13.022 - FALL 2001

QUIZ #3

1. $$\phi_{tt} + g\phi_z + \mu\phi_t = 0, \quad z = 0$$

LET $\phi = Re\left\{ \frac{iSA}{\omega} e^{kz - ikx + i\omega t} \right\}$

BY DEFINITION IT SATISFIES THE LAPLACE EQUATION
SUBSTITUTING IN THE FREE SURFACE CONDITION
WE OBTAIN:

$$-\omega^2 + gk + i\omega\mu = 0$$

FOR KNOWN REAL ω, THE WAVENUMBER K IS
COMPLEX AND GIVEN BY

$$K = \frac{\omega^2 - i\omega\mu}{g} = K_R - iK_I$$

FROM THE DEFINITION OF THE WAVE POTENTIAL
THE COMPLEX FACTOR BECOMES

$$e^{-ikx} = e^{-ik_R x} e^{-K_I x} \quad \text{WHERE} \quad K_I = \frac{\omega\mu}{g}$$

SO THE NEW EFFECTIVE WAVE AMPLITUDE BECOMES:

$$A_{BR} = A e^{-k_I x}$$

THUS THE AMPLITUDE DUE TO BREAKING DECAYS AT AN EXPONENTIAL RATE WITH THE DECAY RATE $k_I = \omega \mu / g$. —

b) IN WATER OF FINITE DEPTH:

$$\phi = \mathrm{Re} \left\{ \frac{igA}{\omega} \frac{\cosh k(z+H)}{\cosh kH} e^{-ikx + i\omega t} \right\}$$

AGAIN ϕ SATISFIES THE LAPLACE EQUATION FOR REAL OR COMPLEX k. UPON SUBSTITUTION IN THE FREE SURFACE CONDITION:

$$-\omega^2 + gk \tanh kH + i\omega\mu = 0$$

$$\Rightarrow gk \tanh kH = \omega^2 - i\omega\mu$$

IN SHALLOW WATER $\tanh kH \cong kH$, SO

$$g k^2 H = \omega^2 - i\mu\omega$$

$$k^2 = \frac{\omega^2 - i\omega\mu}{gH} \Rightarrow k = \left(\frac{\omega^2 - i\omega\mu}{gH} \right)^{1/2}$$

FOR SMALL VALUES OF μ, WE MAY WRITE

$$K = \left(\frac{\omega^2}{gH} - i\frac{\omega\mu}{gH} \right)^{1/2}$$

$$= \frac{\omega}{\sqrt{gH}} \left(1 - i\frac{\omega\mu}{gH} \frac{gH}{\omega^2} \right)^{1/2}$$

$$= \frac{\omega}{\sqrt{gH}} \left(1 - i\frac{\mu}{\omega} \right)^{1/2} = \frac{\omega}{\sqrt{gH}} - \frac{i}{2}\frac{\mu}{\sqrt{gH}}$$

UPON SUBSTITUTION IN e^{-ikx} WE OBTAIN:

$$e^{-ikx} = e^{-i\frac{\omega}{\sqrt{gH}}x} \, e^{-k_I x}$$

WHERE $k_I = \frac{1}{2}\frac{\mu}{\sqrt{gH}}$.

THE RATE OF DECAY IN DEEP WATER IS: $\frac{\omega\mu}{g}$ AND IS LARGER THAN THAT IN SHALLOW WATER WHEN:

$$\frac{\omega\mu}{g} > \frac{1}{2}\frac{\mu}{\sqrt{gH}} \Rightarrow \omega > \frac{1}{2}\sqrt{\frac{g}{H}}.$$

$$p = -\rho \frac{\partial \phi}{\partial t} \Rightarrow -\rho \frac{\partial \phi}{\partial t} + \lambda \frac{\partial \phi}{\partial z} = 0$$

$$-i\omega\rho\,\varphi + \lambda \varphi_z = 0, \qquad z = -H$$

$$\Rightarrow \varphi_z - \frac{i\omega\rho}{\lambda}\varphi = 0, \qquad z = -H$$

ON $z=0$ $\qquad \varphi_z - (\omega^2/g - i\omega\mu)\,\varphi = 0$

IN MORE GENERAL NOTATION:

$$\varphi_z - \alpha^2 \varphi = 0, \quad z = -H$$

$$\varphi_z - \beta^2 \varphi = 0, \qquad z = 0$$

WHERE $\alpha^2 = \dfrac{i\omega\rho}{\lambda}$, $\beta^2 = \dfrac{\omega^2}{g} - i\omega\mu$.

