SECOND-ORDER WAVE EFFECTS

- LINEAR THEORY IS A VERY POWERFUL AND USEFUL MEANS OF MODELING THE RESPONSES OF FLOATING BODIES IN REGULAR AND RANDOM WAVES. YET, FOR CERTAIN TYPES OF FLOATING STRUCTURES AND CERTAIN TYPES OF EFFECTS NONLINEAR EXTENSIONS ARE NECESSARY

- MEAN DRIFT FORCES IN REGULAR WAVES DO NOT REQUIRE AN ACCOUNT OF NONLINEAR EFFECTS. HOWEVER, THE DEFINITION OF THE QUADRATICALLY NONLINEAR SLOW-DRIFT EXCITATION FORCE IN A SEASTATE REQUIRES A SECOND-ORDER THEORY IN PRINCIPLE

- TENSION-LEG PLATFORMS THAT OPERATE IN LARGE WATER DEPTHS FOR THE EXPLORATION AND EXTRACTION OF HYDROCARBONS OFFSHORE OFTEN UNDERGO RESONANT HEAVE OSCILLATIONS WITH PERIODS OF A FEW SECONDS WHICH CAN ONLY BE EXCITED BY NONLINEAR WAVE EFFECTS. THEIR TREATMENT REQUIRES A SECOND-ORDER THEORY AND PEPHAPS A MORE EXACT TREATMENT.

SECOND-ORDER FREE SURFACE CONDITION

THE SECOND-ORDER FREE SURFACE CONDITION OF FREE WAVES WAS DERIVED EARLIER IN THE FORM:

$$\Phi = \phi_1 + \phi_2 + \phi_3 + \cdots$$
$$\zeta = \zeta_1 + \zeta_2 + \zeta_3 + \cdots$$

KINEMATIC CONDITION:

- $$\frac{\partial \zeta_2}{\partial t} - \frac{\partial \phi_2}{\partial t} = \zeta_1 \frac{\partial^2 \phi_1}{\partial z^2} - \underbrace{\nabla \phi_1 \cdot \nabla \zeta_1}_{\phi_{1x}\zeta_{1x} + \phi_{1y}\zeta_{1y}} \, ; \quad z=0$$

DYNAMIC CONDITION:

- $$\zeta_2 + \frac{1}{g}\frac{\partial \phi_2}{\partial t} = -\frac{1}{g}\left(\frac{1}{2}\nabla\phi_1 \cdot \nabla\phi_1 + \zeta_1 \frac{\partial^2 \phi_1}{\partial z \partial t}\right)_{z=0}$$

HYDRODYNAMIC PRESSURE:

- $$p_2 = -\rho\left(\frac{\partial \phi_2}{\partial t} + \frac{1}{2}\nabla\phi_1 \cdot \nabla\phi_1\right)$$

AT SOME FIXED POINT IN THE FLUID DOMAIN

ASSUMING THAT THE LINEAR SOLUTION IS KNOWN AND ELIMINATING ζ_2 FROM THE KINEMATIC CONDITION, WE OBTAIN THE MORE FAMILIAR FORM:

$$\nabla^2 \phi_2 = 0$$

$$\frac{\partial^2 \phi_2}{\partial t^2} + g \frac{\partial \phi_2}{\partial z} = -\frac{\partial}{\partial t}(\nabla\phi_1 \cdot \nabla\phi_1) + \frac{1}{g}\frac{\partial \phi_1}{\partial t}\frac{\partial}{\partial z}\left(\frac{\partial^2 \phi_1}{\partial t^2} + g\frac{\partial \phi_1}{\partial z}\right)_{z=0}$$

$$\zeta_2 = -\frac{1}{g}\left(\frac{\partial \phi_2}{\partial t} + \frac{1}{2}\nabla\phi_1 \cdot \nabla\phi_1 + \zeta_1 \frac{\partial^2 \phi_1}{\partial z \partial t}\right)_{z=0}$$

- THE SOLUTION OF THE ABOVE BOUNDARY-VALUE PROBLEM IS AVAILABLE IN CLOSED FORM WHEN THE LINEAR POTENTIAL ϕ_1 IS THE LINEAR SUPERPOSITION OF REGULAR PLANE PROGRESSIVE WAVES

- VERY USEFUL INSIGHTS FOLLOW FROM THIS SOLUTION EVEN IN THE ABSENCE OF FLOATING BODIES.

