MOMENTUM FLUX IN POTENTIAL FLOW

$$\frac{d\vec{M}(t)}{dt} = \rho \frac{d}{dt} \iiint_{V(t)} \vec{V}\, dv = \rho \iiint_{V(t)} \frac{\partial \vec{V}}{\partial t}\, dv$$

$$+ \rho \oiint_{S(t)} \vec{V} U_n \, ds,$$ BY VIRTUE OF THE TRANSPORT THEOREM

INVOKING EULER'S EQUATIONS IN INVISCID FLOW

$$\frac{\partial \vec{V}}{\partial t} + (\vec{V} \cdot \nabla)\vec{V} = -\frac{1}{\rho} \nabla p + \vec{g}$$

WE MAY RECAST THE RATE OF CHANGE OF THE MOMENTUM ($\equiv$ MOMENTUM FLUX) IN THE FORM

$$\frac{d\vec{M}(t)}{dt} = -\rho \iiint_{V(t)} \left[\nabla\left(p/\rho + gz\right) + (\vec{V}\cdot\nabla)\vec{V} \right] dv$$

$$+ \rho \oiint_{S(t)} \vec{V} U_n \, ds$$

SO FAR $V(t)$ IS AN ARBITRARY CLOSED TIME DEPENDENT VOLUME BOUNDED BY THE TIME DEPENDENT SURFACE $S(t)$. HERE WE NEED TO INVOKE AN IMPORTANT AND COMLEX VECTOR THM.

RECALL FROM THE PROOF OF BERNOULLI'S EQUATION THAT:

$$(\vec{V}\cdot\nabla)\vec{V} = \nabla\left(\tfrac{1}{2}\vec{V}\cdot\vec{V}\right) - \vec{V}\times\left(\underbrace{\cancel{\nabla}\times\vec{V}}_{0}\right)$$

BY VIRTUE OF GAUSS'S VECTOR THM:

$$\iiint_{v(t)} \nabla\left(\tfrac{1}{2}\vec{V}\cdot\vec{V}\right) dv = \frac{1}{2}\oiint_{S(t)} \vec{V}\cdot\vec{V}\,\vec{n}\,ds$$

WHERE IN POTENTIAL FLOW: $\vec{V} = \nabla\phi$

IN POTENTIAL FLOW IT CAN BE SHOWN THAT:

$$\oiint_{S(t)} \tfrac{1}{2}(\vec{V}\cdot\vec{V})\,\vec{n}\,ds = \oiint_{S(t)} \frac{\partial\phi}{\partial n}\nabla\phi\,ds$$

$$= \oiint_{S(t)} V_n \vec{V}\,ds$$

PROOF LEFT AS AN EXERCISE! JUST PROVE THAT FOR $\nabla^2\phi = 0$;

$$\oiint_{S} \tfrac{1}{2}(\nabla\phi\cdot\nabla\phi)\,\vec{n}\,ds \equiv \oiint_{S} \frac{\partial\phi}{\partial n}\nabla\phi\,ds\,. -$$

UPON SUBSTITUTION IN THE MOMENTUM FLUX FORMULA, WE OBTAIN:

$$\frac{d\vec{M}}{dt} = -\rho \oiint_{S(t)} \left[\left(P/\rho + gz \right) \vec{n} + \vec{V} \left(V_n - U_n \right) \right] ds$$

FOLLOWING AN APPLICATION OF THE VECTOR THEOREM OF GAUSS. IN THE EXPRESSION ABOVE $\vec{n}$ IS THE UNIT VECTOR POINTING OUT OF THE VOLUME $\mathcal{V}(t)$, $V_n = \vec{n} \cdot \nabla \phi$ AND U_n IS THE OUTWARD NORMAL VELOCITY OF THE SURFACE $S(t)$. —

THIS FORMULA IS OF CENTRAL IMPORTANCE IN POTENTIAL FLOW MARINE HYDRODYNAMICS BECAUSE THE RATE OF CHANGE OF THE LINEAR MOMENTUM DEFINED ABOVE IS JUST $\pm$ THE FORCE ACTING ON THE FLUID VOLUME. WHEN ITS MEAN VALUE CAN BE SHOWN TO VANISH, IMPORTANT FORCE EXPRESSIONS ON SOLID BOUNDARIES FOLLOW AND WILL BE DERIVED IN WHAT FOLLOWS. —

