2.160 System Identification, Estimation, and Learning

Lecture Notes No. 7

March 1, 2006

4.7. Continuous Kalman Filter

Converting the Discrete Filter to a Continuous Filter

Continuous process $\dot{x} = Fx + Gw(t)$ (49)

Measurement $y = Hx + v(t)$ (50)

Assumptions

$$E\left[w(t)w^T(s)\right] = Q\delta(t-s) \qquad \delta(t-s) = \text{Dirac delta function} \tag{51}$$

$$E\left[v(t)v^T(s)\right] = R\delta(t-s) \tag{52}$$

$$E\left[v(t)w^T(s)\right] = 0 \tag{53}$$

Converting R_t and Q_t in the discrete Kalman filter to Q and R of the above equations, (see Brown and Hwang, Section 7.1 for detail)

$$Q_t = Q\Delta t \qquad R_t = \frac{R}{\Delta t} \qquad \Delta t = \text{sampling interval} \tag{54}$$

From (4)

$$\begin{aligned} K_t &= P_{t|t-1}H_t^T\left(H_t P_{t|t-1}H_t^T + R_t\right)^{-1} \qquad \frac{R}{\Delta t} \\ &= \Delta t P_{t|t-1}H_t^T\left(\Delta t \cdot H_t P_{t|t-1}H_t^T + R\right)^{-1} \\ &\cong \Delta t P_{t|t-1}H_t^T R^{-1} \qquad \text{for } |\Delta t| << 1 \end{aligned} \tag{55}$$

Define $K \overset{\Delta}{=} P_{t|t-1}H_t^T R^{-1}$ (56)

From (8)

$$\begin{aligned} P_{t|t-1} &= A_t P_t A_t^T + G_t Q_t G_t^T \qquad \Delta t K \\ &= A_t\left(I - K_t H_t\right)P_{t|t-1}A_t^T + G_t Q_t G_t^T \end{aligned} \tag{57}$$

$A_t = I + \Delta t F$

Ignoring higher-order small quantities; $O\left(\Delta t^2\right) \cong 0$ (58)

$$P_{t+1|t} = P_{t|t-1} + \Delta t F P_{t|t-1} + \Delta t P_{t|t-1} F^T - \Delta t K H_t P_t + G_t \Delta t Q G_t^T \tag{59}$$

$$\frac{P_{t+1|t} - P_{t|t-1}}{\Delta t} = F P_{t|t-1} + P_{t|t-1} F^T - P_{t|t-1} H_t^T R^{-1} H_t P_t + G_t Q G_t^T \tag{60}$$

$$\Delta t \to 0 \qquad \lim_{\Delta t \to 0} P_{t|t-1} = P_{t-1} \tag{61}$$

$$\dot{P} = FP + PF^T - PH^T R^{-1} HP + GQG^T \tag{62}$$

This is called the Matrix Riccati Equation.

Similarly, we can reduce the discrete time form of state estimation correction to the one of continuous time:

$$\dot{\hat{x}} = F\hat{x} + K\left(y - H\hat{x}\right) \tag{63}$$

where the Kalman gain is given by

$$K = PH^T R^{-1} \tag{64}$$

This is called the Kalman-Bucy Filter

The physical interpretation of the Matrix Riccati Equation

$$\dot{P} = \underbrace{FP + PF^T} - PH^T R^{-1} HP + GQG^T \tag{62}$$

Unforced State Transition: The effect of the unforced system dynamics upon the covariance propagation

The decrease of uncertainty as a result of measurement

The increase of uncertainty due to the process disturbance Q

4.8 The Algebraic Riccati Equation

Assume that the Riccati differential equation has an asymptotically stable solution for $P(t)$:

$$\lim_{t\to\infty} P(t) = P_\infty \tag{65}$$

Then the time derivative vanishes

$$\lim_{t\to\infty} \frac{dP(t)}{dt} = 0 \tag{66}$$

Substituting this into the Riccati equation yields

$$0 = FP_\infty + P_\infty F^T - P_\infty H^T R^{-1} H P_\infty + GQG^T \tag{67}$$

This is called the Algebraic Riccati Equation. This is a nonlinear matrix equation, and need a numerical solver to obtain a solution for P_∞.

