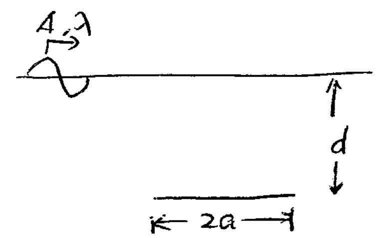
Quiz II Solution

1. a) $\phi_I = Re\left\{ \frac{igA}{\omega} e^{ky} e^{-ikx+i\omega t} \right\}$



$\lambda \gg a$, G.I. Taylor formula

$$F_{surge} = (\rho\forall + a_{11}) \left.\frac{\partial^2 \phi_I}{\partial x \partial t}\right|_{x=0}$$

$$= (\rho\forall + a_{11})\left(\cancel{-i}k \cdot \cancel{i}\omega \cdot \frac{igA}{\omega} e^{-kd} e^{i\omega t} \right\}$$

$$= (\rho\forall + a_{11}) \cdot i\omega^2 A e^{-kd} e^{i\omega t}$$

$$\rho\forall \simeq 0 \qquad a_{11} \simeq 0 \qquad F_{surge} \simeq 0$$

$$F_{heave} = (\rho\forall + a_{33}) \left.\frac{\partial^2 \phi_I}{\partial y \partial t}\right|_{x=0}$$

$$= (\rho\forall + a_{33})\left(k \cdot i\omega \cdot \frac{igA}{\omega} e^{-kd} e^{i\omega t} \right)$$

$$= -(\rho\forall + a_{33}) A\omega^2 e^{-kd} e^{i\omega t}$$

$$a_{33} \simeq \pi a^2 \rho \qquad \forall \simeq 0$$

$$F_{heave} \simeq -\pi a^2 \rho \omega^2 A e^{-kd} e^{i\omega t}$$

$$F_{surge} = Re\left\{ X_1 A e^{i\omega t} \right\}$$

$$F_{heave} = Re\left\{ X_3 A e^{i\omega t} \right\}$$

$$X_1 = 0 \ ; \qquad X_3 = -\pi a^2 \rho \omega^2 e^{-kd}$$

b) $$B_{ii} = \frac{|X_i|^2}{2\rho g V_g}$$

$$B_{11} = 0$$

$$B_{33} = \frac{|X_3|^2}{2\rho g V_g} = \frac{\left(-\pi a^2 \rho \omega^2 e^{-kd}\right)^2}{\rho g \frac{\omega}{k}} = \frac{\pi^2 a^4 \rho \omega^5}{g^2} e^{-2kd}$$

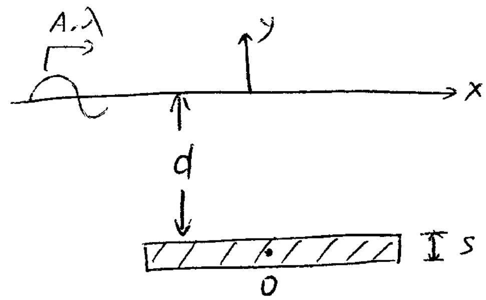
Method 1

c) For roll

$$M_{44} = \frac{1}{12}\rho\cdot 2a\cdot s\,(2a)^2 = \frac{2}{3}\rho s a^3 \qquad s:\ \text{thickness} \ll 1$$

Froude-krylow rolling moment

$$p = -\rho\phi_{It} = \rho g A e^{ky} e^{-ikx+i\omega t}$$



$$F_{FK,roll} = \underbrace{\int_{-a}^{a}}_{\text{bottom}} p\cdot x\cdot \underset{1}{n_y}\,dx + \underbrace{\int_{-a}^{a}}_{\text{top}} p\cdot x\cdot \underset{-1}{n_y}\,dx$$

