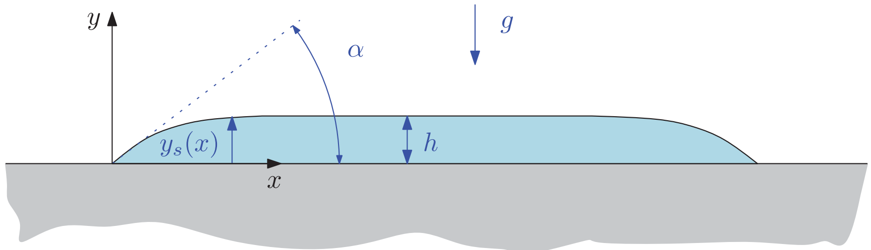
MIT Department of Mechanical Engineering

2.25 Advanced Fluid Mechanics

Problem 2.02

This problem is from "Advanced Fluid Mechanics Problems" by A.H. Shapiro and A.A. Sonin



A liquid of density ρ and surface tension σ has been spilled on a horizontal plate so that it forms a very large puddle whose depth (in the central parts) is h. Consider the region near the edge of the puddle, which can be viewed to good approximation as two-dimensional. If the contact angle is α, derive an expression for the shape of the liquid surface $y_s(x)$.

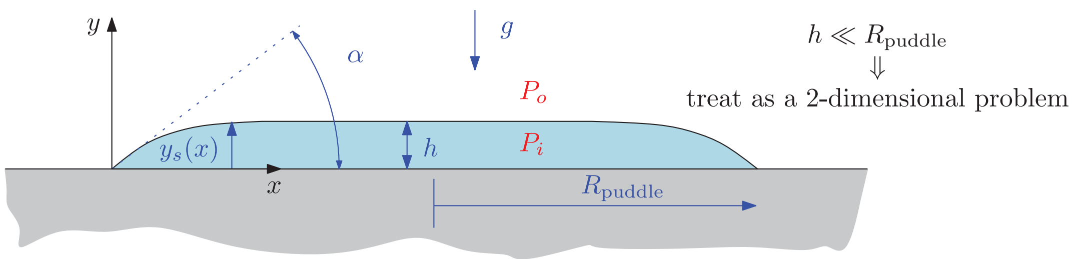
Assume for simplicity that α is small, so that the radius of curvature of the surface is large compared with h and can be approximated by

$$R = \frac{1}{\left|\frac{d^2 y_s}{dx^2}\right|}$$

ans:

$$y_s = h\left[1 - \exp\left(-\sqrt{\rho g/\sigma}\, x\right)\right]$$
$$h = \tan\alpha\sqrt{\sigma/\rho g}$$

Solution:



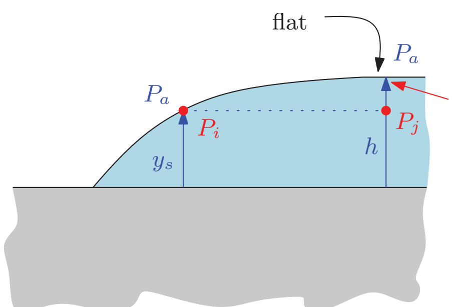
Given: σ, α, ρ

Radius of curvature, $R_1 \approx \dfrac{1}{\frac{d^2y_s}{dx^2}}$ since α is small.

Unknown: y_s, h

Find $P_o - P_i$:

$$P_o - P_i = -\sigma\left(\frac{1}{R_1} + \cancel{\frac{1}{R_2}}\right) \quad \text{since } R_2 = R_{\text{puddle}}\text{, which is assumed to be very large}$$



$P_i = Pa$ at $y_s = h$ since the surface is FLAT (no curvature) $\Rightarrow$ no surface tension!

$$\boxed{P_j = P_a + \rho g(h - y)}$$

For this side of the puddle

$$\frac{d^2y_s}{dx^2} < 0 \quad \Rightarrow \quad R_1 = -\frac{1}{\frac{d^2y_s}{dx^2}} \quad \Rightarrow \quad P_o - P_i = \sigma\frac{d^2y_s}{dx^2}$$

For $y = y_s(x)$: $P_i = P_j = P_o + \rho g(h - y_s) \qquad \Rightarrow \quad -\rho g(h - y_s) = \sigma\frac{d^2y}{dx^2}$

$$\boxed{\frac{d^2y_s}{dx^2} - \frac{\rho g}{\sigma}y_s = -\frac{\rho g}{\sigma}h} \qquad \text{2nd order ODE w/ B.C.} \begin{cases} y_s = 0 & \text{at } x = 0 \\ y_s = h & \text{as } x \to \infty \end{cases} \tag{2.02a}$$

