13.42 HW2 SOLUTIONS SPRING 2005

PROBLEM 1:

$$u_1(t) \longrightarrow \underset{H_1(\omega)}{\boxed{\text{LTI}}} \longrightarrow y_1(t) = u_2(t) \longrightarrow \underset{H_2(\omega)}{\boxed{\text{LTI}}} \longrightarrow y_2(t)$$

Given $u_1(t) = u_0 \cos(\omega_0 t)$ and $H_1(\omega)$ and $H_2(\omega)$.

Can solve this several ways: convolution integrals in time domain or multiplication in frequency domain.

The advantage of going to the frequency domain is the multiplication of input by each filter to get output.

i.e. $Y_2(\omega) = U_1(\omega) H_1(\omega) H_2(\omega)$

Finite exponentials have the unique feature that they can be multiplied by transfer functions directly so rewrite $u_1(t)$ as $u_1(t) = \mathrm{Re}\{u_0 e^{i\omega_0 t}\}$

$y_2(t) = \mathrm{Re}\{u_0 e^{i\omega_0 t} H_1(\omega) H_2(\omega)\}$

where $H_1(\omega)$ and $H_2(\omega)$ have magnitudes $|H_1(\omega)|$ and $|H_2(\omega)|$ and phase $\measuredangle H_1(\omega)$ and $\measuredangle H_2(\omega)$

$$\boxed{y_2(t) = u_0 |H_1(\omega)| |H_2(\omega)| \cos(\omega_0 t + \measuredangle H_1(\omega) + \measuredangle H_2(\omega))}$$

you could also have done convolution integrals or FT's to frequency domain and then taken $\mathrm{IFT}[Y_2(\omega) = U_1(\omega) H_1(\omega) H_2(\omega)]$.

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Problem 2:

a) $\boxed{\text{NOT LTI}}$ b) $\boxed{\text{YES LTI}}$

Problem 3:

a) $f(t) = u_0(t-\tau)$

$$FT(f(t)) = F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt = \int_{-\infty}^{\infty} u_0(t-\tau)\, e^{-i\omega t}\, dt$$

$$\left[\text{USE: } \int_{-\infty}^{\infty} f(t)\, u_0(t-a)\, dt = f(a) \quad \text{where} \quad f(t) = e^{-i\omega t}\right]$$

$$\boxed{F(\omega) = e^{-i\omega\tau}}$$

b) $\tilde{f}(\alpha) = FT[f(x)] = \int_{-\infty}^{\infty} f(x)\, e^{-i\alpha x}\, dx$

$$FT\left[\frac{df}{dx}\right] = \int_{-\infty}^{\infty} \frac{df}{dx}\, e^{-i\alpha x}\, dx$$

let $u = e^{-i\alpha x}$ $du = -i\alpha e^{-i\alpha x} dx$

DO INTEGRATION BY PARTS:

$dv = \frac{df}{dx} dx$ $v = f$

$$= \left[e^{-i\alpha x} \cdot f\right]\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f \cdot \left(-i\alpha e^{-i\alpha x}\, dx\right)$$

$$= 0 + i\alpha \int_{-\infty}^{\infty} f e^{-i\alpha x}\, dx = \boxed{i\alpha \tilde{f}(\alpha)}$$

c) $\int_{-\infty}^{\infty} \frac{d^2 f}{dx^2}\, e^{-i\alpha x}\, dx$

let $u = e^{-i\alpha x}$ $du = -i\alpha e^{-i\alpha x} dx$

AGAIN, DO INTEGRATION BY PARTS:

$dv = \frac{d^2 f}{dx^2} dx$ $v = \frac{df}{dx}$

$$= \left[\frac{df}{dx} \cdot e^{-i\alpha x}\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \frac{df}{dx} \cdot \left(-i\alpha\, e^{-i\alpha x}\, dx\right)$$

$$= 0 + (i\alpha) \underbrace{\int_{-\infty}^{\infty} \frac{df}{dx}\, e^{-i\alpha x}\, dx}_{\text{from part b) this equals } i\alpha \hat{f}(\alpha)} = \boxed{(i\alpha)^2 \hat{f}(\alpha)} = \boxed{-\alpha^2 \hat{f}(\alpha)}$$

Problem 4:

a) LTI system with transfer function $H(\omega)$
and sinuisodal input:

$u(t) = u_0 e^{i(\omega_0 t + \psi_0)}$

if the output is $y(t)$ then:

