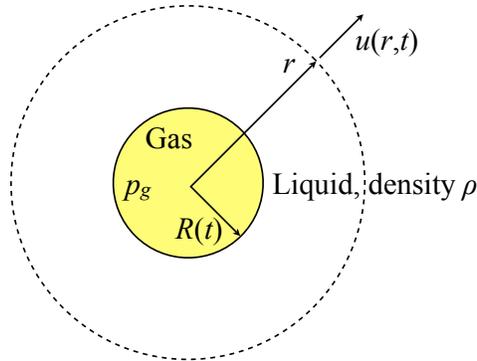


MIT Department of Mechanical Engineering
2.25 Advanced Fluid Mechanics

Problem 4.27

This problem is from “Advanced Fluid Mechanics Problems” by A.H. Shapiro and A.A. Sonin



Consider a bubble of high-pressure gas exploding in an incompressible liquid in a spherically-symmetrical fashion. The gas is not soluble in the liquid, and the liquid does not evaporate into the gas. At any instant R is the radius of the bubble, dR/dt is the velocity of the interface, p_g is the gas pressure (assumed uniform in the bubble), u is the liquid velocity at the radius r , and p_∞ is the liquid pressure at a great distance from the bubble. Gravity is to be neglected. The following questions pertain to the formulation of an analysis which will lead to the details of the pressure and velocity distributions and to the rate of bubble growth in the limit of inviscid liquid flow.

(a) Show that at any instant

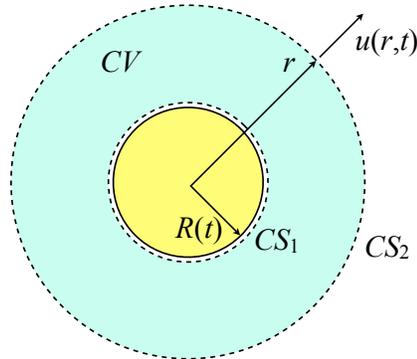
$$u = \frac{R^2}{r^2} \frac{dR}{dt} \quad (4.27a)$$

(b) Show that the rate of growth of the bubble is described by the equation

$$R \frac{d^2 R}{dt^2} + \frac{3}{2} \left(\frac{dR}{dt} \right)^2 + \frac{2\sigma}{\rho R} = \frac{p_g - p_\infty}{\rho} \quad (4.27b)$$

where σ is the surface tension at the gas-liquid interface.

(c) What additional information or assumptions would be necessary in order to establish the bubble radius R as a function of time? Explain how you would use this information.

Solution:

(a) If the control volume is chosen correctly, it is possible to determine u using either Form A or B of the integral mass conservation equation, however, here we will use Form A. Form A is written

$$\frac{d}{dt} \int_{CV} \rho dV + \int_{CS} \rho(\mathbf{u} - \mathbf{u}_{CS}) \cdot \hat{n} dA = 0 \quad (4.27c)$$

We choose a control volume taking the shape of a hollow sphere whose inner control surface, CS_1 , has radius $R(t)$ and whose outer control surface, CS_2 , has arbitrary radius r . Furthermore, the internal surface CS_1 is selected to move radially outward at exactly the rate of expansion of the bubble dR/dt . Let us first evaluate the volume integral term, by noting that the density is constant and the total volume of our CV is $V = \frac{4}{3}\pi(r^3 - R(t)^3)$. Accordingly,

$$\frac{d}{dt} \int_{CV} \rho dV = \frac{4}{3}\pi\rho \frac{d}{dt}(r^3 - R(t)^3) = -4\pi R^2 \rho \frac{dR}{dt} \quad (4.27d)$$

Next, we evaluate the surface flux integrals. At CV_1 we note that the liquid velocity is exactly equal to the velocity of the gas-liquid interface, dR/dt which is also the speed at which CV_1 is moving, hence there is no relative velocity between the liquid and CV_1 so $\mathbf{u} - \mathbf{u}_{CS} = 0$, and

$$\int_{CS_1} \rho(\mathbf{u} - \mathbf{u}_{CS}) \cdot \hat{n} dA = 0 \quad (4.27e)$$

Conversely, at CS_2 , r may be arbitrary, but it is fixed in time so that $\mathbf{u}_{CS_2} = 0$, and provided $R < r$, then the liquid velocity at this r -value is $u(r, t)$, so the surface flux integral at CS_2 is

