PLANE PROGRESSIVE LINEAR WAVES

AS ILLUSTRATED IN THE FIRST FEW SLIDES, REGULAR TIME-HARMONIC PLANE PROGRESSIVE WAVES ARE THE FUNDAMENTAL BUILDING BLOCK IN DESCRIBING THE PROPAGATION OF SURFACE WAVE DISTURBANCES IN A DETERMINISTIC & STOCHASTIC SETTING AND IN PREDICTING THE RESPONSE OF FLOATING STRUCTURES IN A SEASTATE.

THEY SATISFY THE BOUNDARY-VALUE PROBLEM:

$$
\begin{cases}
\dfrac{\partial^2 \phi_1}{\partial t^2} + g \dfrac{\partial \phi_1}{\partial z} = 0, & z = 0 \\[2ex]
\nabla^2 \phi_1 = 0, & -H < z < 0 \\[2ex]
\dfrac{\partial \phi_1}{\partial z} = 0, & z = -H
\end{cases}
$$

THE LAST CONDITION IMPOSES A ZERO NORMAL FLUX CONDITION ACROSS A SEAFLOOR OF CONSTANT DEPTH, H. —

THE RELATION BETWEEN ω AND K IS KNOWN AS THE DISPERSION RELATION OFTEN WRITTEN IN THE FORM:

$$\omega = f(k)$$

$f(k)$ DEPENDS ON THE PHYSICS OF THE WAVE PROPAGATION PROBLEM UNDER STUDY.

IN THE CASE OF SOUND PROPAGATION;

$$\omega = c\,K \equiv f(k)$$

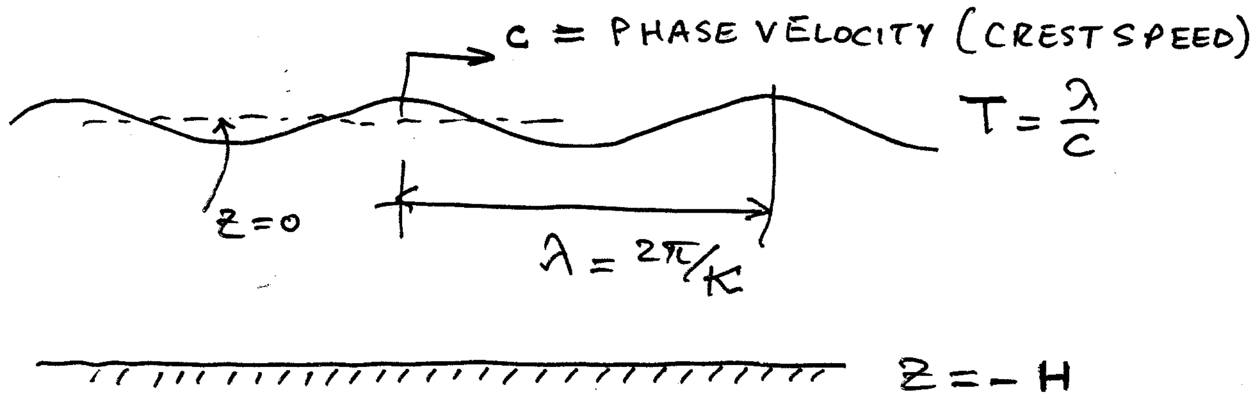
WHERE c IS THE SPEED OF SOUND.
THE ABOVE RELATION SUGGESTS THAT ALL SOUND WAVES IN A MEDIUM WITH ISOTROPIC PROPERTIES PROPAGATE WITH THE SAME PHASE SPEED REGADLESS OF THEIR FREQUENCY OR WAVELENGTH. THESE ARE KNOWN TO BE NON-DISPERSIVE WAVES

SURFACE WAVES ARE DISPERSIVE SINCE $f(k)$ IS A NONLINEAR FUNCTION OF k. IT WILL BE DERIVED BELOW._

BY DEFINITION, REGULAR PLANE PROGRESSIVE WAVES ARE SUCH THAT THEIR FREE SURFACE ELEVATION IS DEFINED <u>A PRIORI</u> AS FOLLOWS:

$$\zeta_1(x,t) = A\cos(\omega t - kx)$$
$$= A\,\mathrm{Re}\{e^{-ikx+i\omega t}\}$$

OR SCHEMATICALLY:



WHERE A IS AN A PRIORI KNOWN WAVE AMPLITUDE AND THE WAVE FREQUENCY AND WAVE NUMBER (ω, k) PAIR HAVE THE SAME DEFINITIONS AS IN ALL WAVE PROPAGATION PROBLEMS, NAMELY:

$$T = \frac{2\pi}{\omega} = \text{WAVE PERIOD}$$

$$\lambda = \frac{2\pi}{k} = \text{WAVE LENGTH}$$

$$c = \frac{\omega}{k} = \text{WAVE PHASE VELOCITY}$$

BY VIRTUE OF THE DEFINITION OF THE WAVE ELEVATION OF A PLANE PROGRESSIVE REGULAR WAVE, WE SEEK A COMPATIBLE DEFINITION OF THE RESPECTIVE VELOCITY POTENTIAL. LET:

$$\Phi_1 = A \, \mathrm{Re} \{ \varphi(x, z) e^{i\omega t} \}$$

THE PROBLEM REDUCES TO THE DEFINITION OF $\varphi(x, z)$ AND THE DERIVATION OF THE APPROPRIATE DISPERSION RELATION BETWEEN ω AND K SO THAT THE LINEAR BOUNDARY VALUE PROBLEM IS SATISFIED.

BEFORE PROCEEDING WITH THE ALGEBRA, CERTAIN UNDERLYING PRINCIPLES ARE ALWAYS AT WORK:

- LINEAR SYSTEM THEORY STATES THAT WHEN THE INPUT SIGNAL IS $e^{i\omega t}$ THE OUTPUT SIGNAL MUST ALSO BE HARMONIC AND WITH THE SAME FREQUENCY.

- WE ASSUME THAT A SOLUTION ALWAYS EXISTS, OTHERWISE THE STATEMENT OF THE PHYSICAL AND/OR MATHEMATICAL BOUNDARY VALUE PROBLEM IS FLAWED. IF WE CAN FIND A SOLUTION IN MOST CASES IT IS THE SOLUTION. SO SIMPLY TRY OUT SOLUTIONS THAT MAY MAKE SENSE FROM THE PHYSICAL POINT OF VIEW.

- IF THE BOUNDARY VALUE PROBLEM IS SATISFIED BY A COMPLEX VELOCITY POTENTIAL THEN IT IS ALSO SATISFIED BY ITS REAL AND IMAGINARY PARTS

IN OUR CASE WE WILL FIRST DERIVE THE BOUNDARY VALUE PROBLEM SATISFIED BY THE COMPLEX POTENTIAL $\varphi(x, z)$ AND THEN WE WILL TRY THE PLAUSIBLE REPRESENTATION:

$$\varphi(x,z) = \psi(z)\, e^{-ikx}.$$

IT FOLLOWS UPON SUBSTITUTION IN THE BOUNDARY VALUE PROBLEM SATISFIED BY $\phi_1(x,z,t)$, THAT $\varphi(x,z)$ IS SUBJECT TO:

$$\begin{cases} -\omega^2\varphi + g\,\varphi_z = 0, & z=0 \\ \nabla^2\varphi = \varphi_{xx}+\varphi_{zz}=0, & -H<z<0 \\ \varphi_z = 0, & z=-H \end{cases}$$

ALLOWING FOR:

$$\varphi(x,z) = \psi(z)\,e^{-ikx}$$

IT FOLLOWS THAT $\psi(z)$ IS SUBJECT TO:

$$\begin{cases} -\omega^2\psi + g\,\psi_z = 0, & z=0 \\ \psi_{zz} - k^2\psi = 0, & -H<z<0 \\ \psi_z = 0, & z=-H \end{cases}$$

VERIFY BY SIMPLE SUBSTITUTION THAT:

$$\psi(z) = \frac{ig}{\omega} \frac{\cosh K(z+H)}{\cosh KH}$$

SATISFIES THE FIELD EQUATION $\psi_{zz} - k^2\psi = 0$, THE SEAFLOOR CONDITION $\psi_z = 0$, $z = -H$ AND THE FREE SURFACE CONDITION, ONLY WHEN

$$\omega^2 = gK \tanh KH$$

OR

$$\omega = \underbrace{\{ g k \tanh KH \}^{1/2}}_{f(k)} .$$

SO BY ENFORCING THE FREE SURFACE CONDITION WE HAVE DERIVED THE DISPERSION RELATION FOR REGULAR WAVES IN FINITE DEPTH.

RECALL THAT: $\cosh x = \dfrac{e^x + e^{-x}}{2}$

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

$$\tanh x = \sinh x / \cosh x$$

THE RESULTING PLANE PROGRESSIVE WAVE VELOCITY POTENTIAL TAKES THE FORM:

$$\phi_1(x,z,t) = A \,\mathrm{Re}\left\{ \frac{ig}{\omega} \frac{\cosh k(z+H)}{\cosh kH} \times \right.$$

$$\left. \times\, e^{-ikx+i\omega t} \right\}$$

VERIFY THAT UPON SUBSTITUTION:

$$\zeta_1 = A\,\mathrm{Re}\left\{ e^{-ikx+i\omega t} \right\}$$

$$= -\frac{1}{g} \frac{\partial \phi_1}{\partial t}\Big|_{z=0}$$

THE CORRESPONDING FLOW VELOCITY AT SOME POINT $\vec{x}=(x,z)$ IN THE FLUID DOMAIN OR ON $z=0$, $z=-H$ IS SIMPLY GIVEN BY:

$$\vec{V}_1 = \nabla \phi_1$$

THE LINEAR HYDRODYNAMIC PRESSURE DUE TO THE PLANE PROGRESSIVE WAVE, WHICH MUST BE ADDED TO THE HYDROSTATIC, IS

$$P_1 = -\rho \frac{\partial \phi_1}{\partial t}$$

$$= \mathrm{Re}\left\{ \rho g A \frac{\cosh k(z+H)}{\cosh kH} e^{-ikx+i\omega t} \right\}$$