CONSIDER SEPARATELY THE TERM IN THE MOMENTUM FLUX EXPRESSION INVOLVING THE HYDROSTATIC PRESSURE:

$$\frac{d\vec{M}_H}{dt} = -\rho \oiint_S g z \vec{n}, \quad \text{WHERE } S = S_F + S^{\pm} + S_B + S_{-\infty}$$

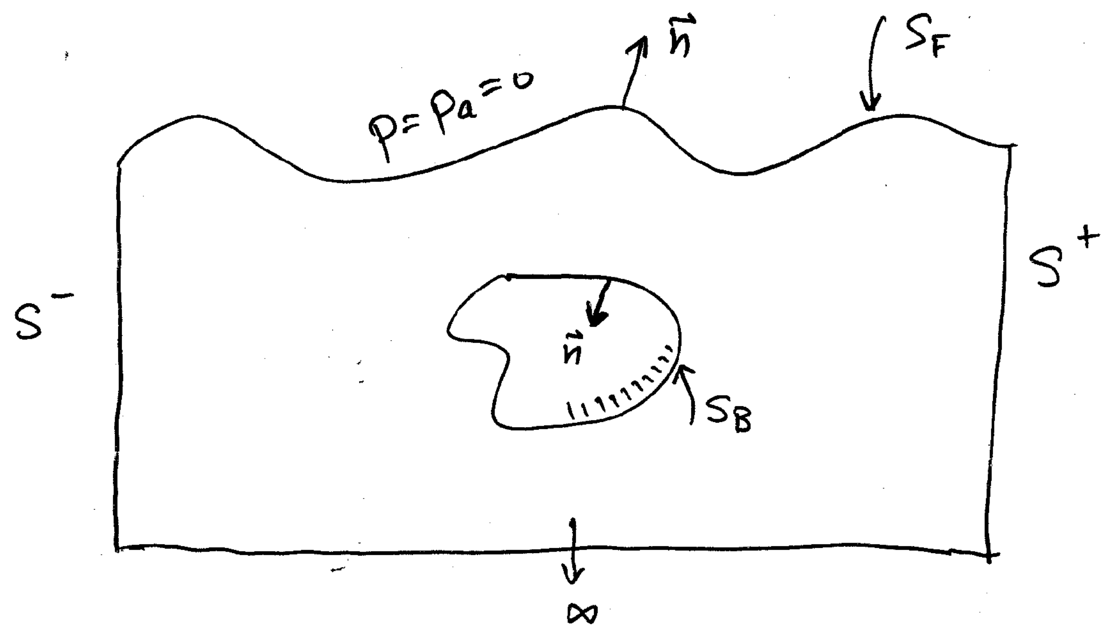
THE INTEGRAL OVER THE BODY SURFACE, ASSUMING A FULLY SUBMERGED BODY IS:

$$\frac{d\vec{M}_{H,B}}{dt} = -\rho \oiint_{S_B} g z \vec{n} = \rho g \forall \vec{k} = \vec{B}$$

BY VIRTUE OF THE VECTOR THEOREM OF GAUSS THIS IS NONE OTHER THAN THE PRINCIPLE OF ARCHIMEDES!

WE MAY THEREFORE CONSIDER THE SECOND PART OF THE INTEGRAL INVOLVING WAVE EFFECTS INDEPENDENTLY AND IN THE ABSENCE OF THE BODY, ASSUMED FULLY SUBMERGED. IN THE CASE OF A SURFACE PIERCING BODY AND IN THE FULLY NONLINEAR CASE MATTERS ARE MORE COMPLEX BUT TRACTABLE!

CONSIDER THE APPLICATION OF THE MOMENTUM CONSERVATION THEOREM IN THE CASE OF A SUBMERGED OR FLOATING BODY IN STEEP WAVES:



HERE WE CONSIDER THE TWO-DIMENSIONAL CASE IN ORDER TO PRESENT THE CONCEPTS. EXTENSIONS TO THREE DIMENSIONS ARE THEN TRIVIAL.