LET φ BE OF THE FORM

$$\varphi = \frac{isA}{\omega} f(z)\, e^{-ikx + i\omega t}$$

WHERE THE RELATION BETWEEN k & ω IS TO BE DETERMINED. BY VIRTUE OF THE LAPLACE EQUATION:

$$f''(z) - k^2 f = 0 \; ; \quad -H < z < 0$$

$$f'(z) - \alpha^2 f = 0, \qquad z = -H$$

$$f'(z) - \beta^2 f = 0, \qquad z = 0$$

$$F(z) = A e^{kz} + B e^{-kz}$$

$$z = -H: \quad AK e^{-KH} - BK e^{+KH} - \alpha^2 \left(A e^{-KH} + B e^{+KH}\right) = 0$$

$$\Rightarrow \quad A e^{-KH}(k - \alpha^2) - B e^{+KH}(k + \alpha^2) = 0$$

$$z = 0: \qquad A(k - \beta^2) - B(K + \beta^2) = 0$$

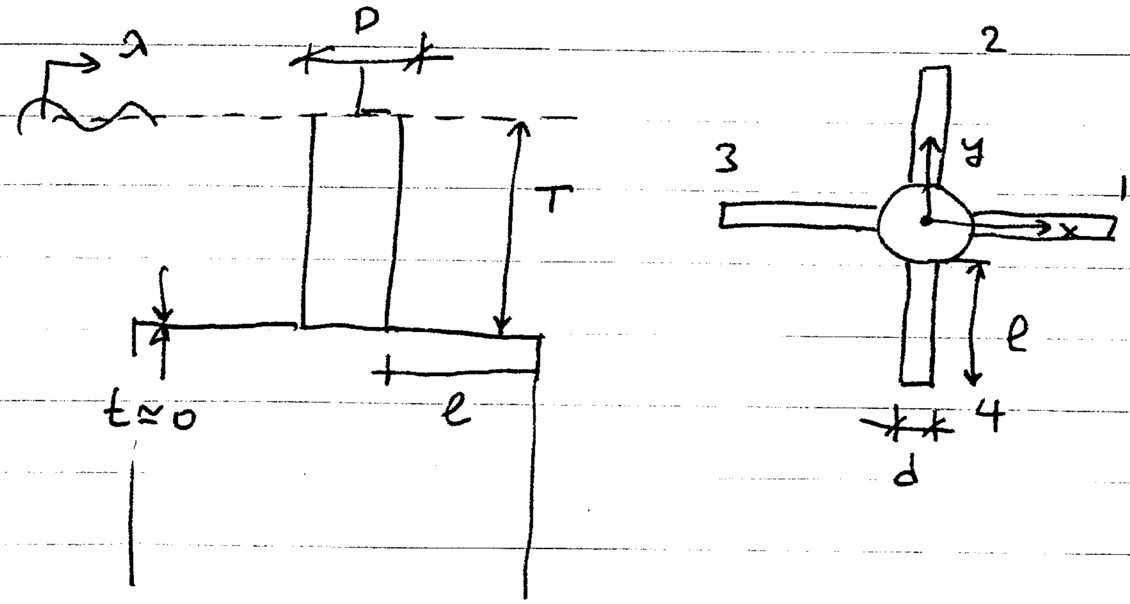
IT FOLLOWS THAT:

$$\frac{A}{B} = \frac{e^{KH}(K + \alpha^2)}{e^{-KH}(K - \alpha^2)} = \frac{K + \beta^2}{K - \beta^2} \Rightarrow$$

$$\frac{K + \beta^2}{K - \beta^2} = e^{2KH} \frac{K + \alpha^2}{K - \alpha^2}$$

WITH : $\alpha^2 = \dfrac{i\omega e}{\lambda}$ AND $\beta^2 = \dfrac{\omega^2}{g} - i\omega\mu$.

