13.42 HW4 SOLUTIONS SPRING 2005

1) $f_X(x) = \begin{cases} \alpha x^2 & , 0 \le x \le 10 \\ 0 & , \text{else} \end{cases}$

a) Find α. By DEFINITION, $\int_{-\infty}^{\infty} f_X(x)dx = 1 = \int_{-\infty}^{0} 0 \cdot dx + \int_{0}^{10} \alpha x^2 dx + \int_{10}^{\infty} 0\, dx$

$$1 = \int_0^{10} \alpha x^2 dx = \frac{\alpha x^3}{3}\Big|_0^{10} = \alpha \cdot \frac{1000}{3}$$

$$\boxed{\alpha = \frac{3}{1000}}$$

b) Find $P(X>5)$. This equals $1 - P(X \le 5)$.

By Definition $P(X \le 5) = F_X(5)$

and $F_X(x) = \int_{-\infty}^{x} f_X(x)dx$

So $F_X(5) = \int_{-\infty}^{0} 0 \cdot dx + \int_0^5 \frac{3}{1000} x^2 dx$

$$F_X(5) = \frac{3}{1000} \cdot \frac{x^3}{3}\Big|_0^5 = \frac{3}{1000} \cdot \frac{125}{3} = \frac{125}{1000} = 0.125$$

So $P(X \le 5) = 0.125$

and $P(X>5) = 1 - 0.125 = \boxed{0.875}$

c) Find μ_X. $\mu_X = \int_{-\infty}^{\infty} x f_X(x)dx = \int_{-\infty}^{0} 0\,dx + \int_0^{10} \frac{3}{1000} x^3 dx + \int_{10}^{\infty} 0\,dx$

$$= \frac{3}{1000} \cdot \frac{x^4}{4}\Big|_0^{10} = \frac{30{,}000}{4\,000} - 0 = \frac{30}{4} = \boxed{7.5}$$

d) Find σ_X^2 and σ_X. $\sigma_X^2 = E[X^2] - \mu_X^2 = \int_{-\infty}^{\infty} x^2 f_X(x)dx - (\mu_X)^2$

$$= \int_{-\infty}^{0} 0\,dx + \int_0^{10} \frac{3}{1000} x^4 dx + \int_{10}^{\infty} 0\,dx - (7.5)^2 = \frac{3}{1000} \cdot \frac{x^5}{5}\Big|_0^{10} - (7.5)^2$$

$$= \frac{3}{1000} \cdot \frac{100{,}000}{5} - 56.25 = 60 - 56.25 = \boxed{3.75 = \sigma_X^2} \quad \boxed{\sigma_X = 1.94}$$

2) GAUSSIAN DISTRIBUTION OF X WITH $\mu_X = 60$ DAYS

$\sigma_X = 15$ DAYS

a) $$f_X(x) = \frac{1}{\sigma_X\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu_x}{\sigma_x}\right)^2} = \boxed{\frac{1}{15\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-60}{15}\right)^2} = f_X(x)}$$

b) $$F_X(x) = \int_{-\infty}^{x} f_X(x)\,dx = \frac{1}{15\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}\frac{(x-60)^2}{15^2}}\,dx$$

[USE THAT $\frac{1}{\sigma_X\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{(x-\mu_x)^2}{2\sigma_x^2}}\,dx = \frac{1}{2}\left[1+\mathrm{erf}\left(\frac{x-\mu_x}{\sigma_x\sqrt{2}}\right)\right]$]

$$\boxed{F_X(x) = \frac{1}{2}\left[1+\mathrm{erf}\left(\frac{x-60}{15\sqrt{2}}\right)\right]}$$

c) $P[40 < x \le 70] = F_X(70) - F_X(40) = 0.7486 - 0.0918 = 0.6568$

TO GET $F_X(70) = \frac{1}{2}\left[1+\mathrm{erf}\left(\frac{70-60}{15\sqrt{2}}\right)\right] = 0.7486$

$F_X(40) = \frac{1}{2}\left[1+\mathrm{erf}\left(\frac{40-60}{15\sqrt{2}}\right)\right] = 0.0918$

} PLUG INTO PROGRAM SUCH AS MATHCAD, PROGRAMMABLE CALCULATOR, ETC.