NOTE THAT UNLIKE THE ENERGY CONSERVATION PRINCIPLE, THE MOMENTUM CONSERVATION THEOREM DERIVED ABOVE IS A VECTOR IDENTITY WITH A HORIZONTAL AND A VERTICAL COMPONENT.

THE INTEGRAL OF THE HYDROSTATIC TERM OVER THE REMAINING SURFACES LEADS TO:

$$\frac{d\vec{M}_{H,S}}{dt} = -\rho \iint_{S_F+S^++S^-+S_\infty} g z \vec{n}\, ds = -\rho g \forall_{FLUID} \vec{k}$$

THIS IS SIMPLY THE STATIC WEIGHT OF THE VOLUME OF FLUID BOUNDED BY S_F, S^+, S^- AND S_∞. WITH NO WAVES PRESENT, THIS IS SIMPLY THE WEIGHT OF THE OCEAN WATER "COLUMN" BOUNDED BY S WHICH DOES NOT CONCERN US HERE. THIS WEIGHT DOES NOT CHANGE IN PRINCIPLE WHEN WAVES ARE PRESENT AT LEAST WHEN S^+, S^- ARE PLACED SUFFICIENTLY FAR AWAY THAT THE WAVE AMPLITUDE HAS DECREASED TO ZERO. SO "IN PRINCIPLE" THIS TERM BEING OF HYDROSTATIC ORIGIN MAY BE IGNORED. HOWEVER, IT IS IN PRINCIPLE MORE "RATIONAL" TO APPLY THE MOMENTUM

CONSERVATION THEOREM OVER THE "LINEARIZED" VOLUME $\mathcal{V}_L(t)$ WHICH IS PERFECTLY POSSIBLE WITHIN THE FRAMEWORK DERIVED ABOVE. IN THIS CASE $\frac{d\vec{M}_H}{dt}$ IS EXACTLY EQUAL TO THE STATIC WEIGHT OF THE WATER COLUMN AND CAN BE IGNORED IN THE WAVE-BODY INTERACTION PROBLEM.

ON S_F; $p = p_a = 0$ AND HENCE ALL TERMS WITHIN THE FREE-SURFACE INTEGRAL AND OVER S_∞ (SEAFLOOR) CAN BE NEGLECTED. IT FOLLOWS THAT:

$$\frac{d\vec{M}}{dt} = -\rho \iint_{S^\pm + S_B} \left[P/\rho \, \vec{n} + \vec{V}(V_n - U_n) \right] ds$$

NOTE THAT THE FREE SURFACE INTEGRALS ALSO VANISH FOR THE HORIZONTAL COMPONENT SINCE THE HYDROSTATIC FORCE IS ALWAYS VERTICAL!

THIS MOMENTUM FLUX FORMULA IS OF CENTRAL IMPORTANCE IN WAVE-BODY INTERACTIONS AND HAS MANY IMPORTANT APPLICATIONS, SOME OF WHICH ARE DISCUSSED BELOW.

- NOTE THAT THE MATHEMATICAL DERIVATIONS INVOLVED IN ITS PROOF APPLY EQUALLY WHEN THE VOLUME $\mathcal{V}$ AND ITS ENCLOSED SURFACE ARE SELECTED TO BE AT THEIR LINEARIZED POSITIONS. IN SUCH A CASE IT IS ESSENTIAL TO SET $U_n = 0$ AND RECALL THAT ON $Z=0$, $p \neq 0$ AND $V_n \neq 0$. LET THE MATH TAKE OVER AND SUGGEST THE PROPER EXPRESSION FOR THE FORCE. IN THE FULLY NONLINEAR CASE, $U_n \neq 0$ ON S_F AND $p = 0$ on S_F!

ON A SOLID BOUNDARY:

$$U_n = V_n$$

AND

$$\vec{F}_B = \iint_{S_B} p \, \vec{n} \, ds$$

WITH $\vec{n}$ POINTING INSIDE THE BODY.