$$= \int_{-a}^{a} \rho g A e^{-k(d+s)}\, x e^{-ikx+i\omega t}\,dx$$

$$+ \int_{-a}^{a} \rho g A e^{-kd}\,(-x) e^{-ikx+i\omega t}\,dx$$

$$= \rho g A\, e^{-kd}\,(e^{-ks}-1)\int_{-a}^{a} x e^{-ikx}\,dx \cdot e^{i\omega t}$$

$$= 2i\rho g A\, e^{-kd}\,(\underset{\approx\, 1-ks+O(ks^2)}{e^{-ks}}-1)\left(\frac{a\cos ka}{k} - \frac{\sin ka}{k^2}\right) e^{i\omega t}$$

$$\simeq -2i\rho g A\, e^{-kd}\cdot s\cdot \left(a\cos ka - \frac{\mathrm{ss}\,ka}{k}\right) e^{i\omega t}$$

Observing G.I. Taylor formula

when $\lambda \gg a$ or $ka \ll 1$

$$F_i = (\rho\forall + a_{ii})\frac{\partial^2\phi_I}{\partial x_i\partial t} = \underbrace{\rho\forall\frac{\partial^2\phi_I}{\partial x_i\partial t}}_{F\text{-}K}\left(1+\frac{a_{ii}}{\rho\forall}\right) = F_{F\text{-}K}\left(1+\frac{a_{ii}}{\rho\forall}\right)$$

We conclude that $\lambda \gg a$, $ka \ll 1$

$$F_4 = F_{F\text{-}K,roll}\left(1+\frac{a_{44}}{\cancel{\rho\forall}M_{44}}\right)$$

$$= -2i\rho g A e^{-kd}\cdot s\cdot\left(a\cos ka - \frac{\mathrm{ss}\,ka}{k}\right) e^{i\omega t}\left(1+\frac{\frac{\pi\rho a^4}{8}}{\frac{2}{3}\rho s a^3}\right)$$

$\cos x \simeq 1 - \frac{x^2}{2} \qquad x \ll 1$

$\mathrm{ss}\,x \simeq x - \frac{x^3}{6} \qquad x \ll 1$

$$F_4 \simeq -2i\rho g A e^{-kd} a\left(1 - \frac{k^2a^2}{2} - \left(1 - \frac{k^2a^2}{6}\right)\right) e^{i\omega t}\left(s + \frac{\frac{\pi\rho a^4}{8}}{\frac{2}{3}\rho a^3}\right)$$

$$= 2i\rho g A e^{-kd}\cdot a\cdot\frac{k^2a^2}{3}\cdot\left(s+\frac{3}{16}\pi a\right) e^{i\omega t} \qquad (s\to 0)$$

$$\simeq \frac{i\pi}{8}\rho g A\, a^4 k^2 e^{-kd} e^{i\omega t} = \frac{i\pi}{8}\rho\omega^2 kA\cdot a^4 e^{-kd}\, e^{i\omega t}$$

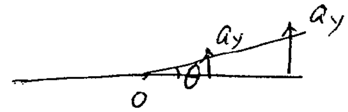
$$= i\cdot\frac{\pi}{8}\rho a^4\cdot\omega^2 kA\cdot e^{-kd} e^{i\omega t}$$

Method 2,

Roll acceleration

$$\ddot{\theta} = \frac{\Delta a_y}{\Delta x} = \frac{d a_y}{dx} = \frac{\partial}{\partial x}\left(\frac{\partial^2 \phi_I}{\partial y \partial t}\right)$$

$$= \frac{\partial^3 \phi_z}{\partial x \partial y \partial t}$$



G. I. Taylor

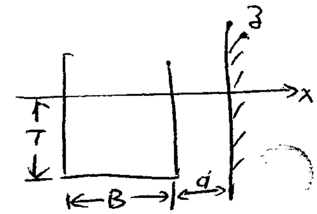
$$F_4 = (M_{44} + a_{44}) \frac{\partial^3 \phi_2}{\partial x \partial y \partial t}$$

