13.42 Design Principles for Ocean Vehicles
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1. STATIONARY AND ERGODIC RANDOM PROCESSES

Given the random process $y(\mathbf{z}, t)$ we assume that the expected value of the random process is zero, however this is not always the case. If the expected value equals some constant x_o we can adjust the random process such that the expected value is indeed zero:

$y(\mathbf{z}, t) = x(t, \mathbf{z}) - x_o$.

Again we note that for the stationary ergodic random process the time statistics and event statistics are equal. We write the autocorrelation $R(\mathbf{t})$:

$$R(\mathbf{t}) = E\{y(t, \mathbf{z}) \, y(t + \mathbf{t}, \mathbf{z})\} = R^t(\mathbf{t}) = \lim_{T \to \infty} \frac{1}{T} \int_0^T y_i(t) \, y_i(t + \mathbf{t}) dt \tag{1}$$

CORRELATION PROPERTIES

1. $R(0) = \text{variance} = \mathbf{s}^2 = (\text{RMS})^2 \geq 0$

2. $R(\mathbf{t}) = R(-\mathbf{t})$

3. $R(0) \geq |R(\mathbf{t})|$

EXAMPLE: Consider the following random process that is a summation of cosines of different frequencies – similar to water waves.

$$y(\mathbf{z}, t) = \sum_{n=1}^{N} a_n \cos(\mathbf{w}_n t + \mathbf{y}_n(\mathbf{z})) \tag{2}$$

where $\mathbf{y}_n(\mathbf{z})$ are all independent random variables in $[0, 2\mathbf{p}]$ with a uniform pdf. This

random process is stationary and ergodic with an expected value of zero.

The autocorrelation $R(t)$ is

$$R(t) = \sum_{n=1}^{N} \frac{a_n^2}{2} \cos(w_n t) \tag{3}$$

2. SPECTRUM

Given a random process that is stationary and ergodic, with an expected value of zero and autocorrelation $R(t)$, the power spectral density, or spectrum, of the random process is defined as the Fourier transform of the autocorrelation.

$$S(w) = \int_{-\infty}^{\infty} R(t) e^{-iwt} dt \tag{4}$$

Conversely, the autocorrelation, $R(t)$, is the inverse FT of the spectrum

$$R(t) = \frac{1}{2p} \int_{-\infty}^{\infty} S(w) e^{iwt} dw \tag{5}$$

Properties of the Spectrum $S(w)$ of $y(z,t)$:

1. $S(w)$ is a real and even function. Since $R(t)$ is real and even.

2. $\int_{-\infty}^{\infty} R(t) e^{-iwt} dt = \int_{-\infty}^{\infty} R(t) \{\cos wt - i \sin wt\} dt$

3. It can be shown that the sine component integrates to zero.

4. The variance of the random process can be found from the spectrum:

5. $s^2 = (RMS)^2 = R(0) = \frac{1}{2p} \int_{-\infty}^{\infty} S(w) dw$

6. The spectrum is positive always: $S(w) \geq 0$

7. With some restrictions it can also be established that

$$S(\boldsymbol{w}) = \lim_{T \to \infty} \frac{1}{2\boldsymbol{p}} \left| \int_{-T}^{T} y(t, \boldsymbol{z}_k) e^{-i\boldsymbol{w}t} dt \right|$$

(Beyond the scope of this course – see Papoulis p. 343 for more info)

A spectrum covers the range of frequencies from minus infinity to positive infinity ($-\infty < \boldsymbol{w} < +\infty$).