WE MAY THEREFORE RECAST THE MOMENTUM CONSERVATION THEOREM IN THE FORM:

$$\vec{F}_B(t) = -\frac{d\vec{M}}{dt} - \rho \iint_{S^\pm} \left[P/\rho \, \vec{n} + \vec{V}(V_n - U_n) \right] ds$$

WHERE $S^\pm$ ARE FLUID BOUNDARIES AT SOME DISTANCE FROM THE BODY.

IF THE VOLUME OF FLUID SURROUNDED BY THE BODY, FREE SURFACE AND THE SURFACES $S^\pm$ DOES NOT GROW IN TIME, THEN THE MOMENTUM OF THE ENCLOSED FLUID CANNOT GROW EITHER, SO $\vec{M}(t)$ IS A STATIONARY PHYSICAL QUANTITY.

IT IS A WELL KNOWN RESULT THAT THE MEAN VALUE IN TIME OF THE TIME DERIVATIVE OF A STATIONARY QUANTITY IS ZERO. SO:

$$\overline{\frac{d\vec{M}}{dt}}^{\,t} = 0$$

PROOF:

$$\overline{\frac{dF}{dt}}^{\,t} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \frac{dF(\tau)}{d\tau}\, d\tau$$

$$= \lim_{T\to\infty} \frac{1}{2T} \left[F(T) - F(-T) \right]$$

SINCE $F(\pm T)$ MUST BE BOUNDED FOR A STATIONARY SIGNAL $F(t)$, IT FOLLOWS THAT $\overline{dF/dt}^{\,t} \to 0$.

TAKING MEAN VALUES, IT FOLLOWS THAT:

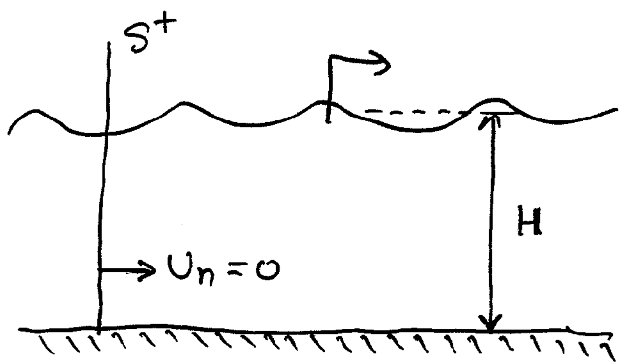
$$\overline{\vec{F}_B(t)}^{\,t} = -\rho\, \overline{\iint_{S\pm} \left[\frac{p}{\rho}\vec{n} + \vec{v}(v_n - u_n) \right] ds}^{\,t}$$

THIS IS THE FUNDAMENTAL FORMULA UNDERLYING THE DEFINITION OF THE MEAN WAVE DRIFT FORCES ACTING ON FLOATING BODIES. SUCH FORCES ARE VERY IMPORTANT FOR STATIONARY FLOATING STRUCTURES AND CAN BE EXPRESSED IN TERMS OF INTEGRALS OF WAVE EFFECTS OVER CONTROL SURFACES $S^{\pm}$ WHICH MAY BE LOCATED AT INFINITY

- THE EXTENSION OF THE ABOVE FORMULA FOR $\bar{\bar{F}}_B^{t}$ IN THREE DIMENSIONS IS TRIVIAL. SIMPLY REPLACE $S^{\pm}$ BY S_∞, A CONTROL SURFACE AT INFINITY. COMMON CHOICES ARE A VERTICAL CYLINDRICAL BOUNDARY OR TWO VERTICAL PLANES PARALLEL TO THE AXIS OF FORWARD MOTION OF A SHIP.

APPLICATIONS

- MEAN HORIZONTAL MOMENTUM FLUX DUE TO A PLANE PROGRESSIVE WAVE



$$\zeta = A \cos(\omega t - kx)$$

$$\omega^2 = gk \tanh kH$$

EVALUATE THE MOMENTUM FLUX ACROSS S^+:

$$\frac{dM_x}{dt} = -\int_{-H}^{\zeta} (p + \rho u^2)\, dz$$

$$p = -\rho \frac{\partial \phi}{\partial t} - \frac{1}{2}\rho(u^2+v^2) - \rho g z \quad ; \text{ BERNOULLI}$$

$$\frac{dM_x}{dt} = \rho \int_{-H}^{\zeta} \left[\frac{\partial \phi}{\partial t} + \frac{1}{2}(v^2 - u^2) + g z \right] dz$$

$$= \rho \left(\int_{-H}^{0} + \int_{0}^{\zeta} \right) \left[\frac{\partial \phi}{\partial t} + \frac{1}{2}(v^2 - u^2) + g z \right] dz$$

