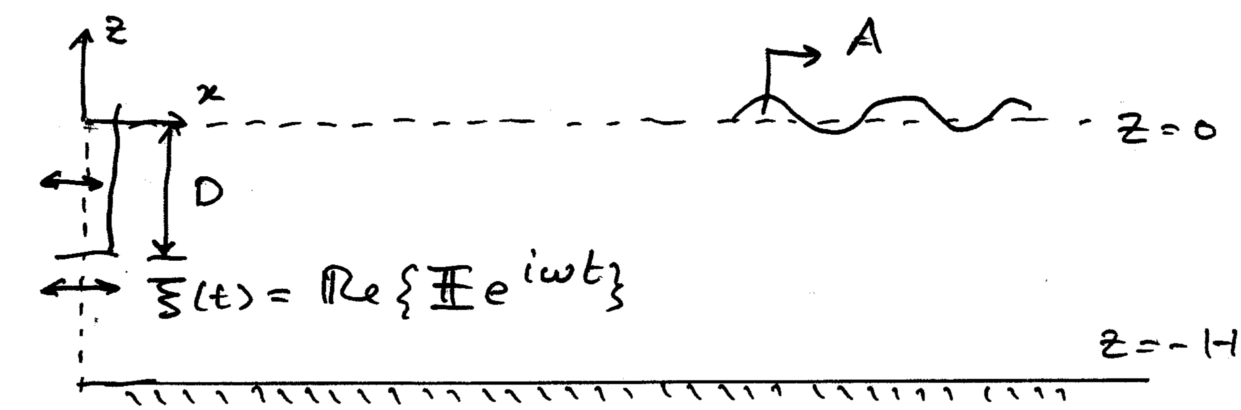
WAVEMAKER THEORY

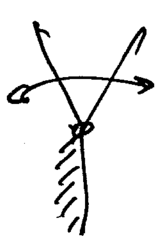
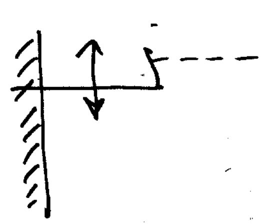


A PADDLE WITH DRAFT D IS UNDERGOING SMALL AMPLITUDE HORIZONTAL OSCILLATIONS WITH DISPLACEMENT

$$\xi(t) = \mathrm{Re}\{\Xi \, e^{i\omega t}\}$$

WHERE Ξ IS ASSUMED KNOWN AND REAL. THIS EXCITATION CREATES PLANE PROGRESSIVE WAVES WITH AMPLITUDE A DOWN THE TANK. THE PRINCIPAL OBJECTIVE OF WAVEMAKER THEORY IS TO DETERMINE A AS A FUNCTION OF ω, Ξ AND H.

OTHER TYPES OF WAVEMAKER MODES, LIKE

 OR  OR  MAY BE TREATED SIMILARLY.

IN GENERAL, THE WAVEMAKER DISPLACEMENT AT $x=0$ MAY BE WRITTEN IN THE FORM

$$\xi(t) = \mathrm{Re}\{\Xi(z)\, e^{i\omega t}\}$$

WHERE $\Xi(z)$ IS A KNOWN FUNCTION OF z. —

LET THE TOTAL VELOCITY POTENTIAL BE:

$$\phi = \mathrm{Re}\{\varphi\, e^{i\omega t}\}$$

WHERE

$$\varphi = \varphi_w + \psi$$

THE FIRST TERM IS A VELOCITY POTENTIAL THAT REPRESENTS A PLANE PROGRESSIVE REGULAR WAVE DOWN THE TANK WITH AMPLITUDE A, YET UNKNOWN. THUS:

$$\varphi_w = \frac{igA}{\omega}\, \frac{\cosh k(z+H)}{\cosh kH}\, e^{-ikx+i\omega t}$$

WITH:

$$\omega^2 = gk \tanh kH . —$$

THE SECOND COMPONENT POTENTIAL ψ IS BY DEFINITION A DECAYING DISTURBANCE AS $x \to \infty$ AND OTHERWISE SATISFIES THE FOLLOWING BOUNDARY VALUE PROBLEM:

$$\begin{cases} \nabla^2 \psi = \psi_{xx} + \psi_{zz} = 0, & -H < z < 0 \\ \psi_z - \dfrac{\omega^2}{g}\psi = 0, & z = 0 \\ \psi_z = 0, & z = -H \\ \psi \to 0, & x \to \infty \end{cases}$$

THE CONDITION ON THE WAVEMAKER $(x=0)$ IS YET TO BE ENFORCED.