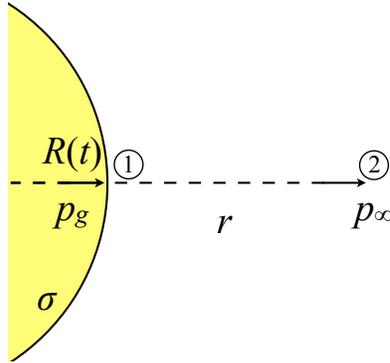
$$\int_{CS_2} \rho(\mathbf{u} - \mathbf{u}_{CS}) \cdot \hat{n} dA = \rho u(r, t) \hat{e}_r \cdot \hat{e}_r 4\pi r^2 = 4\pi r^2 \rho u(r, t) \quad (4.27f)$$

Substituting Eq. (4.27d), (4.27e) and (4.27f) into Eq. (4.27c), we find

$$-4\pi R^2 \rho \frac{dR}{dt} + 4\pi r^2 \rho u(r, t) = 0 \quad (4.27g)$$

and hence

$$\boxed{u(r, t) = \frac{R^2}{r^2} \frac{dR}{dt}} \quad (4.27h)$$



(b) Neglecting gravity, and provided the flow is inviscid and irrotational, we may apply the unsteady Bernoulli equation along a streamline with station 1 located just on the liquid side of the gas-liquid interface and station 2 located at a great distance from the bubble where $p = p_\infty$ and the liquid velocity is approximately zero, $u_2 \approx 0$. The unsteady Bernoulli equation along this streamline is

$$\rho \int_{s_1}^{s_2} \frac{du}{dt} ds + p_2 = p_1 + \frac{1}{2} \rho u_1^2 \quad (4.27i)$$

The pressure p_1 differs from the pressure in the bubble p_g by the Laplace pressure such that $p_1 = p_g - \frac{2\sigma}{R(t)}$, where σ is the surface tension of the gas-liquid interface. Substituting our result for u in Eq. (4.27h) into Eq. (4.27i) and setting $ds = dr$, $s_1 = R(t)$ and $s_2 = r_\infty$, we obtain

$$\rho \int_{R(t)}^{r_\infty} \frac{d}{dt} \left(\frac{R^2}{r^2} \frac{dR}{dt} \right) dr + p_\infty = p_g - \frac{2\sigma}{R} + \frac{1}{2} \rho \left(\frac{dR}{dt} \right)^2 \quad (4.27j)$$

Expanding the time derivative in the integrand, we find

$$\rho \int_{R(t)}^{r_\infty} \left(\frac{2R}{r^2} \left(\frac{dR}{dt} \right)^2 + \frac{R^2}{r^2} \frac{d^2 R}{dt^2} \right) dr + p_\infty = p_g - \frac{2\sigma}{R} + \frac{1}{2} \rho \left(\frac{dR}{dt} \right)^2 \quad (4.27k)$$

Completing the integral, we obtain

$$\rho \left(-\frac{2R}{r} \left(\frac{dR}{dt} \right)^2 - \frac{R^2}{r} \frac{d^2 R}{dt^2} \right) \Big|_{r=R(t)}^{r=r_\infty} + p_\infty = p_g - \frac{2\sigma}{R} + \frac{1}{2} \rho \left(\frac{dR}{dt} \right)^2 \quad (4.27l)$$

which reduces to

$$\rho \left(2 \left(\frac{dR}{dt} \right)^2 + R \frac{d^2 R}{dt^2} \right) + p_\infty = p_g - \frac{2\sigma}{R} + \frac{1}{2} \rho \left(\frac{dR}{dt} \right)^2 \quad (4.27m)$$

Finally, we rearrange this result to obtain the governing equation for R in its desired form

$$\boxed{R \frac{d^2 R}{dt^2} + \frac{3}{2} \left(\frac{dR}{dt} \right)^2 + \frac{2\sigma}{\rho R} = \frac{p_g - p_\infty}{\rho}} \quad (4.27n)$$

(c) Since our governing equation, Eq. (4.27h) is second order in time, we require two initial conditions, such as the initial bubble radius R_0 and interfacial velocity $dR/dt|_{t=0}$. Furthermore, we require a relationship between bubble pressure p_g and bubble radius $R(t)$, which could be obtained from the ideal gas law, $p = \rho RT$ and some reasonable assumption about the nature of the bubble expansion (*e.g.* adiabatic or isothermal).

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