ALTERNATIVELY, YOU CAN USE STANDARD NORMAL TABLES.

where $P(a < x \le b) = \Phi\left(\frac{b-\mu_x}{\sigma_x}\right) - \Phi\left(\frac{a-\mu_x}{\sigma_x}\right)$

PLUG THESE INTO TABLES TO GET Φ VALUES.

$$P(40 < x \le 70) = \Phi\left(\frac{70-60}{15}\right) - \Phi\left(\frac{40-60}{15}\right) = \Phi(0.67) - \Phi(-1.33)$$

[There aren't negatives in the table so use $\Phi(-1.33) = 1 - \Phi(1.33)$]

$$P(40 < x \le 70) = \Phi(0.67) - [1-\Phi(1.33)] = 0.7486 - (1 - 0.9082)$$
$$= 0.6568$$

$$\boxed{P(40 < x \le 70) = 65.7\%}$$

2) d) $P[x \geq 30 \text{ days}] = 1 - F_x(30) = 1 - 0.0228 = 0.9772$

ALTERNATIVELY, USING TABLES
$P(30 < X \leq \infty) = \Phi(\infty) - \Phi\left(\frac{30-60}{15}\right) = 1 - \Phi(-2.00)$
$= 1 - [1 - \Phi(2.00)] = \Phi(2.00) = 0.9772$

$\boxed{P(30 < x) = 0.9772}$

3) Poisson Distribution

$\lambda = 5 \frac{\text{RAINSTORMS}}{\text{YEAR}}$ $P[X=x] = \frac{(\lambda t)^x}{x!} e^{-\lambda t} = \frac{(5t)^x}{x!} e^{-5t}$

a) $P[X=0] = \frac{(5 \cdot 1)^0}{0!} e^{-5 \cdot 1} = e^{-5} = 0.00674$

Next year ∴ t = 1 year

$\boxed{P[0 \text{ storms next year}] = 0.7\%}$

b) $P[X=5] = \frac{(5 \cdot 1)^5}{5!} e^{-5 \cdot 1} = \frac{5^5}{5!} e^{-5} = 0.17547$

Next year ∴ t = 1 year

$\boxed{P[5 \text{ storms next year}] = 17.5\%}$

c) $P[X \geq 3] = 1 - P[X=0] - P[X=1] - P[X=2]$

Next year ∴ t = 1 year

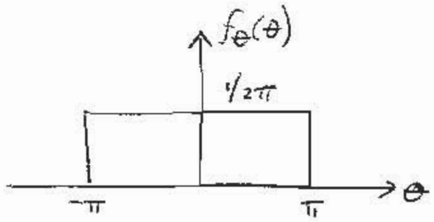
$P[X=1] = \frac{(5 \cdot 1)^1}{1!} e^{-5 \cdot 1} = 0.03369$

$P[X=2] = \frac{(5 \cdot 1)^2}{2!} e^{-5 \cdot 1} = 0.08422$

$P[X \geq 3] = 1 - 0.00674 - 0.03369 - 0.08422 = 0.87535$

$\boxed{P[X \geq 3 \text{ STORMS NEXT YEAR}] = 87.5\%}$

4) $\eta(x,t) = A\sin(\omega_0 t + k_0 x) = A\sin(\omega_0 t + \theta) = \eta(t,\theta)$



a) ENSEMBLE AVERAGE IS CALCULATED AT A CONSTANT TIME, SO LET $t = t_0$

$$\mu_\Theta(t_0) = E\{\eta(t_0,\theta)\} = \int_{-\infty}^{\infty} \eta(t_0,\theta)\cdot f_\Theta(\theta)\,d\theta$$