TAKING MEAN VALUES IN TIME AND KEEPING TERMS OF $O(A^2)$ WE OBTAIN:

$$\overline{\frac{dM_x}{dt}}^{\,t} = \overline{\rho\, \zeta(t) \left.\frac{\partial \phi}{\partial t}\right|_{z=0}}^{\,t} + \frac{1}{2}\rho\, \overline{\int_{-H}^{0} (v^2 - u^2)\, dz}^{\,t} + \frac{1}{2}\rho g \overline{\zeta^2}^{\,t}$$

INVOKING THE LINEARIZED DYNAMIC FREE SURFACE CONDITION:

$$\left.\frac{\partial \phi}{\partial t}\right|_{z=0} = -g\zeta$$

IT FOLLOWS THAT:

$$\overline{\frac{dM_x}{dt}}^{t} = -\frac{1}{2}\rho g \overline{\zeta^2(t)}^{t} + \frac{1}{2}\rho \overline{\int_{-H}^{0} (v^2 - u^2)\, dz}^{t}$$

- IN DEEP WATER $\overline{V^2} = \overline{u^2}$ AND THE SECOND TERM IS IDENTICALLY ZERO, SO

$$\overline{\frac{dM_x}{dt}}^{t} = -\frac{1}{2}\rho g \overline{\zeta^2(t)}^{t} = -\frac{1}{4}\rho g A^2$$

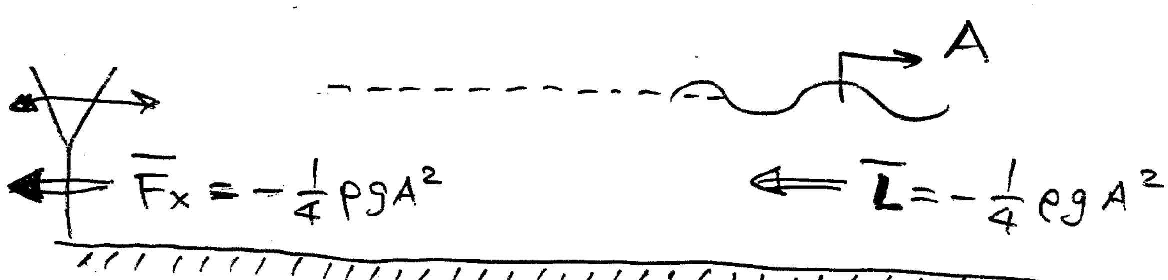
- IN WATER OF FINITE DEPTH THE WAVE PARTICLE TRAJECTORIES ARE ELLIPTICAL WITH THE MEAN HORIZONAL VELOCITIES^2 LARGER THAN THE MEAN VERTICAL VELOCITIES^2 SO;

$$\overline{V_H^2} < \overline{U_H^2}$$

THEREFORE, IN FINITE DEPTH THE MODULUS OF THE MEAN MOMENTUM FLUX IS HIGHER THAN IN DEEP WATER FOR THE SAME A.

SO THE MEAN HORIZONTAL MOMENTUM FLUX DUE TO A PLANE PROGRESSIVE WAVE IS AGAINST ITS DIRECTION OF PROPAGATION AND EQUAL TO $-\frac{1}{4}\rho g A^2$.