NOTE THAT UNLIKE φ_W, ψ IS NOT REPRESENTING A PROPAGATING WAVE DOWN THE TANK SO IT IS CALLED A NON-WAVELIKE MODE. SUCH MODES DO EXIST AS WILL BE SHOWN BELOW. ON THE WAVEMAKER $(x=0)$ THE HORIZONTAL VELOCITY DUE TO φ_W AND THAT DUE TO ψ MUST SUM TO THE FORCING VELOCITY DUE TO $\xi(t)$. —

NOTING THAT $\varphi_w \sim e^{-ikx} \cosh k(z+H)$

WE WILL TRY $\psi \sim e^{-\lambda x} \cos \lambda(z+H)$

ITS CONJUGATE WHICH SATISFIES THE CONDITITION OF VANISHING VALUE AS $x \to \infty$ FOR $\lambda > 0$. —

<u>LAPLACE</u>: $\psi_{xx} + \psi_{zz} = 0$, VERIFY FOR ALL λ

<u>FS CONDITION</u>: $\psi_z - \frac{\omega^2}{g}\psi = 0 \Rightarrow$

$$-\lambda \sin \lambda H - \frac{\omega^2}{g} \cos \lambda H = 0$$

$$\Rightarrow \lambda \tan \lambda H = -\nu \equiv \omega^2/g$$

<u>SEAFLOOR CONDITION</u> $\Big\}$ $\psi_z = 0$, $z = -H$, (VERIFY)

SO FOR THE NON-WAVELIKE MODES ψ, λ MUST SATISFY THE "DISPERSION" RELATION

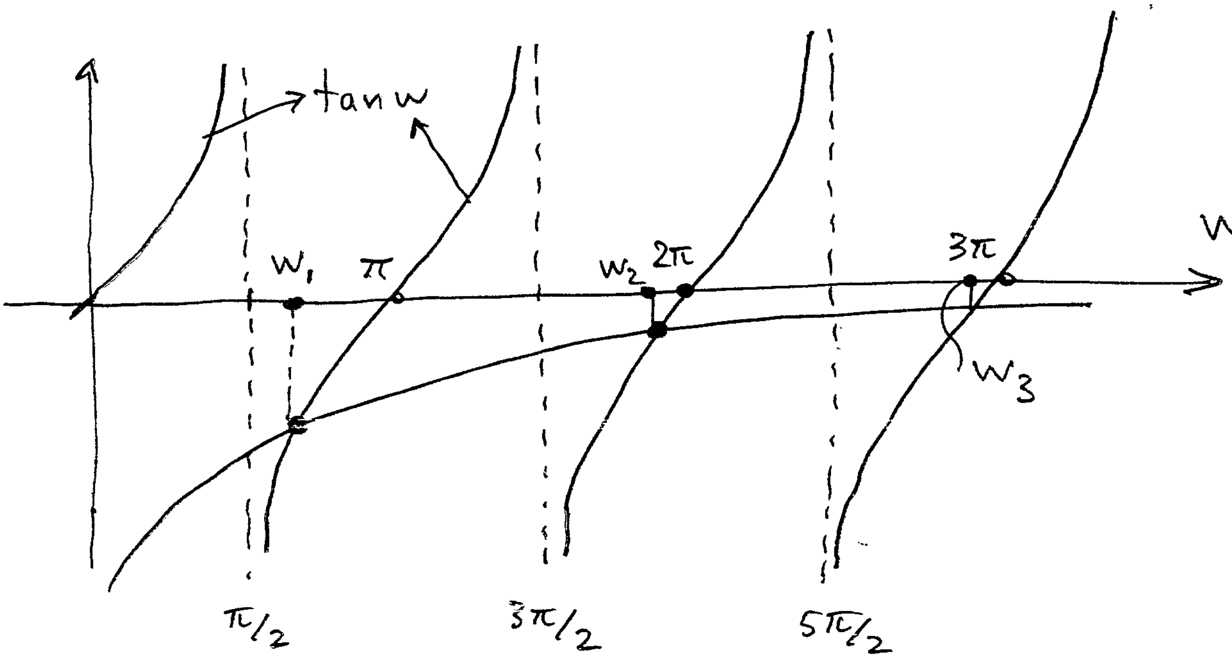
$$\lambda \tan \lambda H = -\nu = -\omega^2/g < 0$$