- THE STATEMENT OF THE SECOND-ORDER PROBLEM WHEN A BODY IS PRESENT RECQUIRES THE DERIVATION OF A SECOND-ORDER BODY BOUNDARY CONDITION WHICH IS INVOLVED. THE SOLUTION OF THE SECOND-ORDER WAVE BODY INTERACTION PROBLEM IS ONLY POSSIBLE NUMERICALLY AND IS IN GENERAL A COMPLEX TIME CONSUMING TASK.

SECOND-ORDER BI-CHROMATIC WAVE THEORY

- THE FORCING IN THE RIGHT-HAND SIDE OF THE SECOND ORDER FREE SURFACE CONDITION IS A QUADRATIC FUNCTION OF THE LINEAR SOLUTION. HENCE, IF THE LINEAR PROBLEM IS WRITTEN AS THE SUM OF MONOCHROMATIC WAVE COMPONENTS, THE TREATMENT OF THE MOST GENERAL SECOND ORDER PROBLEM CAN BE ACCOMPLISHED BY TREATING THE SECOND-ORDER FREE-SURFACE CONDITION BY A BI-CHROMATIC DISTURBANCE (VERIFY)

THE ABOVE CONCLUSION IS ANALOGOUS TO THE FOLLOWING IDENTITY:

$$\mathrm{Re}\{A_1 e^{i\omega_1 t}\}\,\mathrm{Re}\{A_2 e^{i\omega_2 t}\} =$$

$$\frac{1}{2}\mathrm{Re}\Big\{\underbrace{A_1 A_2 e^{i(\omega_1+\omega_2)t}}_{\text{SUM FREQUENCY COMPONENT}} + \underbrace{A_1 A_2^{*} e^{i(\omega_1-\omega_2)t}}_{\text{DIFFERENCE-FREQUENCY COMPONENT}}\Big\}$$

- IT FOLLOWS THAT THE FORCING AND HENCE THE SOLUTION OF THE SECOND-ORDER PROBLEM IS THE LINEAR SUPERPOSITION OF A SUM AND A DIFFERENCE-FREQUENCY COMPONENTS.

- THE VELOCITY POTENTIAL OF A DIRECTIONAL SEASTATE REPRESENTED BY THE LINEAR SUPERPOSITION OF PLANE PROGRESSIVE WAVES OF VARIOUS FREQUENCIES AND WAVE HEADINGS TAKES THE FAMILIAR FORM:

$$\Phi_1 = \mathrm{Re} \sum_k \sum_m \frac{igA_{km}}{\omega_k} e^{\nu_k z - i\nu_k x \cos\theta_m - i\nu_k y \sin\theta_m} \times e^{i\omega_k t + i\phi_{km}}$$

WHERE THE SUMMATIONS ARE OVER A SUFFICIENTLY LARGE NUMBER OF WAVE FREQUENCIES ω_k AND WAVE HEADINGS θ_m NECESSARY TO CHARACTERIZE THE AMBIENT SEASTATE

- THE PHASE ANGLES ϕ_{km} ARE SAMPLED FROM THE UNIFORM DISTRIBUTION OVER $[-\pi, \pi]$ AND THE WAVE AMPLITUDES A_{km} ARE CHOSEN TO CONFORM TO THE AMBIENT WAVE SPECTRUM.