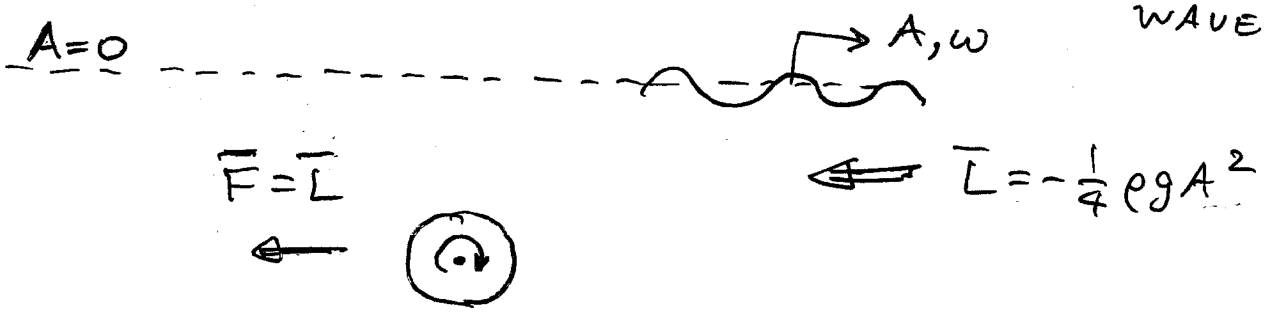
- CONSIDER A WAVEMAKER SHOWN IN THE FIGURE GENERATING A WAVE OF AMPLITUDE A AT INFINITY:



WHAT IS THE MEAN HORIZONTAL FORCE ON THE WAVEMAKER? FROM THE MOMENTUM CONSERVATION THEOREM THE MEAN HORIZONTAL FLUX OF MOMENTUM TO THE LEFT MUST FLOW INTO THE WAVEMAKER. THIS MEAN FLUX TRANSLATES INTO A MEAN HORIZONTAL FORCE IN THE SAME DIRECTION, AS SHOWN IN THE FIGURE. NOT AN EASY CONCLUSION WITHOUT USING SOME BASIC FLUID MECHANICS!

- TWO OTHER ILLUSTRATIONS OF THE MOMENTUM FLUX PRINCIPLE ARE NOTEWORTHY:

CONSIDER A CIRCULAR OBJECT IN TWO DIMENSIONS ROTATING AT AN ANGULAR FREQUENCY ω UNDER A FREE SURFACE AND WITH A SMALL AMPLITUDE. IN OTHER WORDS THE CIRCLE CENTER ROTATES AS A FLUID PARTICLE IN A REGULAR WAVE



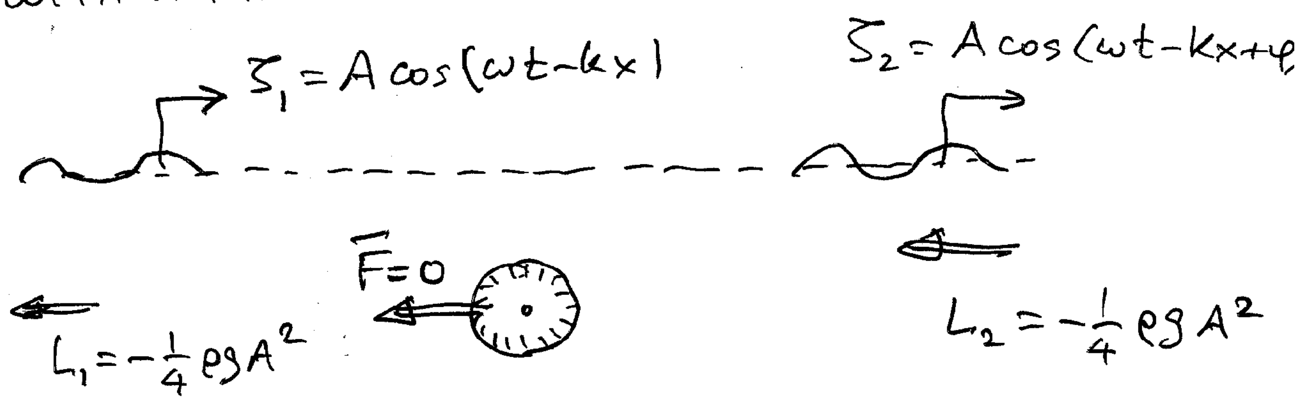
IT CAN BE SHOWN THAT ACCORDING TO LINEAR THEORY THIS ROTATIONAL MOTION CREATES REGULAR WAVES PROPAGATING TO THE RIGHT WITH NO WAVES PROPAGATING TO THE LEFT!

WHAT IS THE MEAN HORIZONTAL FORCE ACTING ON THE CIRCLE ?

