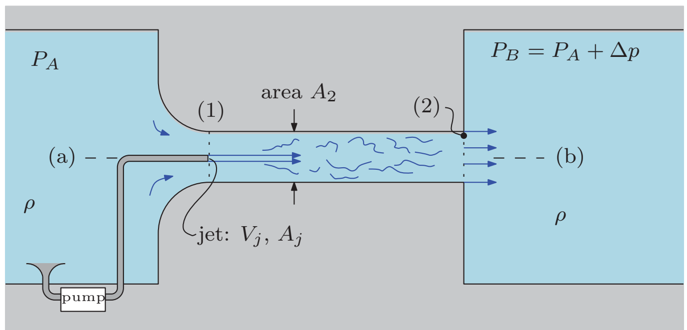
MIT Department of Mechanical Engineering

2.25 Advanced Fluid Mechanics

Problem 5.10

This problem is from "Advanced Fluid Mechanics Problems" by A.H. Shapiro and A.A. Sonin



The device connected between compartments A and B is a simplified version of a jet pump. A jet pump, or ejector, is a simplified device which uses a small, very high-speed jet with relatively low volume flow rate to move fluid at much larger volume flow rates against a pressure differential Δp (see the figure).

The pump in the figure consists of a contoured inlet section leading to a pipe segment of constant area A_2. A small, high velocity jet of speed V_j and area A_j injects fluid, drawn from compartment A, at the entrance plane (1) of the pipe segment. Between (1) and (2), the jet (the "primary" stream) and the secondary fluid flow which is drawn in from compartment A via the contoured inlet section mix in a viscous, turbulent fashion and eventually, at station (2), emerge as an essentially uniform-velocity stream.

We shall assume that the flows are incompressible, that the flow from compartment A to station (1) is inviscid, and that, although viscous forces dominate the mixing process between (1) and (2), the shear force exerted on the walls between those stations is small compared with $\Delta \cdot A_2$. The pump operates in steady state.

Neglect gravitational effects.

(a) Derive an expression for Δp as a function of the total volume flow rate Q from compartment A to compartment B. The given quantities are A_j, A_2, ρ, and V_j. You may assume $A_j \ll A_2$ to simplify your expression.

(b) Sketch the relationship Δp vs. Q (the "pump curve") for positive Δp and Q. Indicate the value of Q when $\Delta p = 0$ (the "short-circuit" volume flow rate). Show that for $A_j \ll A_2$, the latter is large compared with the volume flow rate $V_j A_j$ of the jet.

(c) Sketch the pressure distribution along the line a–b for the case when $\Delta p = 0$ and for a case when $\Delta p > 0$.

(d) Is your formulation in (a) valid when $Q = 0$, *i.e.* when the total flow rate for A to B is zero? Explain. What is the minimum value for Q which your formulation is valid?

Solution:

(a) First we make a table of the relevant parameters

		$\hat{\mathbf{n}}$	$\mathbf{v}$	$\mathbf{v}\cdot\hat{\mathbf{n}}$	Area	Pressure
(1)	jet	$-\hat{\mathbf{e}}_x$	$v_j\hat{\mathbf{e}}_x$	$-v_j$	A_j	P_1
	inflow	$-\hat{\mathbf{e}}_x$	$v_1\hat{\mathbf{e}}_x$	$-v_1$	$(A_2 - A_j)$	P_1
(2)	outflow	$\hat{\mathbf{e}}_x$	$v_2\hat{\mathbf{e}}_x$	v_2	A_2	P_B

Conservation of linear momentum can be stated as

$$\cancelto{0}{\frac{d}{dt}\int_{CV}\rho\mathbf{v}\,dV} + \int_{CS}\rho\mathbf{v}(\mathbf{v}-\cancel{\mathbf{v}_c})\cdot\hat{\mathbf{n}}\,dA = \int_{CS}-p\hat{\mathbf{n}}\,dA + \cancelto{0}{\int_{CS}\underline{\tau}_{\text{visc}}\cdot\hat{\mathbf{n}}\,dA} + \cancelto{0}{\int_{CV}\rho\mathbf{g}\,dV} + \cancelto{0}{F_{\text{ext}}}$$