$y(t) = y_0 e^{i(\omega_0 t + \psi)}$ where $y_0 = |H(\omega)|_{\omega=\omega_0} \cdot u_0$

and $\psi = \psi_0 + \measuredangle H(\omega)|_{\omega=\omega_0}$

$$\therefore \boxed{y(t) = u_0 \, |H(\omega_0)| \, e^{i(\omega_0 t + \psi_0 + \measuredangle H(\omega_0))}}$$

b) Yes, you can find the transfer function by

→ sending a series of monochromatic sinusoidal waves down the tank

→ measuring the platform response at each frequency

→ compute the magnitude and phase of the transfer function at each frequency:

if input is $u(t) = u_0 \cos(\omega_0 t + \psi_0)$
and output is $y(t) = y_0 \cos(\omega_0 t + \psi_1)$,

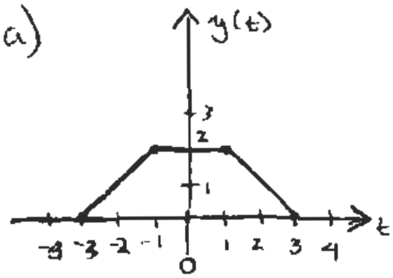
u_0, ω_0, ψ_0 are known inputs and y_0, ψ_1 are measured,

then $|H(\omega)|_{\omega=\omega_0} = y_0/u_0$

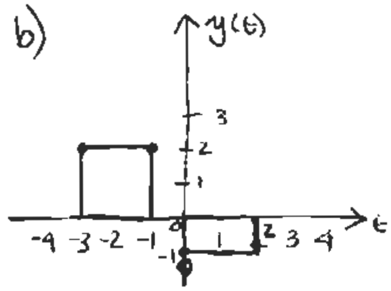
and $\measuredangle H(\omega)|_{\omega=\omega_0} = \psi_1 - \psi_0$

Problem 5: Use graphical "flip & slide" method.

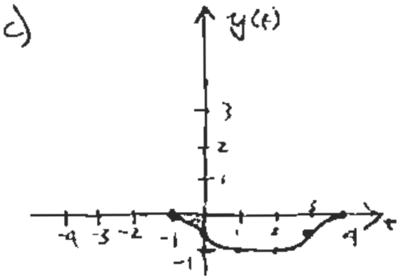
a)



b)

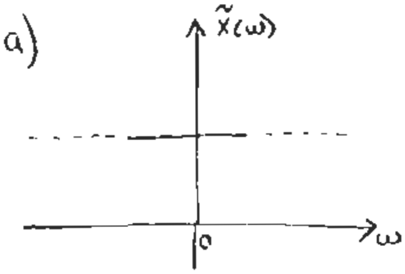


c)



Problem 6:

a)



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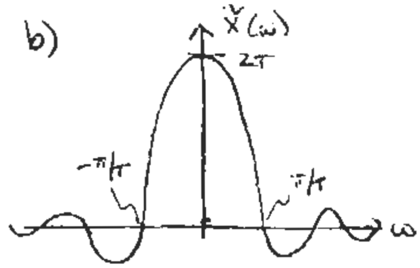
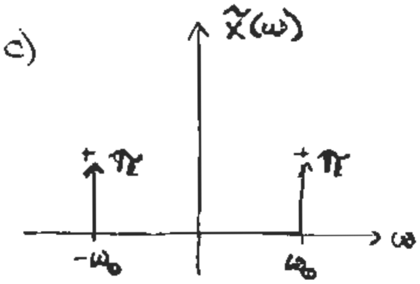
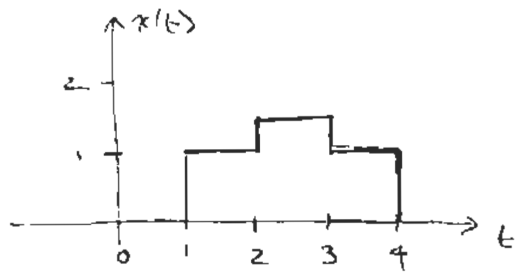
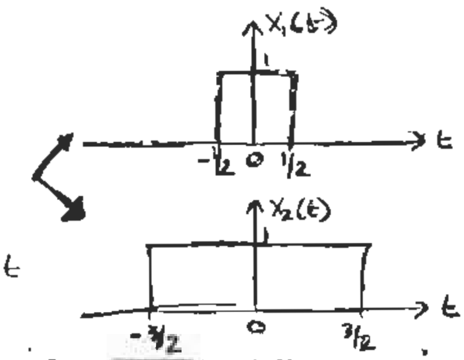


c)



Problem 7:





Decompose $x(t)$ into: $x(t) = \frac{1}{2} x_1(t-2.5) + x_2(t-2.5)$

[USE F.T. OF A RECTANGULAR PULSE SIGNAL $x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & |t| > T_1 \end{cases}$ $X(\omega) = \int_{-T_1}^{T_1} e^{-i\omega t}\, dt = 2\frac{\sin \omega T_1}{\omega}$