FOR POSITIVE VALUES OF λ SO THAT

$e^{-\lambda x} \to 0$, $x \to +\infty$. —

VALUES OF λ_i SATISFYING THE DISPERSION RELATION FOLLOW FROM THE SOLUTION OF THE NON-DIMENSIONAL NONLINEAR EQUATION

$$\tan w = -\nu/w\,, \quad w = \lambda H$$



SOLUTIONS w_i, $i = 1, 2, \ldots$ EXIST AS SHOWN ABOVE WITH $w_i \sim i\pi$ FOR LARGE i. THESE VALUES ARE KNOWN AS THE EIGENVALUES OR EIGEN-WAVENUMBERS OF THE NON-WAVELIKE MODES. THE EIGEN-WAVENUMBER OF THE WAVELIKE SOLUTION k IS GIVEN BY THE DISPERSION RELATION:

$$\frac{\omega^2 H}{g} = kH \tan kH \,.$$

VERIFY THAT BY SETTING $K = i\lambda$; THE DISPERSION RELATION OF THE NON-WAVELIKE MODES FOLLOWS. IN SUMMARY THE PURELY IMAGINARY ROOTS OF THE SURFACE WAVE DISPERSION RELATION AND ITS SINGLE REAL POSITIVE ROOT ENTER THE SOLUTION OF THE WAVEMAKER PROBLEM.

DEFINE THE FOLLOWING ORTHOGONAL EIGENMODES IN THE VERTICAL DIRECTION z:

$$f_0(z) = \frac{\sqrt{2}\cosh K(z+H)}{\left(H + \frac{1}{\nu}\sinh^2 KH\right)^{1/2}}$$

$$f_n(z) = \frac{\sqrt{2}\cos \lambda_n (z+H)}{\left(H - \frac{1}{\nu}\sin^2 \lambda_n H\right)}, \qquad n = 1, 2, \ldots$$

SELECTED TO SATISFY:

$$\begin{cases} \int_{-H}^{0} f_0^2(z)\,dz = \int_{-H}^{0} f_n^2(z)\,dz = 1 \\ \\ \int_{-H}^{0} f_m(z)\,f_n(z)\,dz = 0, \qquad m \neq n \end{cases}$$

SO THE WAVEMAKER VELOCITY POTENTIALS φ_W AND ψ CAN BE EXPRESSED SIMPLY IN TERMS OF THEIR RESPECTIVE EIGENMODES:

$$\varphi_W = a_0 f_0(z) e^{-ikx}$$

$$\psi = \sum_{n=1}^{\infty} a_n f_n(z) e^{-\lambda_n x}$$

AND:

$$\phi = \mathrm{Re}\left\{ (\varphi_W + \psi) e^{i\omega t} \right\}$$

ON $X=0$:

$$\phi_x = \mathrm{Re}\left\{ (\varphi_W + \psi_x)_x e^{i\omega t} \right\}$$

$$= \frac{d\xi}{dt} = \mathrm{Re}\left\{ \Xi(z) i\omega e^{i\omega t} \right\}$$

OR

$$\frac{\partial}{\partial x}(\varphi_W + \psi)_{x=0} = \underset{\uparrow \text{ KNOWN}}{\Xi(z)}\, i\omega$$

$$\left.\frac{\partial \varphi_W}{\partial x}\right|_{X=0} = a_0(-ik) f_0(z)$$

$$\left.\frac{\partial \psi}{\partial x}\right|_{X=0} = \sum_{n=1}^{\infty} a_n(-\lambda_n) f_n(z)$$

IT FOLLOWS THAT:

$$-ik a_0 f_0(z) + \sum_{n=1}^{\infty} a_n (-\lambda_n) f_n(z) = i\omega \Phi(z)$$