Consider a scalar case; $P_\infty \in R^{1\times 1}$, $F, H, Q, R, G \in R^{1\times 1}$. The Algebraic Riccati Equation can be solved analytically

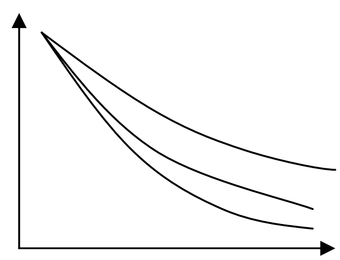
$$\frac{H^2}{R} P_\infty^2 - 2FP_\infty - G^2 Q = 0 \tag{68}$$

$$P_\infty = \frac{R}{H^2}\left(F \pm \sqrt{F^2 + \frac{Q}{R} H^2 G^2} \right) \tag{69}$$

These are two solutions; one positive and the other negative.
Taking the positive solution

$$\lim_{t\to\infty} P(t) = P_\infty = \frac{R}{H^2}\left(F + \sqrt{F^2 + \frac{Q}{R} H^2 G^2} \right) \tag{70}$$

Note that, regardless of the sign of F ($F < 0$ means a stable process dynamics), the above limit P_∞ is positive.



Remarks

1) As the sensor variance R increases, P_∞ increases
2) As the process noise variance Q increases, P_∞ increases
3) When the process noise variance Q is zero, and the process is stable, $F < 0$, P_∞ becomes zero.

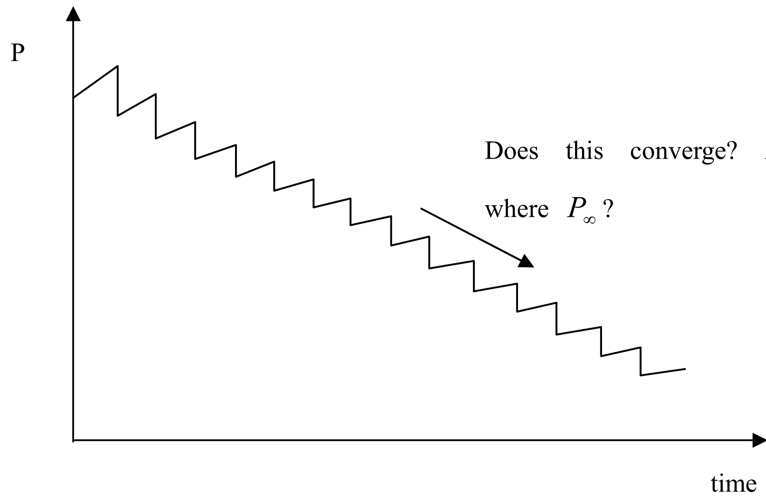
4.8 Convergence Analysis

4.8.1 Transient Response of the Covariance Matrix

The Discrete Kalman Filter is hinged on the covariance matrix update law:

$$P_t = (I - K_t H_t) P_{t|t-1} \tag{41}$$

$$P_{t+1|t} = A_t P_t A_t^T + G_t Q_t G_t^T \tag{45}$$



Continuous Kalman Filter:
The covariance matrix is given by the Riccati Differential equation:

$$\frac{d}{dt} P(t) = FP(t) + P(t)F^T - P(t)H^T R^{-1} HP(t) + GQG^T \tag{62}$$

Where F is a state transition matrix:

$$\frac{d}{dt} x(t) = F(t)x(t) + G(t)w(t) \tag{49}$$

Let us examine the properties of the Riccati differential equation in order to gain insights as to whether the covariance of Kalman filter converges or not.

