13.42 Design Principles for Ocean Vehicles
Homework #4

Out: Thursday, 26 February 2004
Due: Thursday, 4 March 2004

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1. *Probability review problem*.
Consider two bags – with 3 green and 4 red marbles in the first, and with 5 green and 1 red marble in the second. A coin is tossed. If the outcome is "heads," then a marble is taken from the first bag. If "tails," then a marble is chosen from the second bag. Suppose that the outcome of the coin toss is such that a green marble is chosen. Find the probability that the marble came from the first bag.

2. Consider the random process

$$\zeta(t) = A\cos(\omega t + \phi)$$

where ω and ϕ are constants and A is a random variable with probability density function

$$p_A(a) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{a^2}{2\sigma^2}}$$

Determine whether $\zeta(t)$ is a *stationary* process.

3. Consider the following LTI system:

$$U(t) \rightarrow \boxed{\text{LTI}} \rightarrow Y(t)$$

where $U(t)$ is the input random process and $Y(t)$ is the output random process.

a. Show that $\mu_Y(t) = h(t) * \mu_U(t)$, where $h(t)$ is the *impulse response* of the LTI system.

(Hint: Recall that $E[g(X)] = \int_{-\infty}^{\infty} g(x) p_X(x) dx$.)

b. If the impulse response is given as

$$h(t) = e^{-at} S(t)$$

where a is positive and $S(t)$ is the *Heaviside unit step function* defined such that $S(t) = 1$ when $t > 0$ and $S(t) = 0$ when $t < 0$, find the mean of the response $Y(t)$ if $\mu_U(t) = \mu_0$.

4. (*MATLAB* recommended for this problem)
The random wave elevation at a given point in the ocean may be represented as follows:

$$\zeta(t) = \sum_{i=1}^{N} A_i \cos(\omega_i t + \phi_i)$$

where A_i, ω_i, and ϕ_i are the wave amplitude, frequency, and random phase angle (with uniform distribution from 0 to 2π) of wave component number *i*.

The amplitudes A_i corresponding to each frequency ω_i may be found from a known wave energy spectrum, $S_\zeta(\omega)$, where $S_\zeta(\omega_i)\delta\omega = \frac{1}{2} A_i$.

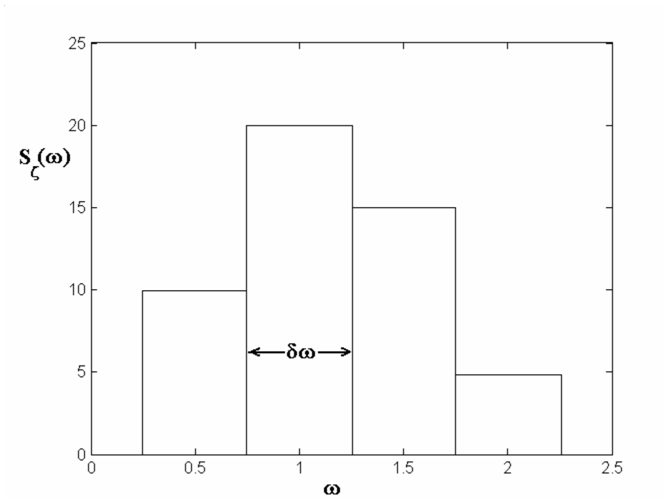

Figure 1 – Wave energy spectrum

a. Using the wave energy spectrum given in Figure 1, generate 10 realizations of the random process $\zeta(t)$. Let t = [0, 100] seconds. (Hint: Use the MATLAB function *rand* to generate a realization of the random variable ϕ_i)

b. Compute the ensemble averages (mean and variance) at t = 20 seconds, and compute the temporal averages (mean and variance) for each realization. Compare the results.

5. Consider the following LTI system:

$$X(t) \rightarrow \boxed{\text{LTI}} \rightarrow Y(t)$$

where $X(t)$ is the input random process and $Y(t)$ is the output random process.

a. If $X(t)$ is stationary, show that

$$E\left\{[X(t+\tau) - X(t)]^2\right\} = 2[R_{XX}(0) - R_{XX}(\tau)]$$

b. Now given its autocorrelation,

$$R_{XX}(\tau) = Ce^{-k|\tau|}$$

where $C > 0$ and $k > 0$, and the impulse response of the system,

$$h(t) = e^{-at} S(t)$$

where $S(t)$ represents the Heaviside unit step function, *find the autocorrelation of the response Y(t).*

(Hint: Use the Wiener-Khinchine Relations)