2.



$\lambda \gg D, d \quad , \quad \lambda \sim T, \ell$.

$$\phi = \mathrm{Re}\left\{ \frac{isA}{\omega} e^{kz - ikx + i\omega t} \right\}$$

THE VERTICAL EXCITING FORCES ON THE TETHERS 1, 2, 3 & 4 DEPEND UPON THE HEAVE EXCITING FORCE AND THE PITCH EXCITING MOMENT ON THE PLATFORM ABOUT THE REFERENCE COORDINATE SYSTEM. LET

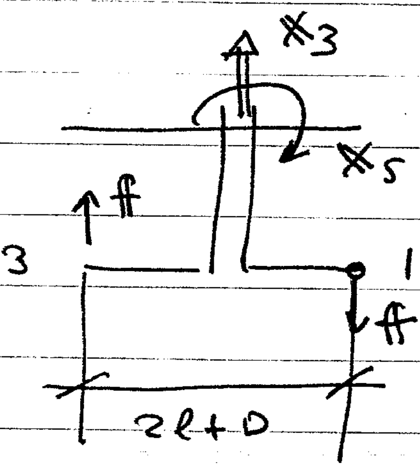
$$\zeta(x,t) = \mathrm{Re}\left\{ A e^{-ikx + i\omega t} \right\}$$

$$F_3(t) = \mathrm{Re}\left\{ X_3 e^{i\omega t} \right\}$$

$$F_5(t) = \mathrm{Re}\left\{ X_5 e^{i\omega t} \right\}.$$

WHERE $\mathbb{X}_3$ AND $\mathbb{X}_5$ ARE THE COMPLEX HEAVE EXCITING FORCE AND PITCH EXCITING MOMENT AMPLITUDES OR RAOS. THEY WILL BE DETERMINED IN A MOMENT.

IN TERMS OF $\mathbb{X}_3$ & $\mathbb{X}_5$ THE VERTICAL EXCITING FORCES ON EACH TETHER ARE OBTAINED AS FOLLOWS:



$$\tilde{\mathbb{X}}_3 = \frac{\mathbb{X}_3}{4}$$

- TETHERS 2 & 4: $F_2(t) = \mathrm{Re}\{\tilde{\mathbb{X}}_3 e^{i\omega t}\}$

THE PITCH MOMENT DOES NOT INDUCE A VERTICAL FORCE UPON TETHERS 2 & 4 TO LEADING ORDER IN THE WAVE AMPLITUDE

- TETHERS 1 & 3:

STATIC EQUILIBRIUM REQUIRES THAT THE PITCH MOMENT $\mathbb{X}_5$ IS BALANCED BY TWO FORCES

$\mathbb{F}$ OF EQUAL AND OPPOSITE MAGNITUDE. WITH THE PITCH MOMENT $\mathbb{X}_5$ DRAWN IN THE FIGURE THE FORCES $\mathbb{F}$ AS DRAWN ACT ON THE TETHERS AND

$$\mathbb{X}_5 = \mathbb{F}\,(2\ell + D)$$

$$\Rightarrow \quad \mathbb{F} = \frac{\mathbb{X}_5}{2\ell + D}$$

SO THE VERTICAL FORCE ON EACH TETHER IS:

TETHER 1: $\quad \mathbb{F}_1 = \tilde{\mathbb{X}}_3 - \dfrac{\mathbb{X}_5}{2\ell + D}, \quad \tilde{\mathbb{X}}_3 = \dfrac{\mathbb{X}_3}{2}$

TETHER 3: $\quad \mathbb{F}_2 = \tilde{\mathbb{X}}_3 + \dfrac{\mathbb{X}_5}{2\ell + D}$. —

THERE REMAINS TO DETERMINE $\mathbb{X}_3$ & $\mathbb{X}_5$. WE WILL USE GI TAYLOR'S THEOREM SINCE λ IS LARGE COMPARED TO THE CROSS-SECTIONAL DIMENSIONS OF THE PLATFORM. —

THE HEAVE EXCITING FORCE IS ARISING FROM THE HEAVE FORCE ON EACH HORIZONTAL PLATE AND THE HEAVE FORCE ON THE BOTTOM OF THE PONTOON MODELED AS A 1/2 SPHERE

