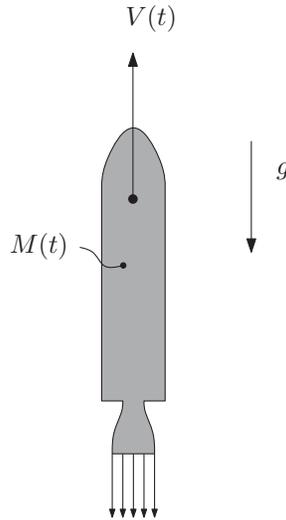


MIT Department of Mechanical Engineering
2.25 Advanced Fluid Mechanics

Problem 5.20

This problem is from “Advanced Fluid Mechanics Problems” by A.H. Shapiro and A.A. Sonin



After its second booster has been fired, a space vehicle finds itself outside the earth's atmosphere, moving vertically upward at a speed V_0 against gravity g . Its total mass at that point is M_0 . At $t = 0$, the vehicle's third stage is turned on and the rocket burns propellant at a mass rate \dot{m}_r kg/s, ejecting gas from the exit plane (area A_e) at speed V_e relative to the rocket.

Show that if the gravitational acceleration remains essentially constant at the vehicle during the rocket firing, the velocity $V(t)$ of the vehicle after time t will be given by

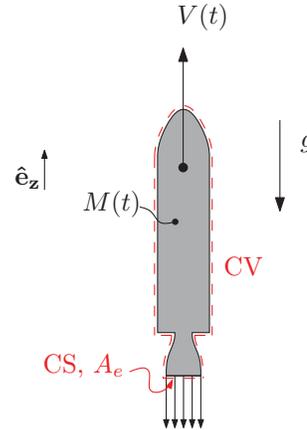
$$V(t) - V_0 = V_e \ln \frac{M_0}{M(t)} - \frac{g[M_0 - M(t)]}{\dot{m}_r}$$

where $M(t)$ is the mass of the system at time t . Assume that although the pressure of the gas at the rocket exit plane is P_e (the rocket exhaust is supersonic, and hence the pressure at the exit is not balanced with the zero pressure of space), the effect of the finite exit plane pressure on the thrust is negligible.

Solution:

Given:

- $V(t = 0) = V_0$
- $M(t = 0) = M_0$
- A_e (area over which gas exits)
- V_e (relative velocity of gas leaving booster)
- \dot{m}_r (rate of mass leaving)



Unknown: $V(t)$

Since the mass flow rate of gas is constant, the mass of the rocket can be expressed as:

$$M(t) = M_0 - \dot{m}_r t \tag{5.20a}$$

By Mass Conservation,

$$\begin{aligned} \frac{d}{dt} \int_{CV} \rho dV + \int_{CS} \rho(\mathbf{v} - \mathbf{v}_c) \cdot \hat{\mathbf{n}} dA &= 0 \\ \Rightarrow \frac{d}{dt} M + \rho V_e A_e &= 0 \end{aligned}$$

Combine the above equation with Eq. (5.20a) to give

$$\frac{dM}{dt} = -\rho V_e A_e = -\dot{m}_r \tag{5.20b}$$

Consider an accelerating frame of reference moving with the rocket:

Linear Momentum in z -direction is

$$\begin{aligned} \underbrace{\frac{d}{dt} \int_{CV} \rho v_z dV}_{\text{Since } v_z = 0 \text{ measured in moving reference frame}} + \int_{CS} \rho \underbrace{v_z}_{-V_e} \underbrace{(\mathbf{v} - \mathbf{v}_c) \cdot \hat{\mathbf{n}}}_{V_e} dA &= - \int_{CV} \rho g dV - \int_{CV} \rho \underbrace{a_{z,\text{ref}}}_{\dot{v}(t)} dV \\ \Rightarrow \underbrace{-\rho V_e^2 A_e}_{-\dot{m}_r V_e} &= -M(t)g - M(t)\dot{v}(t) \\ \Rightarrow \frac{dv}{dt} &= \frac{\dot{m}_r}{M} V_e - g \\ \Rightarrow \int_{V_0}^{V(t)} dV &= \int_0^t \left(\frac{\dot{m}_r}{M} V_e - g \right) dt \end{aligned} \tag{5.20c}$$

where $t = \frac{M_0 - M(t)}{\dot{m}_r}$ from Eq. (5.20a) $\Rightarrow dt = -dM/\dot{m}_r$. Thus the RHS of Eq. (5.20c) becomes

$$\begin{aligned} - \int_{M_0}^{M(t)} \frac{\dot{m}_r V_e}{M} \frac{dM}{\dot{m}_r} - g \int_0^t dt &= V_e \ln \frac{M_0}{M(t)} - gt \\ &= V_e \ln \frac{M_0}{M(t)} - g \frac{M_0 - M(t)}{\dot{m}_r} \end{aligned}$$

$$\Rightarrow V(t) - V_0 = V_e \ln \frac{M_0}{M(t)} - g \frac{M_0 - M(t)}{\dot{m}_r}$$

□

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