Substituting the values from the table into the above equation gives

$$\underbrace{\rho v_j\hat{\mathbf{e}}_x(-v_j)A_j}_{\text{jet}} + \underbrace{\rho v_1\hat{\mathbf{e}}_x(-v_1)(A_2-A_j)}_{\text{inflow}} + \underbrace{\rho(v_2\hat{\mathbf{e}}_x)v_2A_2}_{\text{outflow}}$$
$$= -P_1(-\hat{\mathbf{e}}_x)A_j - P_1(-\hat{\mathbf{e}}_x)(A_2-A_j) - P_B(\hat{\mathbf{e}}_x)(A_2)$$

To solve for P_1 in terms of P_A apply Bernoulli's from compartment A to section (1):

$$P_A + \cancel{\frac{1}{2}\rho v_A{}^2} = P_1 + \frac{1}{2}\rho v_1{}^2 \qquad \Rightarrow \qquad \boxed{P_1 = P_A - \frac{1}{2}\rho v_1{}^2}$$

$$\rho v_2{}^2A_2 - \rho v_j{}^2A_j - \rho v_1{}^2(A_2-A_j) = (P_A - P_B)A_2 - \frac{1}{2}\rho v_1{}^2A_2$$
$$\Rightarrow \boxed{\rho v_2{}^2A_2 - \rho v_j{}^2A_j - \rho v_1{}^2\left(\frac{A_2}{2} - A_j\right) = (P_A - P_B)A_2 = -\Delta pA_2} \tag{5.10a}$$

Now apply conservation of mass:

$$\int_{CV}\frac{\partial\rho}{\partial t}\,dV + \int_{CS}\rho(\mathbf{v}-\cancel{\mathbf{v}_c})\cdot\hat{\mathbf{n}}\,dA = 0$$
$$-v_jA_j - v_1(A_2-A_j) + v_2A_2 = 0$$
$$\boxed{v_1 = \frac{v_2A_2 - v_jA_j}{A_2 - A_j}}$$

Now substitute v_1 into Eq. (5.10a) and recognize that $v_2A_2 = Q$, $A_2 - A_j \approx A_2$

$$\Delta pA_2 = -\rho\frac{Q^2}{A_2} + \rho v_j{}^2A_j + \rho\left(\frac{Q - v_jA_j}{A_2 - A_j}\right)^2\left(\frac{A_2}{2} - A_j\right)$$
$$\Delta p = -\rho\frac{Q^2}{A_2{}^2} + \rho v_j{}^2\frac{A_j}{A_2} + \rho\frac{(Q - v_jA_j)^2}{2A_2{}^2}$$
$$\boxed{\Delta p = -\frac{1}{2}\rho\frac{Q^2}{A_2{}^2} + \rho v_j{}^2\frac{A_j}{A_2} - \rho\frac{Qv_jA_j}{A_2{}^2}} + \cancel{\rho v_j{}^2\frac{A_j^2}{2A_2{}^2}}$$

Note the last term is negligible since $A_j{}^2/A_2{}^2 \ll A_j/A_2$.

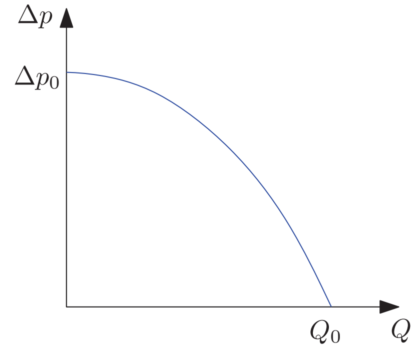
(b) If $\Delta p = 0$ the "short-circuit" volume flow rate Q_0 is given by (assume $Q_0 \gg v_j A_j$):

$$0 = -\rho \frac{{Q_0}^2}{{A_2}^2} + \rho {v_j}^2 \frac{A_j}{A_2} + \rho \frac{(Q_0 - \cancel{v_j A_j})^2}{2{A_2}^2}$$

$$\Rightarrow \boxed{Q_0 = \sqrt{2{v_j}^2 A_j A_2}}$$

If $Q = 0$ the pressure drop Δp_0 is given by:

$$\boxed{\Delta p_0 = \rho {v_j}^2 \frac{A_j}{A_2}}$$



□

Problem Solution by Tony Yu, Fall 2006