$$= \left(\cancel{\tfrac{2}{3}} \cdot \frac{1}{12} \rho \forall (2a)^2 + \frac{\pi}{8} \rho a^4\right) \cdot \left(-ik \cdot k \cdot (i\omega) \frac{igA}{\omega} e^{-kd} e^{i\omega t}\right)$$

$$= \left(\frac{1}{3} \rho \underset{\simeq 0}{\forall} a^2 + \frac{\pi}{8} \rho a^4\right) igAk^2 e^{-kd} e^{i\omega t}$$

$$\simeq i \cdot \frac{\pi}{8} \rho a^4 \cdot gk \cdot kA e^{-kd} e^{i\omega t}$$

$$= i \cdot \frac{\pi}{8} \rho a^4 \cdot \omega^2 \cdot kA e^{-kd} e^{i\omega t}$$



2.

(c) $$\phi_I = Re\left\{ \frac{igA}{\omega} e^{kz} e^{-ikx\cos\beta - iky\sin\beta} e^{i\omega t} \right\}$$

$$\phi_R = Re\left\{ \frac{igA}{\omega} e^{kz} e^{+ikx\cos\beta - iky\sin\beta} e^{i\omega t} \right\}$$

$$\phi_0 \; \cancel{\phi_{total}}_{\text{incident}} = \phi_I + \phi_R = Re\left\{ \frac{2igA}{\omega} e^{kz} \cos(k\cos\beta x)\, e^{-iky\sin\beta} e^{i\omega t} \right\}$$

$$p = -\rho \phi_{0t} = 2\rho g A e^{kz} \cos(k\cos\beta x)\, e^{-iky\sin\beta} e^{i\omega t}$$

At $y =$

$$F_{surge} = \int_c p n_x \, ds = \left[\int_{-T}^{0} 2\rho g A e^{kz} \cos(k\cos\beta(-d-B))\cdot 1 \cdot dz + \int_{-T}^{0} 2\rho g A e^{kz} \cos(k\cos\beta(-d))\cdot(-1)\, dz \right] e^{-iky\sin\beta + i\omega t}$$

$$= 2\rho g A \left[\cos(k\cos\beta(B+d)) - \cos(k\cos\beta d) \right] \cdot \frac{1-e^{-kT}}{k} \; e^{-iky\sin\beta + i\omega t}$$

$$F_{heave} = \int_c p n_z \, ds = \int_{-B-d}^{-d} 2\rho g A e^{-kT} \cos(k\cos\beta x)\, dx \cdot e^{-iky\sin\beta + i\omega t}$$

$$= 2\rho g A e^{-kT} \cdot \frac{\sin(k\cos\beta(B+d)) - \sin(k\cos\beta d)}{k\cos\beta} \; e^{-iky\sin\beta + i\omega t}$$

(a) Beam wave $\beta = 0$

$$F_{surge} = 2\rho g A (\cos k(B+d) - \cos kd)\; e^{-iky\sin\beta + i\omega t} \cdot \frac{1-e^{-kT}}{k}$$

$$F_{heave} = \frac{2\rho g A e^{-kT}}{k} (\sin k(B+d) - \sin kd)\; e^{-iky\sin\beta + i\omega t}$$

(b) $\cancel{B_{33}}$ $\beta = 0$, It's a 2D problem

$$B_{33} = \frac{|X_3|^2}{2\rho g V_g} \qquad B_{33} = 0 \iff |X_3| = 0 \Rightarrow F_{heave} = 0$$