FROM THE MOMENTUM CONSERVATION PRINCIPLE THE MEAN FORCE $\bar{F}$ ACTING ON THE CIRCLE MUST BE EQUAL TO $\bar{L}$, THUS:

$$\bar{F} = -\frac{1}{4}\rho g A^2.$$

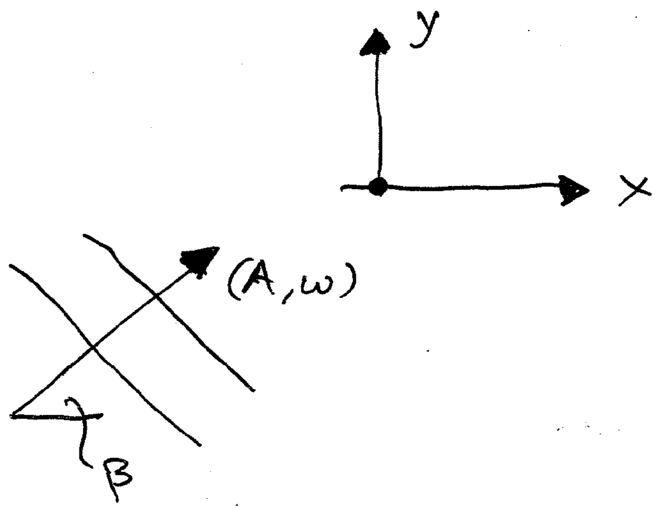
- DEAN (1953) HAS SHOWN THAT REGULAR PLANE PROGRESSIVE WAVES IN DEEP WATER PROPAGATING OVER A CIRCLE FIXED UNDER THE FREE SURFACE ARE UNTIRELY TRANSMITTED WITH A PHASE SHIFT



IN OTHER WORDS THE WAVE REFLECTION IS ZERO. NOTE THIS IS THE ONLY BODY IN 2 DIMENSIONS WITH THIS PROPERTY. REFLECTION IS NON-ZERO IN FINITE DEPTH FOR ALL KNOWN BODIES. SHOW BY APPLYING THE MOMENTUM CONSERVATION PRINCIPLE THAT THE MEAN HORIZONTAL FORCE

$$\bar{F} = 0 .$$

PLANE PROGRESSIVE WAVES IN GENERAL DIRECTIONS.



FREE SURFACE ELEVATION :

$$\zeta = A \cos(\omega t - kx\cos\beta - ky\sin\beta)$$

$$= A\,\mathrm{Re}\left\{ e^{-ikx\cos\beta - iky\sin\beta + i\omega t} \right\}$$

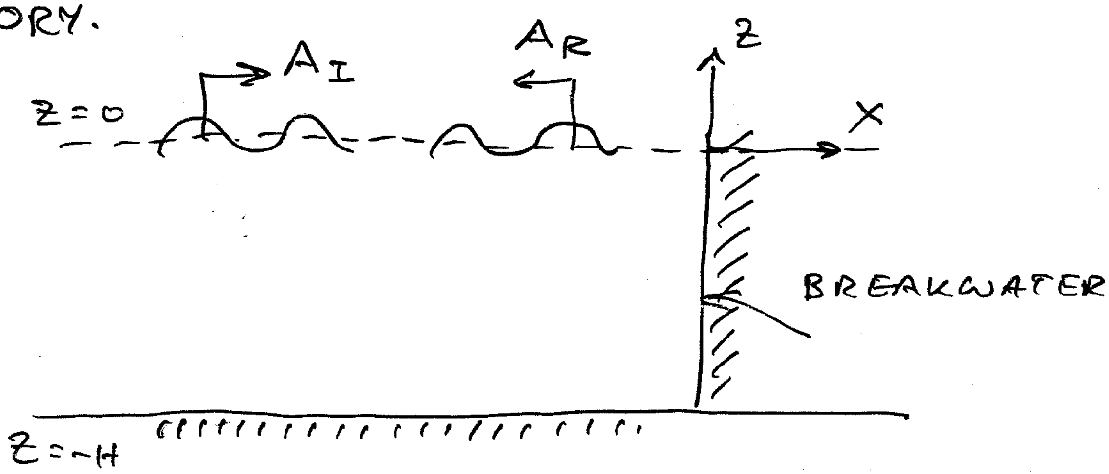
VELOCITY POTENTIAL IN FINITE DEPTH

$$\phi = A\,\mathrm{Re}\left\{ \frac{igA}{\omega} \frac{\cosh k(z+H)}{\cosh kH} \times e^{-ikx\cos\beta - iky\sin\beta + i\omega t} \right\}$$

