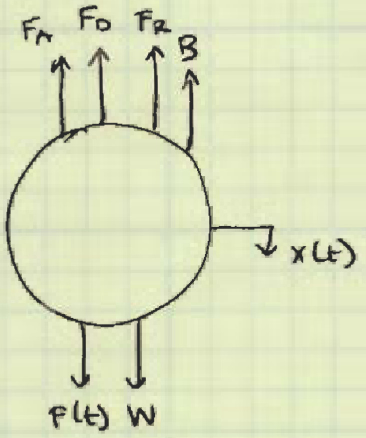
PROBLEM 1:

a)



where: B = BOUYANT FORCE

W = WEIGHT

F_A = ADDED MASS FORCE

F_D = DAMPING FORCE

F_R = RESTORING FORCE

F(t) = FORCE DUE TO OSCILLATIONS

NOTE: ALL FORCES ACTING THROUGH CENTER OF SPHERE.
CYLINDER IS NEUTRALLY BUOYANT SO B=W AND THEY CANCEL OUT

b) $$[m+m_a]\ddot{x}(t) + b\dot{x}(t) + kx(t) = f_o\cos(\omega t)$$

where: m = mass
m_a = added mass coefficient
b = damping coefficient
k = restoring coefficient
f_o = amplitude of forcing function

c) $$\omega_N = \sqrt{\frac{k}{m+m_a}}$$

d) $$x(t) = x_o\cos(\omega t+\psi)$$
$$x(\omega) = x_o e^{i\omega t}e^{i\psi}$$
$$\dot{x}(\omega) = x_o i\omega e^{i\omega t}e^{i\psi}$$
$$\ddot{x}(\omega) = -x_o\omega^2 e^{i\omega t}e^{i\psi}$$

$$f(t) = f_o\cos(\omega t)$$
$$f(\omega) = f_o e^{i\omega t}$$

Note: Both sides have Re{ }, so these terms were omitted.

$$[-(m+m_a)x_o\omega^2 e^{i\psi} + bx_o i\omega e^{i\psi} + kx_o e^{i\psi}]e^{i\omega t} = f_o e^{i\omega t}$$

$$x_o[-(m+m_a)\omega^2 + bi\omega + k]e^{i\psi} = f_o$$

$$\boxed{x_o = \frac{f_o}{-(m+m_a)\omega^2 + bi\omega + k}e^{-i\psi}}$$

PROBLEM 2:

COMPLEX NUMBERS $z = x + iy$

magnitude $= r = |z| = \left|\sqrt{x^2+y^2}\right|$

phase $= \theta = \tan^{-1}\left(\frac{y}{x}\right)$

a) $1 + i\sqrt{3}$ $\quad r = \left|\sqrt{1^2 + (\sqrt{3})^2}\right| = \left|\sqrt{4}\right| = 2$

$\theta = \tan^{-1}\left(\frac{\sqrt{3}}{1}\right) = \frac{\pi}{3}$

b) $\left(\frac{1+i\sqrt{3}}{\sqrt{3}+i}\right)\left(\frac{\sqrt{3}-i}{\sqrt{3}-i}\right) = \frac{\sqrt{3} + 3i - i - i^2\sqrt{3}}{3 - i^2} = \frac{2\sqrt{3}+2i}{4} = \frac{\sqrt{3}+}{2}$

$r = \left|\sqrt{\left(\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{1}{2}\right)^2}\right| = \left|\sqrt{\frac{3}{4}+\frac{1}{4}}\right| = \left|\sqrt{1}\right| = 1$

$\theta = \tan^{-1}\left(\frac{1/2}{\sqrt{3}/2}\right) = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$

c) $-5 - 5i$ $\quad r = \left|\sqrt{(-5)^2 + (-5)^2}\right| = \left|\sqrt{2 \cdot 25}\right| = 5\sqrt{2}$

$\theta = \tan^{-1}\left(\frac{-5}{-5}\right) = \tan^{-1}(1) = \frac{5\pi}{4}$

d) $\underset{(1)}{i}\,\underbrace{(1+i)}_{(2)}\,\underbrace{e^{i\pi/6}}_{(3)}$

REMEMBER: $z = x + iy = re^{i\theta}$

PUT 3 COMPLEX NUMBERS INTO POLAR FORM AND THEN MULTIPLY

$0 + i \longrightarrow r = 1,\ \theta = \frac{\pi}{2},\ z = e^{i\pi/2}$ (1)