We require

$$\sin k(B+d) - \sin kd = 0$$

$$\sin k(B+d) - \sin kd = 2\cos k\left(\tfrac{B}{2}+d\right)\sin\frac{kB}{2} = 0$$

i) $\cos k\left(\frac{B}{2}+d\right) = 0$

$$k\left(\tfrac{B}{2}+d\right) = \frac{\omega^2}{g}\left(\tfrac{B}{2}+d\right) = (2n-1)\cdot\frac{\pi}{2} \qquad n=1,2,\cdots$$

$$\omega_n = \left(\left(n-\tfrac{1}{2}\right)\frac{\pi g}{\frac{B}{2}+d}\right)^{1/2} = \left(\frac{(2n-1)\pi g}{B+2d}\right)^{1/2}$$

ii) $\sin\frac{kB}{2} = 0$

$$\frac{kB}{2} = \frac{\omega^2 B}{2g} = m\pi \qquad m=1,2,\cdots$$

$$\omega_m = \left(\frac{2m\pi g}{B}\right)^{1/2} \qquad m=1,2,\cdots$$

At

$$\omega_n = \left(\frac{(2n-1)\pi g}{B+2d}\right)^{1/2} \qquad n=1,2\cdots$$

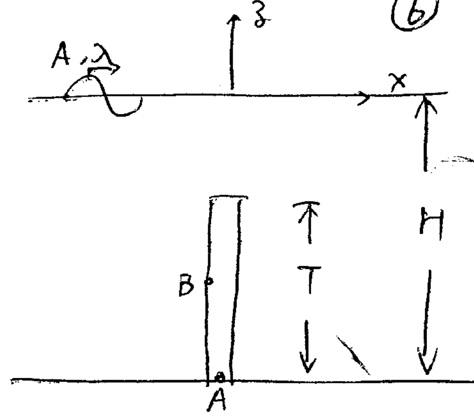
$$\omega_m = \left(\frac{2m\pi g}{B}\right)^{1/2} \qquad m=1,2,\cdots$$

B_{33} vanishes.

c) The first part in page 4

⑥

3. a) $\phi_1 = Re\left\{ \frac{igA}{\omega} \frac{\cosh k(z+H)}{\cosh kH} e^{-ikx+i\omega t} \right\}$



Surge force on a vertical slice of thickness dz

$\lambda >> a$, G. I. Taylor

$$dF_z = (\rho\forall + a_{11}) \frac{\partial^2 \phi_1}{\partial x \partial t} dz \Big|_{x=0}$$

$$= (\rho\pi a^2 + \rho\pi a^2)\left(-ik \cdot i\omega \cdot \frac{igA}{\omega} \frac{\cosh k(z+H)}{\cosh kH} e^{i\omega t}\right) dz$$

$$= 2i\rho\pi a^2 \omega^2 A \cdot \frac{\cosh k(z+H)}{\cosh kH} e^{i\omega t} \cdot dz$$

Pitch moment about A

$$F_5 \cancel{X_5} = \int_{-H}^{-H+T} (z+H) \cdot dF_z$$

$$= \int_{-H}^{-H+T} (z+H) \cdot 2\rho\pi a^2 \cdot i\omega^2 A \cdot \frac{\cosh k(z+H)}{\cosh kH} dz \, e^{i\omega t}$$

$$= 2\rho\pi a^2 \cdot i\omega^2 A \cdot \frac{1}{\cosh kH} \int_{-H}^{-H+T} (z+H)\cosh k(z+H)\, dz \, e^{i\omega t}$$

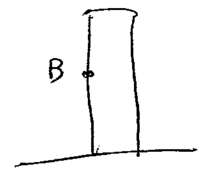
$$\int_{-H}^{-H+T} (z+H)\cosh k(z+H)\, dz = \int_0^T u \cosh ku \, du$$

$$= \frac{1}{k^2}(1 - \cosh kT + kT \sinh kT)$$

$$F_5 = 2\rho\pi a^2 \cdot i\omega^2 A \cdot \frac{1 - \cosh kT + kT\sinh kT}{k^2 \cosh kH} \cdot e^{i\omega t}$$

b) The velocity of incident wave is

$$\vec{V}_I = \left(\frac{\partial \phi_I}{\partial x}, \frac{\partial \phi_I}{\partial z}\right) = (u_I, v_I)$$