A *one-sided spectrum*, $S^+(\boldsymbol{w})$, is a representation of the entire spectrum only in the positive frequency domain. This one-sided spectrum is convenient and used traditionally, but is not *strictly* correct.
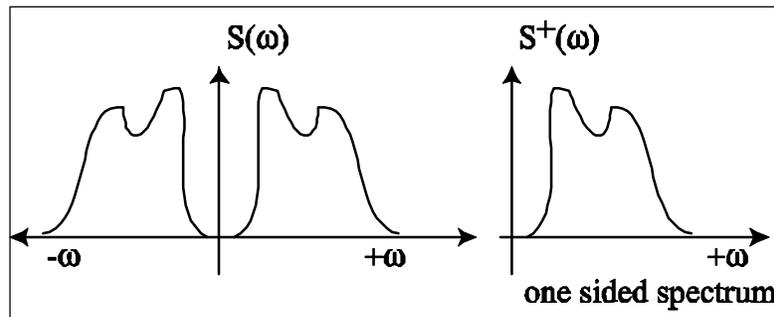
The one sided spectrum is a representation of the entire spectrum only in the positive frequency domain. We "fold" the energy over $\boldsymbol{w} = 0$ and introduce the $\frac{1}{2\boldsymbol{p}}$ factor we get:

$$S^+(\boldsymbol{w}) = \begin{cases} \dfrac{2}{2\boldsymbol{p}} S(\boldsymbol{w}) & \boldsymbol{w} \geq 0 \\ 0 & else \end{cases} \tag{9}$$

This representation for the one-sided spectrum comes from the variance, $R(0)$:

$$R(0) = \boldsymbol{s}^2 = \frac{1}{2\boldsymbol{p}} \int_{-\infty}^{\infty} S(\boldsymbol{w}) d\boldsymbol{w} = \frac{2}{2\boldsymbol{p}} \int_{0}^{\infty} S(\boldsymbol{w}) d\boldsymbol{w} \tag{10}$$

which we can rewrite in terms of the one-sided spectrum

$$\mathbf{s}^2 = \int_0^\infty S^+(\mathbf{w})\,d\mathbf{w} \tag{11}$$

where

$$S^+(\mathbf{w}) = \frac{2}{2\mathbf{p}} S(\mathbf{w}); \quad \text{for } \mathbf{w} \geq 0 \tag{12}$$

The spectrum provides a distributed amplitude, or "probability density" of amplitudes, indicating the energy of the system.

3. Application of Spectrum to LTI systems

u(t) ———————→ | LTI | ———→ y(t)

H(ω)
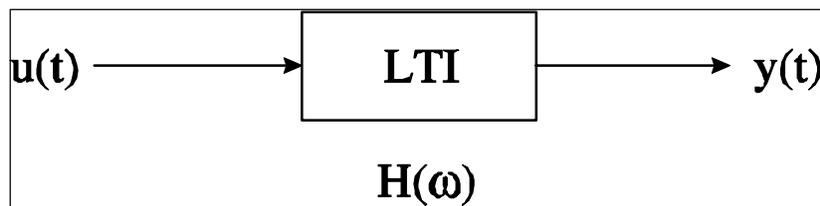
1. Linear time invariant system

We can use the spectrum to help us analyze linear time invariant systems. Since the LTI system is characterized by its impulse response, $h(t)$, given an input, $u(t)$, the output can be found from the convolution of the impulse response and the input:

$$y(t) = u(t) * h(t) \tag{13}$$

or the Fourier transform of the output is equal to the Transfer function, *H(w)*, times the FT of the input:

$$\tilde{y}(\boldsymbol{w}) = H(\boldsymbol{w})\, \tilde{u}(\boldsymbol{w}) \tag{14}$$

For such a LTI system, if $u(t)$ is a stationary and ergodic random process then $y(t)$ is also stationary and ergodic. Defining the spectrum of $u(t)$ as $S_u(\boldsymbol{w})$ and the spectrum of $y(t)$ as $S_y(\boldsymbol{w})$ we can show that the following holds true:

$$S_y(\boldsymbol{w}) = |H(\boldsymbol{w})|^2\, S_u(\boldsymbol{w})$$

where $|H(\boldsymbol{w})|^2$ is square of the magnitude the transfer function of the LTI system. This is known as the Wiener-Khinchine Relation. We would like to use this relationship and properties of the spectrum to gain insight about the system output, essentially the statistics of the output, knowing only the input and the system transfer function.

4. SHORT TERM STATISTICS

Since we are interested in obtaining the statistics associated with the random processes we can use the Spectra to calculate them.