4.8.2 Matrix Fraction Decomposition

The Riccati Differential Equation (62) can be solved by using a technique, called the Matrix Fraction Decomposition

Consider a square matrix $M(t)$ decomposed into two square matrices $A(t)$ and $B(t)$,

$$M(t) = A(t)B^{-1}(t) \tag{71}$$

where B is non-singular and both A and B are differentiable with respect to time t. The above expression is called a fraction decomposition of Matrix M .

Differentiating $B(t)B(t)^{-1} = I$ (identify matrix) with respect to time t,

$$\dot{B}B^{-1} + B\dot{B}^{-1} = 0 \tag{72}$$

Therefore

$$\frac{d}{dt}B^{-1}(t) = -B^{-1} \cdot \frac{d}{dt}B(t) \cdot B^{-1} \tag{73}$$

Now let us represent the covariance matrix $P(t)$ by

$$P(t) = A(t)B^{-1}(t) \tag{74}$$

and applying eq.(73)

$$\begin{aligned}\frac{dP(t)}{dt} &= \dot{A}B^{-1} + A\dot{B}^{-1} \\ &= \dot{A}B^{-1} - AB^{-1}\dot{B}B^{-1}\end{aligned} \tag{75}$$

From the Riccati equation (62)

$$\frac{dP(t)}{dt} = FAB^{-1} + AB^{-1}F^T - AB^{-1}H^T R^{-1} HAB^{-1} + GQG^T \tag{76}$$

Equating the right hand sides of (75) and (76), and post-multiplying B yield

$$\dot{A} - AB^{-1}\dot{B} = \left(FA + GQG^T B\right) - AB^{-1}\left(H^T R^{-1} HA - F^T B\right) \tag{77}$$

Therefore, if we find A and B that satisfy:

$$\dot{A} = FA + GQG^T B \tag{78}$$

$$\dot{B} = H^T R^{-1} HA - F^T B \tag{79}$$

then $P(t) = A(t)B^{-1}(t)$ satisfies the Riccati differential equation. Note that (78) and (79) are linear differential equations with respect to matrices A and B They can be rearranged as

$$\frac{d}{dt}\begin{pmatrix} A(t) \\ B(t) \end{pmatrix} = \overbrace{\begin{bmatrix} F(t) & G(t)Q(t)G^T(t) \\ H^T(t)R^{-1}(t)H(t) & -F(t) \end{bmatrix}}^{\text{The Hamiltonian Matrix}} \begin{pmatrix} A(t) \\ B(t) \end{pmatrix} \tag{80}$$

As for the initial conditions, we can set

$$A(0) = P_0 \quad \text{and} \quad B(0) = I\,. \tag{81}$$

4.8.3 Convergence Properties of a Scalar Case

Consider a scalar case : $A(t) \to a(t)$, and $B(t) \to b(t)$

and assume that the process and measurement equations are time-invariant

$$\left.\begin{aligned} F(t) &= F \\ G(t) &= G \\ Q(t) &= Q \\ R(t) &= R \\ H(t) &= H \end{aligned}\right\} \text{Scalar}$$

Eq.(80) reduces to

$$\begin{pmatrix} \dot{a} \\ \dot{b} \end{pmatrix} = \begin{bmatrix} F & G^2 Q \\ H^2/R & -F \end{bmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \tag{82}$$

This can be solved with initial condition of $a(0) = P_0$ and $b(0) = 1$.

The eigenvalues of the Hamiltonian Matrix are

$$\lambda_{1,}\lambda_2 = \pm\sqrt{F^2 + \frac{Q}{R} G^2 H^2} = \pm\lambda \tag{83}$$

Using λ_1 and λ_2

$$a(t) = \frac{1}{2\lambda}\left\{[P_0(\lambda + F) + q]e^{\lambda t} + [P_0(\lambda - F) + q]e^{-\lambda t}\right\}$$

$$b(t) = \frac{1}{2\lambda q}\left\{(\lambda - F)[P_0(\lambda + F) + q]e^{\lambda t} - (\lambda + F)[P_0(\lambda - F) - q]e^{-\lambda t}\right\} \tag{84}$$