$1 + i \longrightarrow r = \sqrt{2},\ \theta = \frac{\pi}{4},\ z = \sqrt{2}\,e^{i\pi/4}$ (2)

$z = e^{i\pi/6}$ (3)

$$z = \left(e^{i\pi/2}\right)\left(\sqrt{2}\,e^{i\pi/4}\right)\left(e^{i\pi/6}\right) = \sqrt{2}\,e^{i\frac{11\pi}{12}}$$

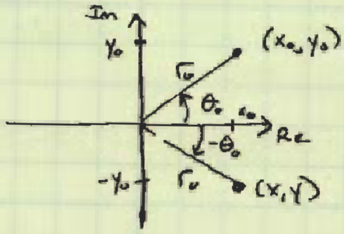
$r = \sqrt{2}$

$\theta = \frac{11\pi}{12}$

PROBLEM 3:

$z_0 = x_0 + iy_0 = r_0 e^{i\theta_0}$ $\quad z = x + iy \quad$ FIND (x,y) IN TERMS OF (x_0, y_0)

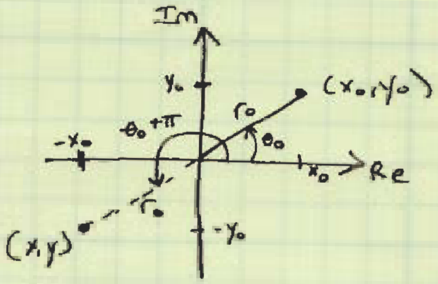
a) $z = r_0 e^{-i\theta_0}$ $\quad r = r_0 \quad \theta = -\theta_0$



$x = x_0$

$y = -y_0$

b) $z = r_0 e^{i(\theta_0 + \pi)}$ $\quad r = r_0 \quad \theta = \theta_0 + \pi$

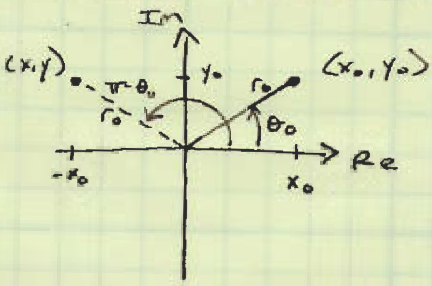


$x = -x_0$

$y = -y_0$

c) $z = r_0 e^{-i(\theta_0 - \pi)}$

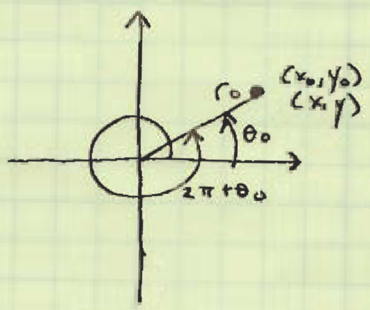
$r = r_0 \quad \theta = \pi - \theta_0$



$x = -x_0$

$y = y_0$

d) $z = r_0 e^{i(\theta_0 + 2\pi)}$ $\quad r = r_0 \quad \theta = \theta_0 + 2\pi$



$x = x_0$

$y = y_0$

PROBLEM 4:

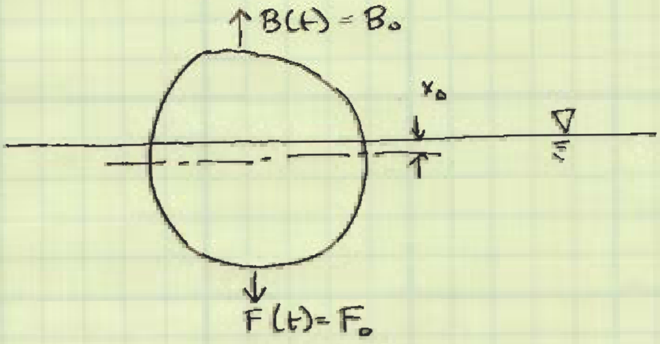
EQUATION OF MOTION IN HEAVE: $\ddot{x}(t)\left[M+A_{33}\right]+\dot{x}(t)\left[B_{33}\right]+x(t)\left[C_{33}\right]=F(t)$