As an example, lets look at a spectrum, $S_u(\boldsymbol{w})$, of sea elevations which consists of *many* harmonic components. The central limit theorem from probability says, given that there are many events, the sea elevation will have a gaussian distribution. If we assume that the input function, $u(t)$, is a stationary and ergodic random process with a gaussian pdf, then the the output function, $y(t)$ is also stationary and ergodic with a gaussian pdf. This assumption is good for "short" time intervals, on the order of a storm or an afternoon, but not necessarily over weeks or decades.
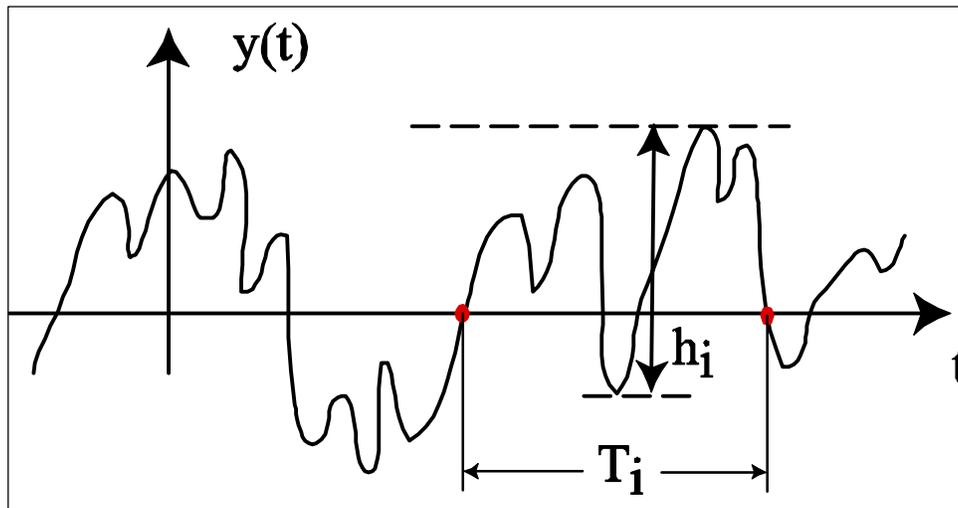
We are interested in gathering the statistics of $y(t)$ given the spectrum $S_y(w)$. The waveheights, h_i, and wave periods of interest T_i are the random variables in this problem. We have already given that this is a stationary and ergodic random process thus we know that the time statistics are equivalent to the event statistics, we can also show that if $y(t)$ is a realization of the random process $y(t, z)$ (which is stationary and ergodic) then ergodicity says that h_i and T_i will provide the statistics on $y(t, z)$ and vice versa.

Often we need to know how often is a certain level is exceeded by the process, in this case the wave height. In order to determine this, we can look at the occurrences of *UPCROSSINGS* only in a certain time period of interest. (There is further information on this subject in section 3 of the supplemental notes by Traintafyllou and Chryssotomidis.)

We can use the *moments* of the spectrum as follows:

Zeroth Moment:

$$M_o = \int_0^\infty S^+(w)\, dw = s^2 = VARIANCE \qquad (15)$$

Second Moment:

$$M_2 = \int_0^\infty S^+(\boldsymbol{w})\,\boldsymbol{w}^2 d\boldsymbol{w} \qquad (16)$$

Fourth Moment:

$$M_4 = \int_0^\infty S^+(\boldsymbol{w})\,\boldsymbol{w}^4 d\boldsymbol{w} \qquad (17)$$

Note, it can be shown that $M_1, M_3,$ etc... are zero (for n odd).

Lets define $\bar{\boldsymbol{h}}(A)$ as the average frequency of upcrossings past a certain level A (crossings above A per time) and $\bar{\boldsymbol{h}}(0)$ as the average frequency of all upcrossings (past a zero level), such that

$$\bar{\boldsymbol{h}}(A) = \frac{1}{2\boldsymbol{p}}\sqrt{\frac{M_2}{M_o}}\, e^{-A^2/2M_o} \qquad (18)$$

and

$$\bar{\boldsymbol{h}}(0) = \frac{1}{2\boldsymbol{p}}\sqrt{\frac{M_2}{M_o}}. \qquad (19)$$