- A_{km} MAY BE SELECTED TO BE DETERMINISTIC OR MAY BE DRAWN FROM A RAYLEIGH DISTRIBUTION INDEPENDENTLY FROM THE PHASE ANGLE ϕ_{km}. THE LATTER CHOICE HAS BEEN FOUND TO BE SUPERIOR IN PRACTICE WHEN LONG RECORDS ARE NEEDED WITH VERY LONG PERIODICITY.—

UPON SUBSTITUTION OF Φ_1 INTO THE RIGHT-HAND SIDE OF THE SECOND-ORDER FREE SURFACE CONDITION, SUM- AND DIFFERENCE-FREQUENCY TERMS ARISE AS ILLUSTRATED ABOVE.

WE THEREFORE DEFINE THE YET UNKNOWN SECOND-ORDER POTENTIAL AS FOLLOWS:

$$\Phi_2 = \mathrm{Re} \left\{ \varphi_2^+ e^{i(\omega_k+\omega_\ell)t} + \varphi_2^- e^{i(\omega_k-\omega_\ell)t} \right\} .$$

- SO THE PRINCIPAL TASK OF SECOND-ORDER THEORY IN THE ABSENCE OF FLOATING BODIES IS TO DETERMINE THE COMPLEX SECOND-ORDER POTENTIALS φ_2^+ AND φ_2^-.

THEY SATISFY SECOND-ORDER FREE-SURFACE CONDITIONS STATED BELOW. USE IS MADE OF THE FOLLOWING IDENTITY SATISFIED BY REGULAR PLANE PROGRESSIVE WAVE COMPONENTS:

$$\frac{d}{dz}\left(\frac{\partial^2 \phi_1}{\partial t^2} + g \frac{\partial \phi_1}{\partial z} \right) = 0, \quad \text{ANY } z \le 0$$

(VERIFY!)

MAKING USE OF THE ABOVE IDENTITY IT IS EASY TO SHOW THAT:

- $$-\Omega^{+^2}\varphi_2^+ + g\frac{\partial\varphi_2^+}{\partial z} = -i\Omega^+ \nabla\varphi_{1km}\cdot\nabla\varphi_{2\ell n}\Big|_{z=0}$$

AND

- $$-\Omega^{-^2}\varphi_2^- + g\frac{\partial\varphi_2^-}{\partial z} = -i\Omega^- \nabla\varphi_{1km}\cdot\nabla\varphi_{2\ell n}^*\Big|_{z=0}$$

WHERE φ_{1km} AND $\varphi_{2\ell n}$ ARE THE COMPLEX VELOCITY POTENTIALS OF THE REGULAR PLANE PROGRESSIVE WAVES THE AMBIENT SEASTATE CONSISTS OF.

IT FOLLOWS THAT $\varphi_2^\pm$ EACH DEPEND ON 4 INDICES, TWO FOR THE FREQUENCIES ω_k & ω_ℓ AND TWO FOR THE RESPECTIVE HEADINGS θ_m AND θ_ℓ.—

- THE SOLUTION OF THE ABOVE FORCED FREE SURFACE PROBLEMS TOGETHER WITH $\nabla^2\varphi_2^\pm=0$ IN THE FLUID DOMAIN, IS A STRAIGHTFORWARD EXERCISE IN FOURIER THEORY (LEFT AS AN EXERCISE). THE SOLUTIONS FOR $\varphi_2^\pm$ ARE:

$$\varphi_2^{\pm} = \frac{\pm i}{2} \Omega^{\pm} A_k A_\ell \frac{\omega_k \omega_\ell}{g} \left[1 \mp \cos(\theta_m - \theta_\ell)\right] \cdot$$

$$\cdot \frac{e^{z(\alpha^{\pm 2} + \beta^{\pm 2})}}{(\alpha^{\pm 2} + \beta^{\pm 2})^{1/2} - \Omega^{\pm 2}/g} e^{-i\alpha^{\pm}x - i\beta^{\pm}y + i(\phi_k \pm \phi_\ell)}$$

(VERIFY BY SUBSTITUTION)

WHERE:

$$\alpha^{\pm} = \nu_k \cos\theta_m \pm \nu_\ell \cos\theta_n$$

$$\beta^{\pm} = \nu_k \sin\theta_m \pm \nu_\ell \sin\theta_n . -$$

THE REAL SECOND-ORDER POTENTIAL FOLLOWS BY QUADRUPLE SUMMATION OVER ALL PAIRS OF FREQUENCIES AND WAVE HEADINGS AND IS NOT STATED HERE.—

- A NUMBER OF SPECIAL CASES ARE NOTEWORTHY TO DISCUSS SINCE THEY CONVEY THE NEW PHYSICS INTRODUCED BY THE SECOND-ORDER POTENTIALS.