$$\omega^2 = gk \tanh kH.$$

- WAVES REFLECTING OFF A VERTICAL WALL

THIS IS ONE OF THE FEW VERY IMPORTANT ANALYTICAL SOLUTIONS OF REGULAR WAVES INTERACTING WITH SOLID BOUNDARIES SEEN IN PRACTICE. THE OTHER IS WAVEMAKER THEORY.



LET THE INCIDENT WAVE ELEVATION BE:

$$\zeta_I = \mathrm{Re}\left\{ A_I e^{-ikx + i\omega t} \right\}$$

AND THE REFLECTED BE:

$$\zeta_R = \mathrm{Re}\left\{ A_R e^{+ikx + i\omega t} \right\}$$

WHERE THE SIGN OF THE e^{ikx} TERM HAS BEEN REVERSED TO DENOTE A WAVE PROPAGATING TO THE LEFT. A_R CAN BE COMPLEX IN ORDER TO ALLOW FOR PHASE DIFFERENCES. —

THE CORRESPONDING VELOCITY POTENTIALS ARE:

$$\Phi_I = \mathrm{Re}\left\{ \frac{igA_I}{\omega} \frac{\cosh K(z+H)}{\cosh kH} e^{-ikx+i\omega t} \right\}$$

$$\Phi_R = \mathrm{Re}\left\{ \frac{igA_R}{\omega} \frac{\cosh k(z+H)}{\cosh kH} e^{ikx+i\omega t} \right\}$$

ON $x=0$:

$$\frac{\partial}{\partial x}(\phi_I + \phi_R) = 0$$

OR

$$-ikA_I + ikA_R \Rightarrow A_R = A_I \equiv A$$

THE RESULTING TOTAL VELOCITY POTENTIAL IS:

$$\Phi = \Phi_I + \Phi_R = \mathrm{Re}\left\{ \frac{igA}{\omega} \frac{\cosh K(z+H)}{\cosh kH} e^{i\omega t} \underbrace{\left(e^{ikx} + e^{-ikx}\right)}_{2\cos kx} \right\}$$

$$= 2A\,\mathrm{Re}\left\{ \frac{ig}{\omega} \frac{\cosh K(z+H)}{\cosh kH} \cos kx\, e^{i\omega t} \right\}$$

THE RESULTING STANDING-WAVE ELEVATION IS:

$$\zeta = \zeta_I + \zeta_R = 2A \cos kx \cos \omega t.$$

(VERIFY)

• EXERCISES

- APPLY THE MOMENTUM CONSERVATION PRINCIPLE TO DETERMINE THE MEAN HORIZONTAL FORCE ON THE BREAKWATER. VERIFY YOUR RESULT WITH THE ANSWER FROM DIRECT PRESSURE INTEGRATION

- DERIVE AN EXPRESSION FOR THE ENERGY DENSITY OF THE STANDING WAVE. WHAT IS THE ENERGY FLUX ?

- DERIVE EXPRESSIONS FOR THE ENERGY DENSITY, ENERGY FLUX AND MOMENTUM FLUX OF BI-CHROMATIC WAVES PROPAGATING IN THE SAME DIRECTION

$$\zeta = A_1 \cos(\omega_1 t - k_1 x) + A_2 \cos(\omega_2 t - k_2 x)$$

- REPEAT THE EXERCISE FOR WAVES PROPAGATING IN THE OPPOSITE DIRECTIONS

$$\zeta = A_1 \cos(\omega_1 t - k_1 x) + A_2 \cos(\omega_2 t + k_2 x)$$

ASSUME DEEP WATER

- DERIVE THE "STANDING" WAVE OFF A BREAKWATER FOR A WAVE INCIDENT AT AN ANGLE β.