We now have an equation for the upcrossing frequency which can be easily determined if we know the average period, $\bar{T}$. The average period can be thought of here as the expected value of all periods, T_i, such that

$$\frac{1}{\bar{T}} \simeq \frac{1}{2\boldsymbol{p}}\sqrt{\frac{M_2}{M_o}} \qquad (20)$$

and

$$\bar{\boldsymbol{h}}(A) = \frac{1}{\bar{T}}\, e^{-A^2/2M_o} \qquad (21)$$

APPLICATION EXAMPLE: An offshore platform is exposed to a storm with waves of standard deviation, $s = 2\,meters$ and an average wave period, $\overline{T} = 8\,sec$. We want to design h, the platform height, so that the deck is flooded only once every 10 minutes. Here we will neglect diffraction of the waves, thus the incoming waves are not effected by the presence of the structure and the magnitude of the transfer function is 1. The input wave train $u(t)$ equals $y(t)$, the wave height at the platform. Following from the previous lecture we have the relationship between the input spectrum and the output spectrum.

$$S_u(\boldsymbol{w}) = |H(\boldsymbol{w})|^2 \, S_y(\boldsymbol{w}) = 1 \cdot S_y(\boldsymbol{w}) \tag{22}$$

We want the frequency of upcrossings above the deck height, h, to be one every ten minutes:

$$\overline{\boldsymbol{h}(h)} = \frac{1}{10^{min} * 60^{sec/min}} = 1/600\,(times/sec). \tag{23}$$

We also have the equation for the number of upcrossings above height h as

$$\overline{\boldsymbol{h}(h)} = \frac{1}{\overline{T}} e^{-h^2/2Mo} \tag{24}$$

So we can equate these two and solve for h:

$$\tfrac{1}{\overline{T}} e^{-h^2/2Mo} = 1/600$$

$$\tfrac{\overline{T}}{600} = e^{-h^2/2Mo}$$

$$2M_o \ln\left(\tfrac{8}{600}\right) = h^2$$

$$\underline{\underline{h = 5.877\ meters}}$$

This number of times above a level is sufficient for deck wetting frequencies. However for structural design calculations where peak stresses are needed to check for structural safety, we need to consider the probability of a maximum level, x_m, reaching a certain design level, A. For this it is ideal to define here the bandwidth of the spectrum in terms of the moments of the Spectrum.

4.1. BANDWIDTH OF THE SPECTRUM

The bandwidth of the spectrum describes how "wide" the spectrum is. For a harmonic signal with one frequency the bandwidth is nearly zero and a tight spectral peak appears. However in a signal that contains multiple frequencies the bandwidth increases. For white noise the bandwidth approaches 1.
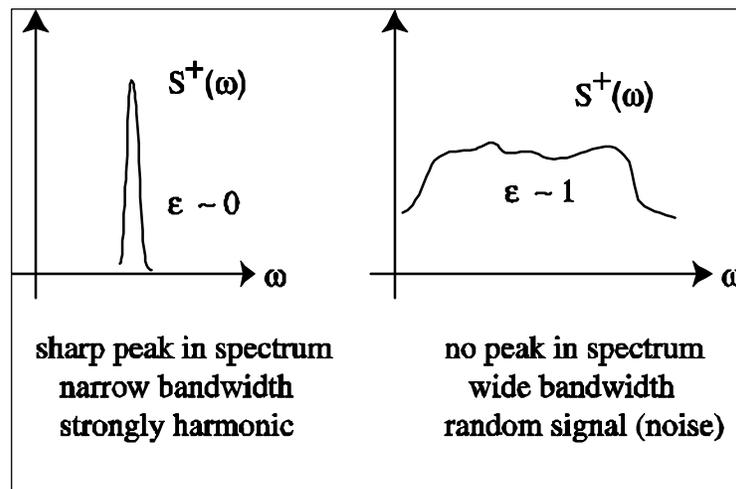
Bandwidth is defined as

$$e^2 = 1 - \frac{M_2^2}{M_o M_4} \tag{25}$$

The value of e is between 0 and 1. In the ocean, a bandwidth between 0.6 and 0.8 is common.