[USE EQN (32), pg 5 of R.V. LECTURE: $E[h(x)] = \mu_{h(x)} = \int_{-\infty}^{\infty} h(x) f_x(x)\,dx$]

$$\mu_\Theta(t_0) = \int_{-\infty}^{-\pi} \eta(t_0,\theta)\cdot 0\cdot d\theta + \int_{-\pi}^{\pi} \eta(t_0,\theta)\cdot\frac{1}{2\pi}\cdot d\theta + \int_{\pi}^{\infty} \eta(t_0,\theta)\cdot 0\cdot d\theta$$

$$= \int_{-\pi}^{\pi} \eta(t_0,\theta)\frac{1}{2\pi}\,d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} A\sin(\omega_0 t_0 + \theta)\,d\theta$$

$$= \frac{A}{2\pi}\underbrace{\int_{-\pi}^{\pi} \sin(\omega_0 t_0 + \theta)\,d\theta}_{0} = \boxed{0 = \text{ENSEMBLE AVERAGE}}$$

ENSEMBLE VARIANCE. AGAIN LET $t = t_0$.

$$VAR(t_0) = E\{[\eta(t_0,\theta) - \mu_\theta(t_0)]^2\} = E\{\eta^2(t_0,\theta)\}$$

$$= \int_{-\infty}^{\infty} \eta^2(t_0,\theta) f_\theta\theta\,d\theta = \int_{-\pi}^{\pi} \frac{1}{2\pi}\cdot A^2\sin^2(\omega_0 t_0 + \theta)\,d\theta$$

$$= \frac{A^2}{2\pi}\underbrace{\int_{-\pi}^{\pi} \sin^2(\omega_0 t_0 + \theta)\,d\theta}_{\pi} = \frac{A^2}{2\pi}\cdot\pi = \boxed{\frac{A^2}{2} = \text{ENSEMBLE VARIANCE}}$$

CORRELATION. LET $t = t_0$ and τ be a CONSTANT TIME LAG.

$$R_{\theta\theta}(t_0, t_0+\tau) = E\{\eta(t_0,\theta)\cdot\eta(t_0+\tau,\theta)\} = \int_{-\infty}^{\infty} \eta(t_0,\theta)\,\eta(t_0+\tau,\theta) f_\theta\theta\,d\theta$$

$$= \int_{-\pi}^{\pi} A\sin(\omega_0 t_0 + \theta)\,A\sin(\omega_0(t_0+\tau)+\theta)\frac{1}{2\pi}\,d\theta$$

4a cont.) $R_{\theta\theta}(t_0, t_0+\tau) = \frac{A^2}{2\pi}\int_{-\pi}^{\pi} \sin(\omega_0 t_0+\theta)\sin(\omega_0(t_0+\tau)+\theta)\,d\theta$

$$R_{\theta\theta} = \frac{A^2}{2\pi}\cdot\pi\cos(\omega_0\tau) = \frac{A^2}{2}\cos(\omega_0\tau)$$

$$\boxed{R_{\theta\theta} = \frac{A^2}{2}\cos(\omega_0\tau)}$$

b) TEMPORAL AVERAGE:

$$M^t\{\eta_i(t)\} = \bar{\eta}^t = \lim_{T\to\infty}\frac{1}{T}\int_0^T \eta_i(t)\,dt$$

$$= \lim_{T\to\infty}\frac{1}{T}\int_0^T A\sin(\omega_0 t+\theta_i)\,dt = \lim_{T\to\infty}\frac{1}{T}\cdot\frac{-A}{\omega_0}\left[\cos(\omega_0 t+\theta_i)\right]_0^T$$

$$= \lim_{T\to\infty}\frac{-A}{T\omega_0}\left[\cos(\omega_0 T+\theta_i)\right] = \boxed{0 = \bar{\eta}^t}$$

TEMPORAL VARIANCE

$$V^t\{\eta_i(t)\} = \lim_{T\to\infty}\frac{1}{T}\int_0^T\left[\eta_i(t) - \underbrace{M\{\eta_i(t)\}}_{0}\right]^2 dt = \lim_{T\to\infty}\frac{1}{T}\int_0^T \eta_i^2(t)\,dt$$

$$= \lim_{T\to\infty}\frac{1}{T}\int_0^T A^2\sin^2(\omega_0 t+\theta_i)\,dt = \lim_{T\to\infty}\frac{A^2}{T}\underbrace{\int_0^T\sin^2(\omega_0 t+\theta_i)\,dt}_{T/2}$$

$$= \lim_{T\to\infty}\frac{A^2 T}{T2} = \lim_{T\to\infty}\frac{A^2}{2} = \boxed{\frac{A^2}{2} = V^t\{\eta_i(t)\}}$$