$\therefore \; X_1(\omega) = \dfrac{2\sin(\omega/2)}{\omega}$ and $X_2(\omega) = \dfrac{2\sin(3\omega/2)}{\omega}$

USING THE LINEARITY, AND TIME SHIFTING PROPERTIES OF THE F.T.

$$\boxed{X(\omega) = \left\{ \frac{\sin(\omega/2)}{\omega} + \frac{2\sin(3\omega/2)}{\omega} \right\} e^{-i\frac{5\omega}{2}}}$$

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PROBLEM 8: ASSUME DEEP WATER

$$\eta(x,t) = a\cos(kx-\omega t)$$

$$\phi(x,z,t) = \frac{ga}{\omega} e^{kz}\sin(kx-\omega t)$$

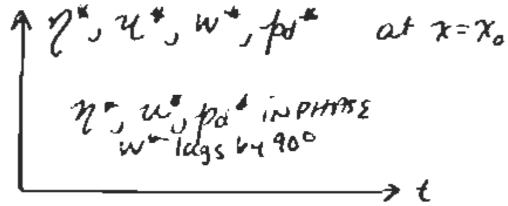
$$\left[\text{USE } \eta(x,t) = -\frac{1}{g}\frac{\partial\phi}{\partial t}\Big|_{z=0}\right]$$

$$u(x,z,t) = \frac{\partial\phi}{\partial x} = \omega a e^{kz}\cos(kx-\omega t)$$

$$w(x,z,t) = \frac{\partial\phi}{\partial z} = \omega a e^{kz}\sin(kx-\omega t)$$

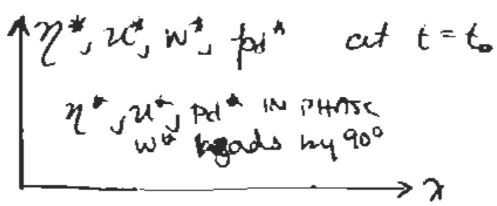
$$p_d = -\rho\frac{\partial\phi}{\partial t} = \rho g a e^{kz}\cos(kx-\omega t)$$

a) See attached plots of the form



"*" indicates NONDIMENSIONALIZED VALUE

b) See attached plots of the form



c) $P_{TOTAL} = P_d + P_{STATIC} = -\rho\frac{\partial\phi}{\partial t} - \rho g z = \rho g\left[a e^{kz}\cos(kx-\omega t) - z\right]\Big|_{z=z_n}$

where $z_n = -16n$ meters

WAVE CREST: $\cos(kx-\omega t) = 1$ $\therefore$ $p_{TOT} = \rho g\left[a e^{kz_n} - z_n\right]$ @ CREST

WAVE TROUGH: $\cos(kx-\omega t) = -1$ $\therefore$ $p_{TOT} = \rho g\left[-a e^{kz_n} - z_n\right]$ @ TROUGH

WAVE NODE: $\cos(kx-\omega t) = 0$ $\therefore$ $p_{TOT} = -\rho g z_n$ @ NODE

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PROBLEM 8 CONT.:

d) FLUID ACCELERATION $\frac{D\vec{V}}{Dt} = \frac{\partial \vec{V}}{\partial t} + (\vec{V}\cdot\nabla)\vec{V}$

BY LINEAR APPROXIMATION $\frac{D\vec{V}}{Dt} \simeq \frac{\partial \vec{V}}{\partial t}$

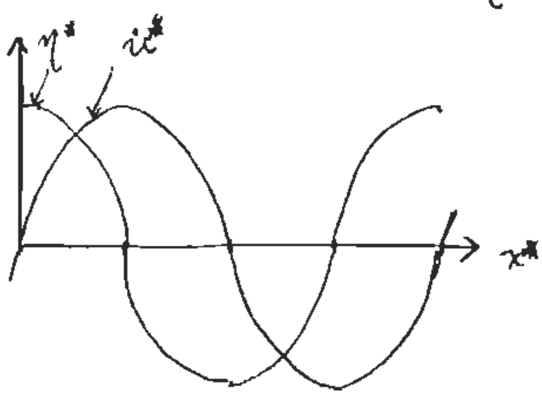
$F_{A.M} \propto$ FLUID ACCELERATION

$\therefore$ HORIZONTAL ADDED MASS FORCE IS $\begin{Bmatrix} \text{MAXIMUM} \\ \text{MINIMUM} \\ \text{ZERO} \end{Bmatrix}$ WHEN FLUID HORIZONTAL ACCELERATION IS.