4.2. Maxima Above a Level

The probability of a maximum, X_m, reaching a level A can be found using the following pdf:

$$f(x_m) = \frac{2/\sqrt{M_o}}{1+\sqrt{1-e^2}} \left\{ \frac{e}{\sqrt{2p}} \text{EXP}\left[-\frac{1}{e^2}\frac{A^2}{2M_o} \right] \right.$$
$$\left. + \sqrt{1-e^2}\frac{A^2}{M_o}\text{EXP}\left[-\frac{A^2}{2M_o} \right]\left[1-f\left(-\frac{\sqrt{1-e^2}}{e}\frac{A^2}{\sqrt{M_o}} \right) \right] \right\} \tag{26}$$

which holds for any e where the function $f(x)$ is given by

$$f(x) = \frac{1}{2p}\int_{-\infty}^{x} e^{-u^2/2}\,du. \tag{27}$$

This can be approximated for very large events x_m as

$$f(x_m) \simeq \frac{2\sqrt{1-e^2}}{1+\sqrt{1-e^2}}\left(\frac{A}{\sqrt{M_o}} \right)\text{EXP}\left\{ -\frac{A^2}{2M_o} \right\} \tag{28}$$

Note that this approximation is not a true pdf as it does not integrate to one.

IF $e = 0$

$$f(x_m = A) = \frac{A}{\sqrt{M_o}}e^{-A^2/2M_o};\ 0 < A < \infty \tag{29}$$

IF $e = 1$

$$f(x_m = A) = \sqrt{\frac{1}{2p}}e^{-A^2/2M_o};\ 0 < A < \infty \tag{30}$$

Typically the bandwidth of an ocean spectrum is $e = 0.6$.

For most studies of structures and ships we are *most* interested in the probability that a wave height exceeds a certain value. It can be shown that for large values of the wave maxima equation 26 reduces to the approximate pdf

$$f(X_m = A) \approx \frac{2\sqrt{1-e^2}}{1+\sqrt{1-e^2}} \frac{A}{\sqrt{M_o}} e^{-\frac{A^2}{2M_o}} \tag{31}$$

This is not a "true" pdf since it will not integrate to 1. However the approximation holds over a range bandwidths. It is especially good for values of e below 0.6 with $A/\sqrt{M_o}$ greater than 1.4. For larger e we need larger values of $A/\sqrt{M_o}$. Since $e = 0.6$ is typical in the ocean we can justify this approximation.

Let's define some height $h = A/\sqrt{M_o}$, where h is large, and rewrite the approximate pdf as

$$f(h_m = h) \simeq \frac{2\sqrt{1-e^2}}{1+\sqrt{1-e^2}} h \, \mathrm{EXP}\{-h^2/2\} \tag{32}$$

Since we know that a pdf must (technically) integrate to one and that the probability that a random variable is below a value A is the integral of the pdf from zero to the height A, then we can readily calculate the probability that the wave height is above the level A using the pdf given above. The probability that a wave height is above a level a is

$$P(x_m \geq A) = 1 - \int_0^A f(X_m = a)da = \int_A^\infty f(X_m = a)da \tag{33}$$

For the most part this integral cannot be easily evaluated without using integral tables or

numerical integration.

Using the non-dimensional amplitude, $\boldsymbol{h}_o$ which is always greater than 1. Recall:

$$\boldsymbol{h}_o = \frac{A}{\sqrt{M_o}} > 1 \tag{34}$$

To find the probability that the wave maxima is greater than some amplitude, lets look at the probability that the non-dimensional wave amplitude is greater than $\boldsymbol{h}_o$:

$$P(\boldsymbol{h} \geq \boldsymbol{h}_o) = \int_{\boldsymbol{h}_o}^{\infty} f(\boldsymbol{h} = u)du \approx \frac{2\sqrt{1-\boldsymbol{e}^2}}{1+\sqrt{1-\boldsymbol{e}^2}} e^{-\boldsymbol{h}_o^2/2} \tag{35}$$

5. Useful References

- Devore, J "Probability and Statistics for Engineering and Sciences"
- Triantafyllou and Chryssostomidis, (1980) "Environment Description, Force Prediction and Statistics for Design Applications in Ocean Engineering" Course Supplement.