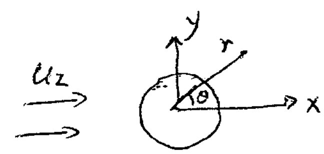


$\lambda \gg a$, the ~~ff~~ incident flow near point B ~~is~~ can be treated as uniform.

$$\phi_{Inc,local} = u_I x + v_I z$$

A vertical pile will not disturb the incident flow if the incident flow is vertical, i.e., $(u_I = 0, v_I \neq 0)$

The inflow component u_I is disturbed by the tank. It's a problem of uniform flow passing through a circle.



$$\phi_{disturb} = u_I \frac{a^2 \cos\theta}{r} = u_I \frac{a^2 x}{x^2+y^2}$$

$$\phi_{total} = \phi_{Inc,local} + \phi_{disturb}$$

$$= u_I x + u_I \frac{a^2 x}{x^2+y^2} + v_I z$$

$$= u_I \cos\theta \left(r + \frac{a^2}{r}\right) + \cancel{u_I r \sin\theta}\, v_I z$$

$$= \underbrace{\frac{igA}{\omega}\cdot(-ik)\frac{\cosh k(-H+\frac{T}{2}+H)}{\cosh kH} e^{-ik(-a)+i\omega t}}_{u_I} \cdot \left(-a + \frac{-a^3}{a^2}\right)$$

$$+ \underbrace{\frac{igA}{\omega}\cdot k \cdot \frac{\sinh k(-H+\frac{T}{2}+H)}{\cosh kH} e^{-ik(-a)+i\omega t}}_{v_I} \cdot \left(-H + \frac{T}{2}\right)$$

$$p = -\rho(\phi_{total})_t = -\rho i\omega\, \phi_{total}$$

$$= -\rho g A k \frac{\sinh\frac{kT}{2}(H-\frac{T}{2}) - i\cosh\frac{kT}{2}\cdot 2a}{\cosh kH} e^{ika+i\omega t}$$

Assume deep water

c) If there is a current, the total velocity potential is ⑧

$$\phi = Ux + Re\left\{ \frac{igA}{\omega_0} e^{kz} e^{-ikx+i\omega t} \right\} = Ux + \psi$$

in which

$$\omega = \omega_0 + kU$$

$$p = -\rho\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\psi$$

$$= \rho g A e^{kz} e^{-ikx+i\omega t}$$

For a)

$|F_5|$ is the same with encounter frequency $\omega = \omega_0 + kU$.

For b)

$$u_1 = \phi_x = U + \frac{gAk}{\omega_0} e^{kz} e^{-ikx+i\omega t}$$

$$= U + A\omega_0 e^{kz} e^{-ikx+i\omega t}$$

$$v_1 = \phi_z = iA\omega_0 e^{kz} e^{-ikx+i\omega t}$$

$$\phi_{total} = \left(U + A\omega_0 e^{kz} e^{-ikx+i\omega t}\right)\left(x + \frac{xa^2}{x^2+y^2}\right) + iA\omega_0 e^{kz} e^{-ikx+i\omega t} z$$

$$= U\left(x + \frac{x}{x^2+y^2}\right) + \psi$$

$$p = \rho\left(\frac{U^2}{2} - \frac{\partial \phi_{total}}{\partial t} - \frac{1}{2}\nabla\phi_{total}\cdot\nabla\phi_{total}\right)$$

At B $\quad (\phi_{total})_x = 0$

$$p = \frac{\rho U^2}{2} - \rho\frac{\partial \phi_{total}}{\partial t} - \cancel{\frac{1}{2}(\phi_{total,z})^2} \quad \text{omit}$$

$$= \frac{\rho U^2}{2} - \rho i\omega\,\psi_{total}$$

$$= \frac{\rho U^2}{2} - \underbrace{\rho\omega\omega_0 A}_{\rho\frac{\omega}{\omega_0} gAk} \cdot e^{-k(H-\frac{T}{2})}\left[(H-\tfrac{T}{2}) - 2ia\right] e^{ika+i\omega t}$$

The amplitude of p also changes.