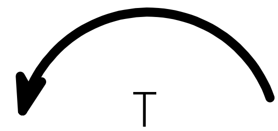
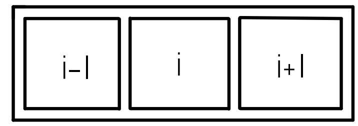
St. Venant Torsion Multi-cell

applied torque

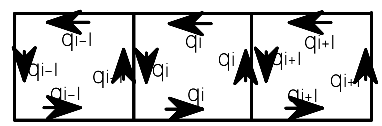


this is a combination of nomenclature from Hughes section 6.1 Multiple Cell Sections and Mixed Sections page 227 ff and Kollbrunner sectin 2.3 Multicelluar Box Section Members

cross section (partial)



shear flow (concept)



first recall ... torque in a cell = $M_{x_i} = 2 \cdot A_i \cdot q_i$

shear flow constant, integral of shear stress*h*dA = torque etc...

=> total torque $M_x = T = \sum_i M_{x_i} = \sum_i 2 \cdot A_i \cdot q_i$ H: 6.1.26

angle of twist (same for all cells) $\phi' = \frac{T}{G \cdot J}$

i.e. ... the derivative of the twist angle wrt the axial coordinate is = applied torque/torsional stiffness H: 6.1.21

and from our development of pure twist (closed section)

$$u = \int_0^s \frac{\tau}{G} ds - \frac{\delta\phi}{\delta x} \cdot \int h_D ds + u_0(x)$$ &H: 6.1.23a

if this quantity is integrated entirely around a closed section ...

$$\int_0^s \frac{\tau}{G} ds - \frac{\delta\phi}{\delta x} \cdot \int h_D ds = 0$$

the axial displacement returns to the starting point

rearranging and substituting q = τ*t....

$$\int_0^s \frac{q}{t} ds = \frac{\delta\phi}{\delta x} \cdot G \cdot \int h_D ds$$

and since $\int h_D ds = 2 \cdot A$

$$\frac{1}{2} \cdot \text{base(ds)} \cdot \text{altitude(h)} = dA$$

$$\int_0^s \frac{q}{t} ds = \frac{\delta\phi}{\delta x} \cdot G \cdot 2 \cdot A = \frac{T}{G \cdot J} \cdot G \cdot 2 \cdot A = \frac{2T}{J} \cdot A$$

now we have enough pieces to assemble the puzzle ...

for each cell ....

$$\left(\int_0^s \frac{q}{t}\,ds\right)_i = \frac{2T}{J}\cdot A_i$$

the shear flow q in cell i is composed of qi (reference direction) and segments of qi-1 and qi=1 in opposite direction ON COMMON BOUNDARY.

i.e. .....

$$\left(\int_0^s \frac{q}{t}\,ds\right)_i = \int_0^b \frac{q_i}{t}\,ds - \int_{s0(i\_and\_i-1)}^{s1(i\_and\_i-1)} \frac{q_{i-1}}{t}\,ds - \int_{s0(i\_and\_i+1)}^{s1(i\_and\_i+1)} \frac{q_{i+1}}{t}\,ds$$

the individual shear flows are constant .... "normalize" the shear flow by 2*T/J Hughes retains G but the approach is the same

$$q\_bar = \frac{q}{\left(\frac{2\cdot T}{J}\right)}$$

$$\left(\int_0^s \frac{q\_bar}{t}\,ds\right)_i = -q\_bar_{i-1}\int_{s0(i\_and\_i-1)}^{s1(i\_and\_i-1)} \frac{1}{t}\,ds + q\_bar_i\int_0^b \frac{1}{t}\,ds - q\_bar_{i+1}\int_{s0(i\_and\_i+1)}^{s1(i\_and\_i+1)} \frac{1}{t}\,ds = A_i$$

this is a set of equations, one for each cell corresponding to qi, like shear but with different rhs.

$$q\_bar_1\cdot\int_{1} \frac{1}{t}\,ds - q\_bar_2\cdot\int_{1,2} \frac{1}{t}\,ds = A_1$$

$$-q\_bar_1\cdot\int_{1,2} \frac{1}{t}\,ds + q\_bar_2\cdot\int_{2} \frac{1}{t}\,ds - q\_bar_3\cdot\int_{2,3} \frac{1}{t}\,ds = A_2$$

$$-q\_bar_2\cdot\int_{2,3} \frac{1}{t}\,ds + q\_bar_3\cdot\int_{3} \frac{1}{t}\,ds - q\_bar_4\cdot\int_{3,4} \frac{1}{t}\,ds = A_3$$

.... = ...

$$q\_bar_{n-1}\cdot\int_{n-1,n} \frac{1}{t}\,ds - q\_bar_n\cdot\int_{n} \frac{1}{t}\,ds = A_n$$

as we did for shear due to bending; let each element of the matrix $\int \frac{1}{t}\,ds$ be expressed by η

where $\eta_{ik} = \int_{i,k} \frac{1}{t}\,ds$ integral along wall separating i and k

and $\eta_{ii} = \int \frac{1}{t}\,ds$ integral around cell i etc

if wall thickness is piecewise constant walls =>

$$\eta_{ik} = \frac{s_{ik}}{t_{ik}} \quad \text{and} \quad \eta_{ii} = \sum_{j=1}^{4} \frac{s_{ij}}{t_{ij}} = \frac{s_{i1}}{t_{i1}} + \frac{s_{i2}}{t_{i2}} + \frac{s_{i3}}{t_{i3}} + \frac{s_{i4}}{t_{i4}}$$

where s_{ij}, t_{ij} is the length and thickness of wall j of cell i

and thus we have .... for the three cell arranged as above ...

$$\begin{pmatrix} \eta_{11} & -\eta_{12} & 0 \\ -\eta_{21} & \eta_{22} & -\eta_{23} \\ 0 & -\eta_{32} & \eta_{33} \end{pmatrix} \cdot \begin{pmatrix} q\_bar_1 \\ q\_bar_2 \\ q\_bar_3 \end{pmatrix} = \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix}$$

solve for q_bar_i $\qquad q_i = q\_bar_i \cdot \frac{2 \cdot T}{J}$

$$T = \sum_i 2 \cdot A_i \cdot q_i = \sum_i 2 \cdot A_i \cdot q\_bar_i \cdot \frac{2 \cdot T}{J} \quad \Rightarrow \quad J = \sum_i 4 \cdot A_i \cdot q\_bar_i$$

$$q_i = T \cdot \frac{q\_bar_i}{\left(\sum_i 2 \cdot A_i \cdot q\_bar_i \right)}$$