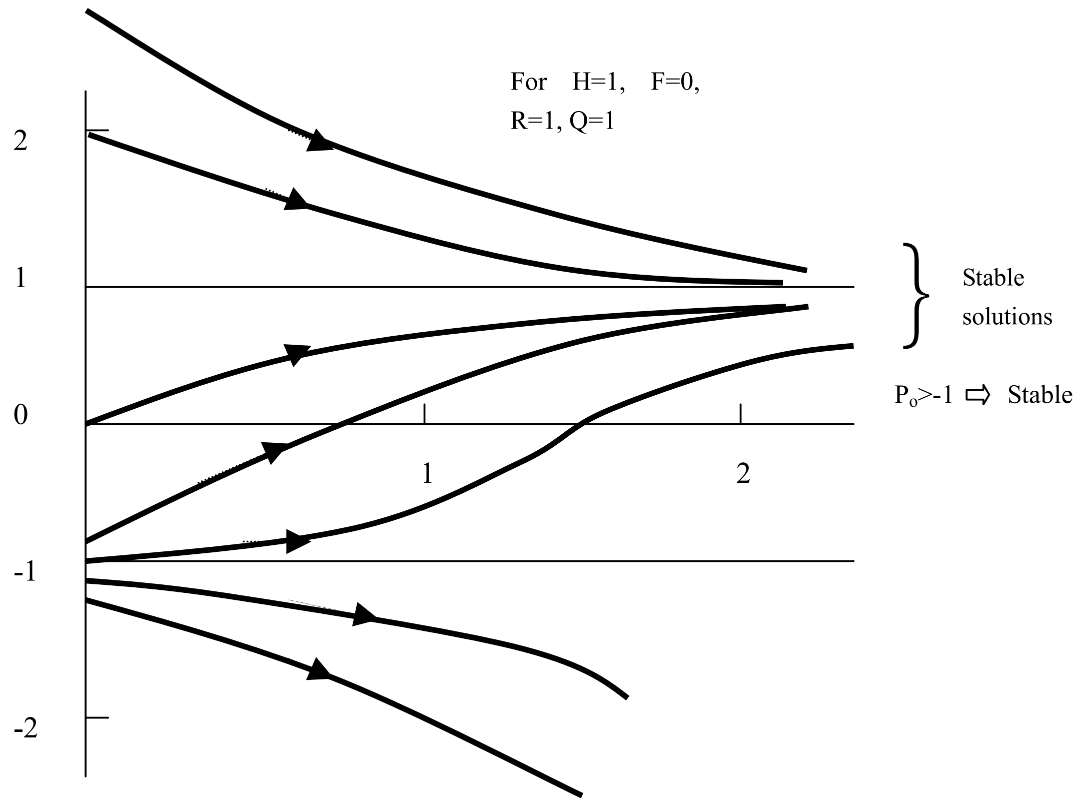
where $q = G^2 Q$. Therefore, the covariance is given by

$$P(t) = \frac{a(t)}{b(t)} = q\,\frac{[P_0(\lambda + F) + q] + [P_0(\lambda - F) - q]e^{-2\lambda t}}{(\lambda - F)[P_0(\lambda + F) + q] - (\lambda + F)[P_0(\lambda - F) - q]e^{-2\lambda t}} \tag{85}$$

The steady-state solution is given by

$$P_\infty = \lim_{t\to\infty} P(t) = \frac{q}{\lambda - F} = \frac{R}{H^2}\left(F + \sqrt{F^2 + \frac{Q}{R} H^2 G^2} \right)$$

This agrees with the previous result, eq.(70).



An important property of the Riccati Differential Equation (RDE):

If the system is observable, i.e. (F, H): Observable Pair, then the RDE has a positive-definite, symmetric solution for an arbitrary positive-definite initial value of matrix P_o>0;

$$P(t) > 0 \ p.d., \quad P(t) = P^T(t) \in R^{n\times n}, \quad \forall t > 0 \tag{86}$$