- ALL EFFECTS DISCUSSED BELOW ARE ADDITIVE TO THE LINEAR SOLUTION BY VIRTUE OF PERTURBATION THEORY AND OF $O(A_k A_\ell) = O(A^2)$ AN IN DEPTH DISCUSSION OF PERTURBATION THEORY ITS EXTENSIONS AND LIMITATIONS IN DEEP AND SHALLOW WATERS IS PRESENTED IN ME1.—

SPECIAL CASES

- $\Theta_m = \Theta_n = \Theta$: UNIDIRECTIONAL WAVES

$$\varphi_2^+ \equiv 0$$

$$\varphi_2^- = -i\Omega^- A_k A_\ell \frac{\omega_k \omega_\ell}{g} \frac{e^{z|N^-|}}{|N^-| - \Omega^{-2}/g} \times$$

$$\times\, e^{-iN^-(x\cos\theta + y\sin\theta) + i(\phi_k - \phi_\ell)}$$

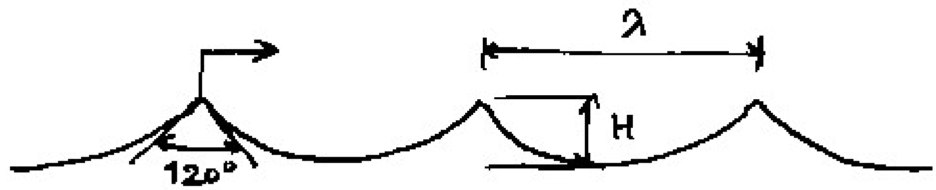
WHERE: $N^- = \nu_k - \nu_\ell \,, \quad \Omega^- = \omega_k - \omega_\ell$

- VERIFY THAT $|N^-| \neq \Omega^{-2}/g$.
- IN THE LIMIT $\omega_k = \omega_\ell$ VERIFY THAT $\varphi_2^- \equiv 0$

SO THE SECOND-ORDER POTENTIAL IS IDENTICALLY ZERO IN UNIDIRECTIONAL WAVES OF THE SAME FREQUENCY.

- VERIFY THAT THIS IS NOT HOWEVER THE CASE FOR THE SECOND-ORDER WAVE ELEVATION AND SECOND-ORDER PRESSURE.
- DERIVE AS AN EXERCISE THE SECOND-ORDER WAVE ELEVATION WHEN $\omega_k = \omega_\ell = \omega$ AND SHOW THAT IT CONTRIBUTES A CORRECTION TO THE LINEAR SINUSOIDAL SOLUTION WHICH PRODUCES MORE STEEPNESS AT THE CRESTS AND LESS AT TROUGHS.—

- CARRYING THE PERTURBATION SOLUTION WHEN $\omega_k = \omega_\ell = \omega$ TO ARBITRARILY HIGH ORDER (IT HAS BEEN APPLIED NUMERICALLY WITH OVER 100 TERMS!) THE FAMOUS SOLUTION BY STOKES IS RECOVERED:

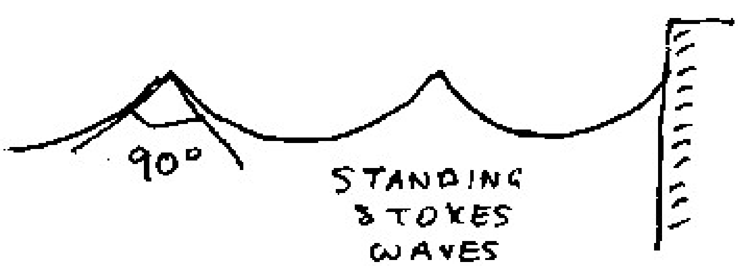


STOKES WAVES OF LIMITING STEEPNESS

THE CRESTS IN THE LIMIT BECOME CUSPED WITH AN ENCLOSED ANGLE OF 120° AND THE LIMITING VALUE OF THE H/λ RATIO IS ABOUT 1/7.