FROM THE LAGRANGIAN KINEMATIC RELATION

$$\frac{d\vec{\xi}_1}{dt} = \vec{V}(\vec{\xi}_1, t) = \nabla\phi_1(\vec{\xi}_1, t)$$

WE MAY OBTAIN ORDINARY DIFFERENTIAL EQUATIONS GOVERNING $\vec{\xi}_1(t)$. MARKING A PATICULAR PARTICLE WITH THE FLUID AT REST, SO THAT $\vec{\xi}_1(0) = \vec{X}$, WE MAY WRITE:

$$\vec{\xi}_1(t) = \vec{X} + \underbrace{\vec{\Delta\xi}(t)}_{O(\varepsilon)}$$

WHERE $\vec{X}$ IS THE PARTICLE POSITION AT REST AND $\vec{\Delta\xi}$ IS ITS DISPLACEMENT DUE TO THE "ARRIVAL" OF A PLANE PROGRESSIVE WAVE.

UPON SUBSTITUTION IN THE EQUATION OF MOTION

$$\frac{d\vec{\Delta\xi}}{dt} = \underbrace{\nabla\phi_1}_{O(\varepsilon)}(\vec{x} + \underbrace{\vec{\Delta\xi}}_{O(\varepsilon)}, t)$$

SINCE $\frac{d\vec{x}}{dt} = 0$. KEEPING TERMS OF $O(\epsilon)$ ON BOTH SIDES, IT FOLLOWS THAT

$$\frac{d\vec{\Delta\xi}}{dt} = \nabla\phi_1(\vec{x}, t) + O(\varepsilon^2)$$

THIS EQUATION WHEN FORCED BY THE VELOCITY VECTOR THAT CORRESPONDS TO THE PLANE PROGRESSIVE WAVE SOLUTION DERIVED ABOVE, LEADS TO A HARMONIC SOLUTION FOR THE PARTICLE DISPLACED TRAJECTORIES $\Delta\vec{\xi}(t) = (\Delta\xi_1, \Delta\xi_3)$ WHICH ARE CIRCULAR [LEFT AS AN EXERCISE].

IF SECOND-ORDER EFFECTS ARE INCLUDED, THE PARTICLES UNDER A PLANE PROGRESSIVE WAVES ALSO UNDERGO A STEADY-STATE DRIFT KNOWN AS THE STOKES DRIFT. IT CAN BE EASILY MODELED BASED ON THE APPROACH DESCRIBED ABOVE BY SUBSTITUTING SECOND-ORDER EFFECTS CONSISTENTLY INTO THE RIGHT-HAND SIDE OF THE EQUATION OF MOTION (SEE MH).

DISPERSION RELATION IN DEEP & SHALLOW WATERS

IN FINITE DEPTH:

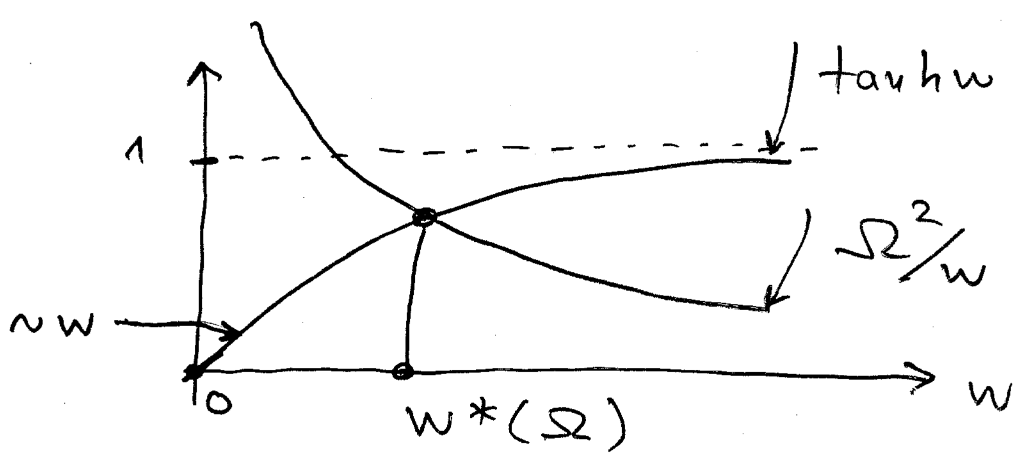
$$\omega^2 = g K \tanh KH$$

OR

$$\Omega^2 = \frac{\omega^2 H}{g} = \underbrace{KH}_{w} \tanh \underbrace{KH}_{w}$$