- ONE OF THE PRIMARY OBJECTIVES OF WAVEMAKER THEORY IS TO DETERMINE a_0 (OR THE FAR-FIELD WAVE AMPLITUDE A) IN TERMS OF $\Phi(z)$. MULTIPLYING BOTH SIDES BY $f_0(z)$, INTEGRATING FROM $-H \to 0$ AND USING ORTHOGONALITY WE OBTAIN:

$$-ik a_0 = i\omega \int_{-H}^{0} dz\, f_0(z) \Phi(z)$$

$$\Rightarrow \quad a_0 = -\frac{\omega}{k} \int_{-H}^{0} dz\, f_0(z)\, \Phi(z)$$

THE FAR-FIELD WAVE COMPONENT REPRESENTING PROPAGATING WAVES IS GIVEN BY:

$$\varphi_W = a_0 \frac{\sqrt{2} \cosh k(z+H)}{(H + \frac{1}{\nu} \sinh^2 kH)^{1/2}} e^{-ikx}$$

$$\equiv \frac{igA}{\omega} \frac{\cosh k(z+H)}{\cosh kH} e^{-ikx}$$

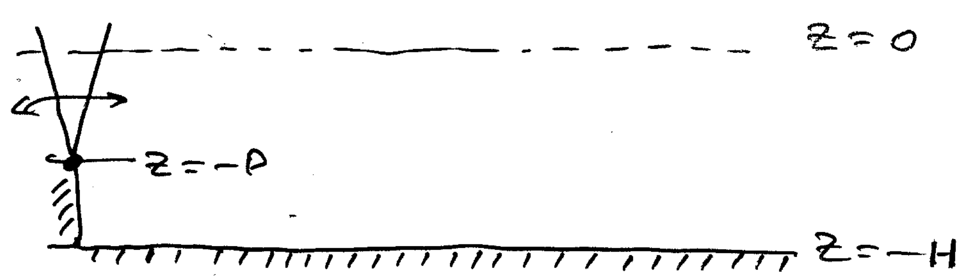
PLUGGING IN a_0 AND SOLVING FOR A WE OBTAIN THE COMPLEX AMPLITUDE OF THE PROPAGATING WAVE AT INFINITY, NAMELY MODULUS AND PHASE, IN TERMS OF THE WAVE MAKER DISPLACEMENT $\Xi(z)$ AND THE OTHER FLOW PARAMETERS.

EXERCISES

- TRY $\Xi(z) = A \begin{cases} 1, & -D < z < 0 \\ 0, & -H < z < -D \end{cases}$

  PADDLE-TYPE WAVEMAKER. DETERMINE THE AMPLITUDE A_W AND PHASE OF THE FAR-FIELD WAVE-TRAIN.

- REPEAT ABOVE EXERCISE FOR A HINGE TYPE WAVEMAKER:



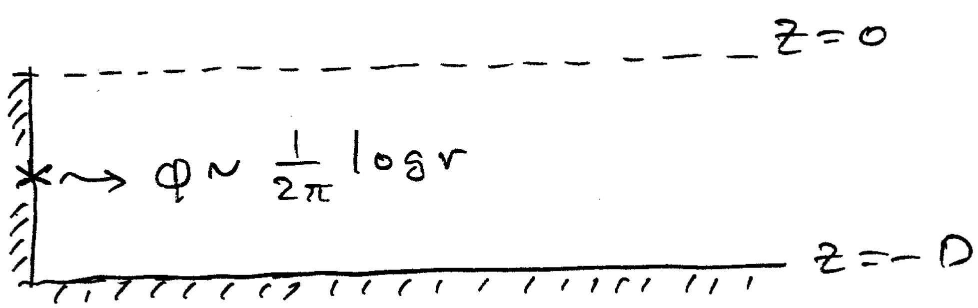
- FOR WHAT TYPE OF $\Xi(z)$ ARE THE NON-WAVELIKE MODES $\psi \equiv 0$?

IT IS EASY TO VERIFY BY VIRTUE OF ORTHOGONALITY THAT:

$$\Xi(z) \propto f_0(z)$$

UNFORTUNATELY THIS IS NOT A "PRACTICAL" DISPLACEMENT SINCE $f_0(z,k)$ DEPENDS ON K; THUS ON ω. SO ONE WOULD NEED TO BUILD A FLEXIBLE PADDLE!

- WHAT IS THE WAVE AMPLITUDE AT INFINITY GENERATED BY A POINT SOURCE LOCATED AT $z = -D$?



MORE DETAILS ON THE ABOVE THEORY AND EXTENSIONS TO THE NONLINEAR CASE MAYBE FOUND IN W&L AND MEI.—