- SO TO THE EXTENT THAT PERIODIC WAVES OF THIS FORM CAN BE CREATED OR OBSERVED THESE LIMITING PROPERTIES ARE USEFUL TO REMEMBER AND KNOW THEY CAN BE OBTAINED BY PERTURBATION THEORY (AS WELL.)

- WHEN STOKES WAVES ARE REFLECTED OFF A WALL THE LIMITING ENCLOSED ANGLE IS 90°



(VERIFY NEXT TIME YOU ARE OFF FISHING OFF A PIER...)

- $\theta_m + \pi = \theta_n = \theta$

IN THIS CASE WE HAVE TWO WAVES PROPAGATING IN OPPOSITE DIRECTIONS.

IT CAN BE SHOWN THAT NOW:

$$\varphi_2^- \equiv 0$$

YET THE CORRESPONDING COMPONENTS OF THE WAVE ELEVATION AND PRESSURE ARE AGAIN NON-ZERO.

THE SUM-FREQUENCY POTENTIAL BECOMES:

$$\varphi_2^+ = i\Omega^+ A_k A_\ell \frac{\omega_k \omega_\ell}{g} \frac{e^{z|N^-|}}{|N^-| - \Omega^{+2}/g} \times$$

$$e^{iN^-(x\cos\theta + y\sin\theta) + i(\phi_k + \phi_\ell)}$$

WHERE $N^- = \omega_k - \omega_\ell$.

A NOTEWORTHY PROPERTY OF φ_2^+ IS THAT WHEN $\omega_k = \omega_\ell$, OR $N^- = 0$, THE VELOCITY POTENTIAL AND THE TERM IN THE SECOND ORDER HYDRODYNAMIC PRESSURE PROPORTIONAL TO $\partial\phi_2/\partial t$, DOES NOT ATTENUATE WITH DEPTH!

- THIS IS A NONLINEAR WAVE PROPERTY WHEN WAVES PROPAGATE IN OPPOSITE DIRECTIONS AND HAS BEEN MEASURED IN OCEAN PRESSURE FIELDS AT LARGE DEPTHS. IT HAS BEEN CONSIDERED RESPONSIBLE OF MICROSEISMS ON THE OCEAN FLOOR.

- THIS LACK OF ATTENUATION OF THE HYDRODYNAMIC SECOND-ORDER PRESSURE ALSO ARISES WHEN DIFFRACTION OCCURS OFF LARGE VOLUME FLOATING PLATFORMS, LIKE TLP'S. THE POSSIBLE RESULT IS A LARGE ENOUGH HIGH-FREQUENCY EXCITATION IN HEAVE & PITCH WHICH COULD CONTRIBUTE APPRECIABLY TO THE RESONANT RESPONSES OF TLP TETHERS.

- IN ORDER TO ASSESS THE FULL EFFECT OF SECOND-ORDER NONLINEARITIES ON FLOATING PLATFORMS (AND CERTAIN LARGE SHIPS THAT MAY UNERGO FLEXURAL VIBRATIONS) THE COMPLETE SOLUTION OF THE SUM-FREQUENCY SECOND-ORDER PROBLEM IS NECESSARY.

- IN RANDOM WAVES, IT IS ALSO BELIEVED THAT THE THIRD-ORDER SOLUTION IS NECESSARY. THIS BEING A FORMIDABLE TASK, FULLY NONLINEAR SOLUTIONS MAY BE APPROPRIATE.—