ACCORDING TO G I TAYLOR:

$$F_z = -\rho\left(\frac{a}{\rho} + \cancel{\rho}\forall\right)\frac{\partial p}{\partial z}$$

ON $z = -H$:

$$\frac{\partial p}{\partial z} = \frac{\partial}{\partial z}\left[-\rho\frac{\partial \phi}{\partial t}\right] = -\rho\frac{\partial^2}{\partial z\,\partial t}\,\mathrm{Re}\left\{\frac{igA}{\omega}e^{kz-ikx+i\omega t}\right\}$$

$$= -\rho\,\mathrm{Re}\left\{\frac{igA}{\omega}k(i\omega)\,e^{-ikx}\,e^{-kH}\,e^{i\omega t}\right\}$$

THE ADDED MASS OF A PLATE OF WIDTH d IS:

$$a_{33}\Big|_{\text{PLATE}} = \pi\rho\frac{d^2}{4}$$

ITS DISPLACEMENT IS ASSUMED TO BE ZERO FOR A SMALL THICKNESS. THE ADDED MASS OF A SPHERE IS 1/2 ITS DISPLACED VOLUME

$$a_{33}\Big|_{\text{SPHERE}} = \frac{1}{2}\rho\forall = \frac{1}{2}\rho\,\frac{4}{3}\pi\left(\frac{D}{2}\right)^8$$

SO THE HEAVE EXCITING FORCE ON THE TLP BECOMES —

$$F_3(t) \cancel{X_3} = -\rho\left(\frac{a_{33}}{\ell} + \frac{\forall}{2}\right)_{\text{SPHERE}} \frac{\partial p}{\partial z}\bigg|_{z=-H}$$

$$-\rho \int_{\substack{c_1+c_2\\+c_3+c_4}} d\ell \left(\frac{a_{33}}{\ell}\right)_{\text{PLATE}} \frac{\partial p}{\partial z}\bigg|_{z=-H} = \mathrm{Re}\{X_3 e^{i\omega t}\}.$$

WHERE THE CONTOUR INTEGRALS ARE OVER THE AXES (RECTILINEAR) OF THE FOUR HORIZONTAL PLATES.

THE PITCH MOMENT $F_5(t) = \mathrm{Re}\{X_5 e^{i\omega t}\}$ WILL BE CONTRIBUTED FROM THE TWO HORIZONTAL PLATES 1 & 3 IN LINE WITH THE DIRECTION OF PROPAGATION OF THE WAVES. HENCE

$$F_5(t) = -\rho \int_{c_1+c_2} dx \left(\frac{a_{33}}{\ell}\right)_{\text{PLATE}} \frac{\partial p}{\partial z}\bigg|_{z=-H} (-x)$$

WHERE $(-x)$ IS THE MOMENT ARM OF THE DIFFERENTIAL ELEMENT dx ALONG PLATES 1 & 3. —

C) HEAVE IS RESTORED EQUALLY BY ALL FOUR TETHERS. SO THE EFFECTIVE SPRING CONSTANT IS:

$$C_{33} = \rho g A_w + 4k$$

WHERE $A_w = \pi D^2/4$ WITH D BEING THE DIAMETER OF THE VERTICAL CYLINDER

THE EFFECTIVE MASS IS GIVEN BY THE MASS OF THE PLATFORM, THE HEAVE ADDED MASS OF THE 4 HORIZONTAL PLATES + ½ SPHERE, AND THE EFFECTIVE INERTIA OF EACH TETHER:

$$\mathcal{M}_{TOT} = A_{33} + M + 4m$$

WHERE $A_{33} = 4A_{PLATE} + \frac{1}{2} A_{SPHERE}$

$$A_{PLATE} = \ell \frac{\pi d^2}{4}, \quad A_{SPHERE} = \frac{1}{2} \forall = \frac{1}{2} \frac{4}{3} \pi \left(\frac{D}{2}\right)^3$$

SO THE HEAVE NATURAL PERIOD OF THE TLP IS:

$$\omega^2 = \frac{\rho g A_w + 4k}{A_{33} + M + 4m}.$$