HORIZONTAL FLUID ACCELERATION $= \frac{\partial u}{\partial t} = +\omega^2 a e^{kz} \sin(kx - \omega t)$

THIS ACHIEVES A $\begin{Bmatrix} \text{MAX} \\ \text{MIN} \\ \text{ZERO} \end{Bmatrix}$ WHEN $\sin(kx-\omega t)$ IS $\begin{Bmatrix} 1 \\ -1 \\ 0 \end{Bmatrix}$ AND $\cos(kx-\omega t)$ is $\begin{Bmatrix} 0 \\ 0 \\ \pm 1 \end{Bmatrix}$

A SNAPSHOT IN TIME:



$\dot{u}$ and η are 90° out of phase

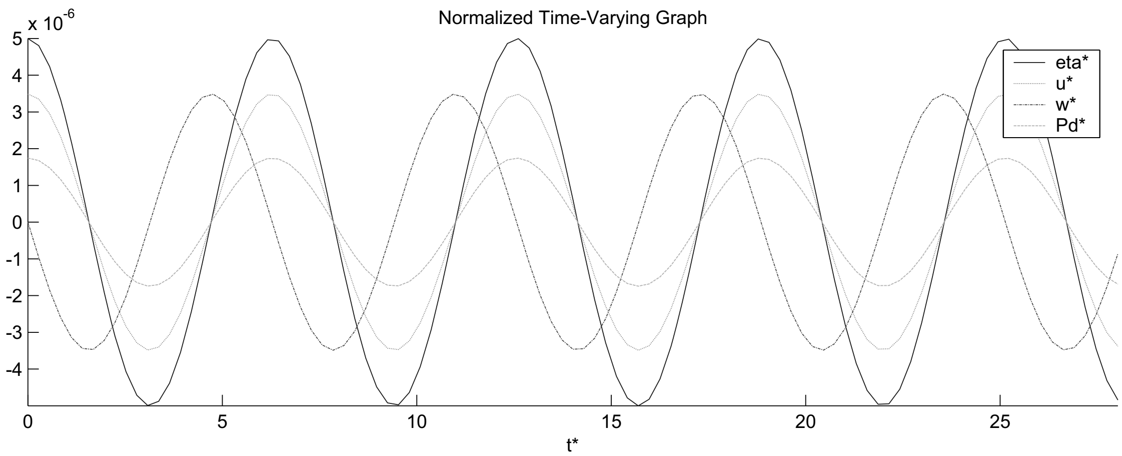
THE HORIZONTAL ADDED MASS FORCE IS ZERO AT BOTH CREST AND TROUGH OF THE WAVE ELEVATION.

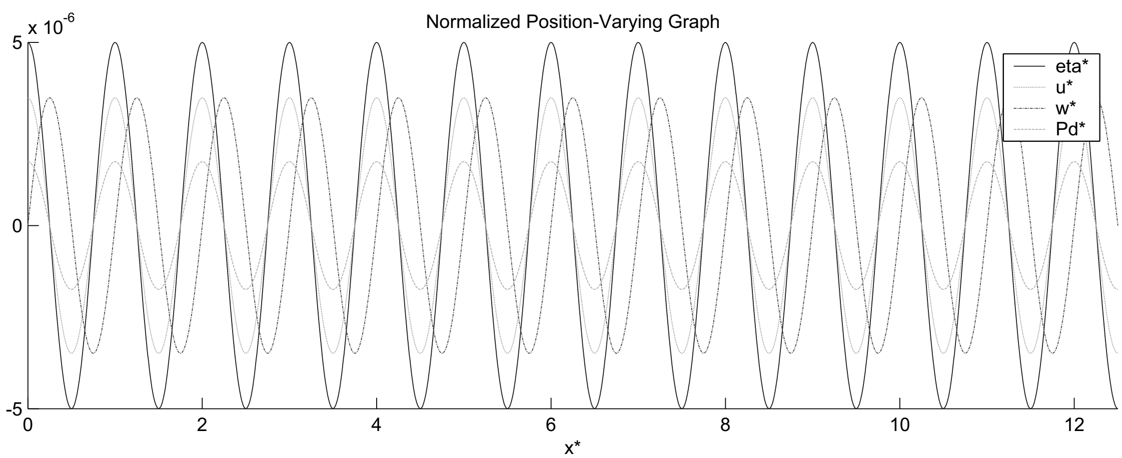
THE HORIZONTAL ADDED MASS FORCE ALTERNATES BETWEEN MAXIMUM AND MINIMUM AT CONSECUTIVE NODES.

e) HORIZONTAL VISCOUS FORCE $\propto u|u|$.

u IS IN PHASE WITH η SO |HORIZ. VISCOUS FORCE| IS MAXIMUM UNDER CRESTS & TROUGHS.

VERTICAL VISCOUS FORCES $\propto w|w|$, AND w IS 90° OUT OF PHASE WITH η SO |VERT. VISCOUS FORCE| IS MAXIMUM UNDER NODES.





Student Version of MATLAB