CORRELATION

$$R^t\{\eta_i(t)\} = \lim_{T\to\infty}\frac{1}{T}\int_0^T\left[\eta_i(t) - \underbrace{M^t\{\eta_i(t)\}}_{0}\right]\left[\eta_i(t+\tau) - \underbrace{M^t\{\eta_i(t+\tau)\}}_{0}\right]dt$$

$$= \lim_{T\to\infty}\frac{1}{T}\int_0^T\eta_i(t)\,\eta_i(t+\tau)\,dt = \lim_{T\to\infty}\frac{1}{T}\int_0^T A^2\sin(\omega_0 t+\theta_i)\sin(\omega_0(t+\tau)+\theta_i)\,dt$$

$$= \lim_{T\to\infty}\frac{A^2}{T}\int_0^T\sin(\omega_0 t+\theta_i)\sin(\omega_0(t+\tau)+\theta_i)\,dt$$

$$\boxed{R^t\{\eta_i(t)\} = \frac{A^2}{2}\cos(\omega_0\tau)}$$

∴ ENSEMBLE STATISTICS EQUAL TEMPORAL STATISTICS.

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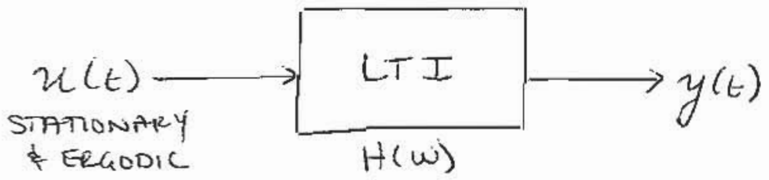
5)



a) Yes. If the input to an LTI system is stationary and ergodic, the output is as well.

b) If a random process has ensemble statistics that do not depend on time, it is stationary.

c) Yes. You can have a stationary, non-ergodic process.
No. You cannot have an ergodic, non-stationary process.

d) Given $S_x(\omega)$ and $H(\omega)$, $S_y(\omega) = |H(\omega)|^2 S_x(\omega)$.

e) Given $S_x(\omega)$ and $S_y(\omega)$, $|H(\omega)| = \sqrt{\dfrac{S_y(\omega)}{S_x(\omega)}}$.

You can find the magnitude of $H(\omega)$, but you would need more information to find the phase of $H(\omega)$.

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6) $\sigma = M_0 = 1.25m$ $\quad \bar{T} = 12.4\, sec$ $\quad$ PLATFORM HEIGHT, $h = A = 3m$

a) $$\bar{\eta}(A) = \frac{1}{\bar{T}} \cdot e^{-A^2/2M_0}$$

$$\bar{\eta}(3m) = \frac{1}{12.4 sec} \cdot e^{\frac{-(3m)^2}{2\cdot(1.25m)^2}} = 0.00453\, s^{-1}$$

$\bar{\eta}(3m) = 0.00453$ upcrossings per second
$= 0.2716$ upcrossings per minute
$= 16.3$ upcrossings per hour

b) $$\bar{\eta}(h) = \frac{1}{12.4sec} \cdot e^{-h^2/2(1.25m)^2} = \frac{1}{1\,hour} = \frac{1}{60min} = \frac{1}{3600\,sec}$$

$$e^{\frac{-h^2}{3.125m^2}} = \frac{12.4sec}{3600sec} = 0.003444$$

$$\frac{-h^2}{3.125m^2} = \ln[0.003444] = -5.67099$$

$$h^2 = (3.125m^2)(5.67099)$$

$$h^2 = 17.722\, m^2$$

$$h = 4.21m$$

c) $$\bar{\eta}(h) = \frac{1}{12.4sec} e^{-h^2/3.125m^2} = \frac{1}{1day} = \frac{1}{24hr} = \frac{1}{86400sec}$$

$$e^{-h^2/3.125m^2} = \frac{12.4sec}{86400sec} = 0.000144$$

$$-h^2/3.125m^2 = \ln[0.000144] = -8.849$$

$$h^2 = (3.125m^2)(8.849) = 27.6533\, m^2$$

$$h = 5.26m$$