PUSH OBJECT DOWN AND HOLD IT AT DISPLACEMENT $x(t)=x_o$

THEREFORE $\ddot{x}(t)=0$ $\dot{x}(t)=0$

AND $x_o\left[C_{33}\right]=F(t)$

$F(t)$ IS THE FORCE REQUIRED TO DISPLACE THE EXTRA WATER



Therefore, $F_o = B_o =$ BUOYANCY FORCE OF DISPLACING EXTRA WATER

FOR SPHERE: $B_o=\rho_F \cdot g \cdot \forall_b$ where $\forall_b \cong x_o \cdot A_{wp} = x_o \cdot \pi \cdot \frac{d^2}{4}$

$$B_o \cong \rho_F \cdot g \cdot x_o \cdot A_{wp}$$

THEREFORE $x_o\left(C_{33}\right) \cong F_o \cong \rho_F \cdot g \cdot x_o \cdot A_{wp}$

$$C_{33} \cong \rho_F \cdot g \cdot A_{wp}$$

NOTE: "$\cong$" is used because there is a slight curvature of the body at the waterline.

FOR CYLINDER: $B_o=\rho_F \cdot g \cdot \forall_b$ where $\forall_b \cong x_o \cdot A_{wp} = x_o \cdot d \cdot L$

$$B_o \cong \rho_F \cdot g \cdot x_o \cdot A_{wp}$$

THEREFORE $x_o\left(C_{33}\right) = F_o \cong \rho_F \cdot g \cdot x_o \cdot A_{wp}$

$$C_{33} \cong \rho_F \cdot g \cdot A_{wp}$$

PROBLEM 5:

a) $y(t) = \int_0^{t+\alpha} u(s)\,ds$

→ FIRST CHECK LINEARITY:

SUBSTITUTE $a_1 u_1(s) + a_2 u_2(s)$ for $u(s)$

$$\int_0^{t+\alpha} [a_1 u_1(s) + a_2 u_2(s)]\,ds = a_1 \int_0^{t+\alpha} u_1(s)\,ds + a_2 \int_0^{t+\alpha} u_2(s)$$

$$= a_1 y_1(t) + a_2 y_2(t)$$ ∴ IT IS LINEAR

→ NOW CHECK TIME INVARIANCE

SUBSTITUTE $u(s+\tau)$ for $u(s)$ ON THE R.H.S.

$$\int_0^{t+\alpha} u(s+\tau)\,ds$$

LET $\xi = s + \tau$
$d\xi = ds$

WHEN $s = t + \alpha \longrightarrow \xi = t + \alpha + \tau$
$s = 0 \longrightarrow \xi = \tau$

$$= \int_\tau^{t+\alpha+\tau} u(\xi)\,d\xi$$

NOW SUBSTITUTE $y(t+\tau)$ FOR $y(t)$ ON L.H.S AND $(t+\tau)$ FOR (t) ON R.H.S

$$y(t+\tau) = \int_0^{t+\tau+\alpha} u(s)\,ds$$

NOW CHECK TO SEE IF THESE ARE EQUAL

$$\int_\tau^{t+\alpha+\tau} u(\xi)\,d\xi \;?\; \int_0^{t+\tau+\alpha} u(s)\,ds$$

NO THEY ARE NOT EQUAL BECAUSE OF DIFFERENT LIMITS OF INTEGRATION

∴ IT IS NOT TIME INVARIANT

PROBLEM 5, CONT:

b) $y(t) = \int_{t-\alpha}^{t+\alpha} [u(s)]^2 ds$

→ FIRST CHECK LINEARITY:

SUBSTITUTE $a_1 u_1(s) + a_2 u_2(s)$ FOR $u(s)$

$$\int_{t-\alpha}^{t+\alpha} [a_1 u_1(s) + a_2 u_2(s)]^2 ds$$

BY INSPECTION THIS IS NOT EQUAL TO

$a_1 y_1(t) + a_2 y_2(t)$ WHICH $= a_1 \int_{t-\alpha}^{t+\alpha} [u_1(s)]^2 ds + a_2 \int_{t-\alpha}^{t+\alpha} [u_2(s)]^2 d$

∴ IT IS NOT LINEAR

→ NOW CHECK TIME INVARIANCE

SUBSTITUTE $u(s+\tau)$ FOR $u(s)$ ON THE R.H.S.