$$\Rightarrow \quad \frac{\Omega^2}{w} = \tanh w$$

THIS IS A NONLINEAR ALGEBRAIC EQUATION FOR $w = KH$ WHICH HAS A UNIQUE SOLUTION AS CAN BE SHOWN GRAPHICALLY



$\tanh w$: MONOTONICALLY INCREASING WITH AN ASYMPTOTIC VALUE OF 1

Ω^2/w : HYPERBOLA MONOTONICALLY DECREASING

UNIQUE REAL ROOT $W^*(\Omega)$ CAN ONLY BE FOUND NUMERICALLY. YET IT ALWAYS EXISTS AND THE ITERATIVE METHODS THAT MAY BE IMPLEMENTED ALWAYS CONVERGE RAPIDLY.

GIVEN $(\omega, H) \rightarrow W^*\left(\Omega^2 = \frac{\omega^2 H}{g}\right) \equiv W^* = KH$

$$\Rightarrow \quad \frac{2\pi}{\lambda} H = W^*(\Omega) \Rightarrow \lambda = \frac{2\pi H}{W^*(\Omega)} \cdot -$$

SO GIVEN THE WAVE FREQUENCY $\omega \rightarrow \lambda$

IN DEED WATER; $H \rightarrow \infty$

$$\tanh KH \rightarrow 1$$

$$\Rightarrow \quad \omega^2 = g k \quad ;$$ DEEP WATER DISPERSION RELATION

PHASE SPEED: $C = \frac{\omega}{K} = \frac{\omega}{\omega^2/g} = \frac{g}{\omega}$

OR $C = \frac{g}{2\pi/T} = \frac{gT}{2\pi}$

SO THE SPEED OF THE CREST OF A WAVE WITH PERIOD $T = 10$ secs IS APPROXIMATELY $15.6\,\frac{m}{s}$ OR ABOUT 30 KNOTS!

OFTEN WE NEED A QUICK ESTIMATE OF THE LENGTH OF A DEEP WATER WAVE THE PERIOD OF WHICH WE CAN MEASURE ACCURATELY WITH A STOPWATCH. WE PROCEED AS FOLLOWS:

$$C = \frac{\omega}{K} = \frac{\lambda}{T}$$

; BY DEFINITION THE PHASE SPEED IS THE RATIO OF THE WAVELENGTH OVER THE PERIOD, OR THE TIME IT TAKES FOR A CREST TO TRAVEL THAT DISTANCE

$$\Rightarrow \frac{\lambda}{T} = \frac{g}{\omega} = \frac{g}{2\pi/T}$$

$$\Rightarrow \lambda = \frac{gT^2}{2\pi} \simeq T^2 + \frac{1}{2}T^2$$

SO THE WAVELENGTH OF A DEEP WATER WAVE IN m IS APPROXIMATELY THE SQUARE OF ITS PERIOD IS SECONDS PLUS HALF THAT AMOUNT.

SO A WAVE WITH PERIOD $T = 10$ secs IS ABOUT 150 m LONG OR ABOUT 492 ft.

IN THE LIMIT OF SHALLOW WATER; $KH \to 0$

$$\tanh kH \simeq kH$$

AND $\omega^2 = gk(kH) \Rightarrow \frac{\omega^2}{k^2} = gH$

OR $$\frac{\omega}{k} = C = \sqrt{gH}$$

THUS, ACCORDING TO LINEAR THEORY SHALLOW WATER WAVES BECOME NONDISPERSIVE AS IS THE CASE WITH ACOUSTIC WAVES.

UNFORTUNATELY, NONLINEAR EFFECTS BECOME IMPORTANT IN THE LIMIT OF SHALLOW WATER AND MUST BE CAREFULLY BE TAKEN INTO ACCOUNT. SOLITONS AND WAVE BREAKING ARE SOME MANIFESTATIONS OF NONLINEARITY. (SEE MEI).

THE TRANSITION FROM DEEP TO FINITE DEPTH WAVE EFFECTS OCCURS FOR VALUES OF $KH \leq \pi$. CHECK THAT:

$$\tanh \pi \simeq 1$$

AND FOR $KH = \pi \Rightarrow \frac{2\pi H}{\lambda} = \pi$

$$\Rightarrow \frac{H}{\lambda} = \frac{1}{2}$$

SO FOR $H/\lambda > 1/2$ OR $KH > \pi$ WE ARE EFFECTIVELY DEALING WITH DEEP WATER.