-1}\cdot F_{red}$$

$$\begin{pmatrix}\Delta_3\\ \Delta_4\end{pmatrix} \rightarrow \begin{pmatrix}\dfrac{a44 + c22}{a33\cdot a44 + a33\cdot c22 + c11\cdot a44 + c11\cdot c22 - a34\cdot a43 - a34\cdot c21 - c12\cdot a43 - c12\cdot c21}\cdot F_3 + \dfrac{}{a33\cdot a44 + a33\cdot c22} \\ \dfrac{-a43 - c21}{a33\cdot a44 + a33\cdot c22 + c11\cdot a44 + c11\cdot c22 - a34\cdot a43 - a34\cdot c21 - c12\cdot a43 - c12\cdot c21}\cdot F_3 + \dfrac{}{a33\cdot a44 + a33\cdot c22}\end{pmatrix}$$

$$\Delta_{ii} := 0 \qquad \begin{pmatrix}\Delta_3\\ \Delta_4\end{pmatrix} := \begin{pmatrix}\Delta 3\\ \Delta 4\end{pmatrix} \qquad F := K\cdot\Delta$$

$$F \rightarrow \begin{bmatrix} a13\cdot\Delta 3 + a14\cdot\Delta 4 \\ a23\cdot\Delta 3 + a24\cdot\Delta 4 \\ (a33 + c11)\cdot\Delta 3 + (a34 + c12)\cdot\Delta 4 \\ (a43 + c21)\cdot\Delta 3 + (a44 + c22)\cdot\Delta 4 \\ c31\cdot\Delta 3 + c32\cdot\Delta 4 \\ c41\cdot\Delta 3 + c42\cdot\Delta 4 \end{bmatrix}$$

to obtain the element forces the transformation matrix can be applied to Δ in structure coordinates to obtain δ and then ke used to obtain f from which stress is determined ...

$$top = \begin{pmatrix}1 & 2 & 3 & 4\\ 1 & 2 & 5 & 6\\ 3 & 4 & 5 & 6\end{pmatrix} \qquad ie := 1 .. n\_elements$$

$$\Delta e_{ie,i} := \Delta_{top_{ie,i}} \qquad \Delta e \rightarrow \begin{pmatrix}0 & 0 & \Delta 3 & \Delta 4\\ 0 & 0 & 0 & 0\\ \Delta 3 & \Delta 4 & 0 & 0\end{pmatrix}$$

$$\delta e = T_e\cdot\Delta e \qquad fe = k_e\cdot\delta e \qquad \sigma = Se\cdot\delta e$$