$\int_{t-\alpha}^{t+\alpha} [u(s+\tau)]^2 ds$ LET $\xi = s+\tau$ $d\xi = ds$

WHEN $s = t+\alpha \longrightarrow \xi = t+\alpha+\tau$

$s = t-\alpha \longrightarrow \xi = t-\alpha+\tau$

$$= \int_{t-\alpha+\tau}^{t+\alpha+\tau} [u(\xi)]^2 d\xi$$

NOW SUBSTITUTE $y(t+\tau)$ FOR $y(t)$ ON THE L.H.S. AND $(t+\tau)$ FOR (t) ON R.H.S

$$y(t+\tau) = \int_{t+\tau-\alpha}^{t+\tau+\alpha} [u(s)]^2 ds$$

NOW CHECK TO SEE IF THESE ARE EQUAL

$$\int_{t-\alpha+\tau}^{t+\alpha+\tau} [u(\xi)]^2 d\xi \ ? \ \int_{t+\tau-\alpha}^{t+\tau+\alpha} [u(s)]^2 ds$$

YES THEY ARE EQUAL SO

IT IS TIME INVARIANT

PROBLEM 5, CONT:

c) $y(t) = \alpha \frac{du(t)}{dt} \left| \frac{du(t)}{dt} \right|$

→ FIRST CHECK LINEARITY

SUBSTITUTE $a_1 \frac{du_1}{dt} + a_2 \frac{du_2}{dt}$ FOR $\frac{du}{dt}$

$$\alpha \left[a_1 \frac{du_1}{dt} + a_2 \frac{du_2}{dt} \right] \left| \left[a_1 \frac{du_1}{dt} + a_2 \frac{du_2}{dt} \right] \right|$$

BY INSPECTION, THIS DOES NOT EQUAL

$a_1 y_1 + a_2 y_2$ WHICH $= a_1 \alpha \frac{du_1}{dt} \left| \frac{du_1}{dt} \right| + a_2 \alpha \frac{du_2}{dt} \left| \frac{du_2}{dt} \right|$

∴ IT IS NOT LINEAR

→ NOW CHECK TIME INVARIANCE

SUBSTITUTE $\frac{du(t+\tau)}{dt}$ FOR $\frac{du(t)}{dt}$ ON THE R.H.S.

$$\alpha \frac{du(t+\tau)}{dt} \left| \frac{du(t+\tau)}{dt} \right|$$

NOW SUBSTITUTE $y(t+\tau)$ FOR $y(t)$ ON R.H.S.
AND $(t+\tau)$ FOR (t) ON L.H.S

$$y(t+\tau) = \alpha \frac{du(t+\tau)}{dt} \left| \frac{du(t+\tau)}{dt} \right|$$

NOW CHECK IF THESE ARE EQUAL:

$$\alpha \frac{du(t+\tau)}{dt} \left| \frac{du(t+\tau)}{dt} \right| = \alpha \frac{du(t+\tau)}{dt} \left| \frac{du(t+\tau)}{dt} \right|$$

∴ IT IS TIME INVARIANT

PROBLEM 5, CONT:

d) $\alpha \ddot{y}(t) + \beta \dot{y}(t) + \gamma y(t) = u(t)$

→ FIRST CHECK LINEARITY

SUBSTITUTE $a_1 u_1 + a_2 u_2$ FOR u ON THE R.H.S.

$$a_1 u_1 + a_2 u_2 = a_1\left[\alpha \ddot{y}_1 + \beta \dot{y}_1 + \gamma y_1\right] + a_2\left[\alpha \ddot{y}_2 + \beta \dot{y}_2 + \gamma y_2\right]$$

$$= a_1 y_1 + a_2 y_2$$

∴ IT IS LINEAR

→ NOW CHECK TIME INVARIANCE

SUBSTITUTE $u(t+\tau)$ FOR $u(t)$ ON R.H.S.

$u(t+\tau)$

SUBSITUTE $y(t+\tau)$ FOR $y(t)$ ON LHS AND $(t+\tau)$ FOR (t) ON R.H.S

$$\alpha \ddot{y}(t+\tau) + \beta \dot{y}(t+\tau) + \gamma y(t+\tau) = u(t+\tau)$$

NOW CHECK IF THESE ARE EQUAL

$$u(t+\tau) = u(t+\tau)$$

THEREFORE IT IS TIME INVARIANT

⇒ NOTICE THAT THE DERIVATIVE OF A FUNCTION IS A LINEAR, TIME INVARIANT OPERATOR ON THAT FUNCTION. THIS IS AN IMPORTANT CONCEPT TO REMEMBER.