Solve Eq. (2.02a) by assuming the form of the solution to be: $y_s = Ae^{Bx} + C$

$$\Rightarrow \quad \frac{d^2y_s}{dx^2} = AB^2e^{Bx} \quad \rightarrow \quad AB^2e^{Bx} - \frac{\rho g}{\sigma}Ae^{Bx} - \frac{\rho g}{\sigma}C = -\frac{\rho g}{\sigma}h$$

$$\therefore C = h$$

$$\therefore B = \pm\sqrt{\frac{\rho g}{\sigma}} \rightarrow B = -\sqrt{\frac{\rho g}{\sigma}} \qquad (\text{so that } y_s = h \text{ at } x \to \infty)$$

$$\Rightarrow y_s = A\exp\left(-\sqrt{\frac{\rho g}{\sigma}}x\right) + h$$

Solve for A by invoking the first boundary condition:

$$y_s = 0 = A + h \qquad \Rightarrow \qquad A = -h$$

$$\Rightarrow \boxed{y_s = h\left[1 - \exp\left(-\sqrt{\frac{\rho g}{\sigma}}x\right)\right]} \tag{2.02b}$$

<u>Find h</u> by letting $\frac{dy_s}{dx} = \tan\alpha$ at $x = 0$:

$$\frac{dy_s}{dx} = h\sqrt{\frac{\rho g}{\sigma}}\exp\left(-\sqrt{\frac{\rho g}{\sigma}}x\right) \qquad \Rightarrow \qquad \left.\frac{dy_s}{dx}\right|_{x=0} = h\sqrt{\frac{\rho g}{\sigma}} = \tan\alpha$$

$$\Rightarrow \boxed{h = \sqrt{\frac{\sigma}{\rho g}}\tan\alpha}$$

Alternatively, starting from Young-Laplace, (assuming that α is small) such that curvature is equal to $\frac{1}{\frac{d^2y_s}{dx^2}}$ $\Rightarrow \Delta P = -\sigma\frac{d^2y_s}{dx^2}$, $\Rightarrow$

$$P_i(x, y_s(x)) - P_o = -\sigma\frac{d^2y_s}{dx^2}, \tag{2.02c}$$

besides, $P_i(x,y) = P_i(x,y_s) + \rho g(y_s - y)$, from hydrostatics. Then, combining the information from hydrostatics and Young Laplace,

$$\Rightarrow P_i(x,y) = -\sigma\frac{d^2y_s}{dx^2} + P_o + \rho g(y_s - y). \tag{2.02d}$$

Now, since it's hard to work with so many functions, let's use one more property from hydrostatics, $\frac{dP}{dx} = 0$, at any y. Then, differentiating all the expression,

$$0 = -\sigma\frac{d^3y_s}{dx^3} + \rho g\frac{dy_s}{dx}, \tag{2.02e}$$

and setting $L_c = \sqrt{\frac{\sigma}{\rho g}}$, $\Rightarrow$ for $0 < x < \infty$ (large puddle),

$$\boxed{0 = L_c^2\frac{d^3y_s}{dx^3} - \frac{dy_s}{dx},} \tag{2.02f}$$

which is a 3rd order differential equation with the boundary conditions

$$y_s(0) = 0, \tag{2.02g}$$

$$\frac{dy_s(0)}{dx} = \tan\alpha, \tag{2.02h}$$

$$\frac{dy_s}{dx} \to 0 \quad as \quad x \to \infty. \tag{2.02i}$$

Solving, we first introduce $G = \frac{dy_s}{dx}$, so

$$L_c\frac{d^2G}{dx^2} - G = 0, \qquad \Rightarrow \qquad G = C_1e^{-\frac{x}{L_c}} + \underbrace{C_2e^{\frac{x}{L_c}}}_{C_2=0,\ unbounded\ \ term} \tag{2.02j}$$

then,

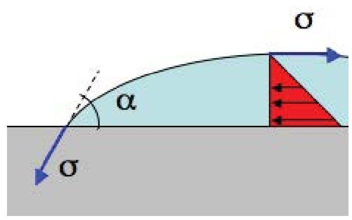
$$y_s = C_1'e^{-\frac{x}{L_c}} + C_3. \tag{2.02k}$$

From the boundary condition $y_s(0) = 0$, $\Rightarrow C_1' = C_3$, $\Rightarrow y_s = C_3(1 - e^{\frac{-x}{L_c}})$.

From the boundary condition $\frac{dy_s(0)}{dx} = \tan\alpha$, $\Rightarrow \frac{C_3}{L_c} = \tan\alpha$, $\Rightarrow y_s = L_c\tan\alpha(1 - e^{\frac{-x}{L_c}})$.
Finally,

$$\boxed{y_s(x) = L_c\tan\alpha(1 - e^{\frac{-x}{L_c}}),} \tag{2.02l}$$

from which we notice that L_c measures the extent of the effect of interface tension on the surface profile.



Note that we can deduce h (for all x) by a force balance as done in class:

$$\frac{\rho g h^2}{2} + \sigma \cos\alpha = \sigma, \qquad \Rightarrow \qquad h^2 = \frac{2\sigma(1-\cos\alpha)}{\rho g} \tag{2.02m}$$

then, as $\alpha \to 0$

$$\frac{2\sigma(1-\cos\alpha)}{\rho g} \approx 2\sigma(1-(1-\alpha^2+...))\frac{1}{\rho g} = \frac{\sigma\alpha^2}{\rho g}. \tag{2.02n}$$

□

Problem Solution by Sungyon Lee, MC (Updated), Fall 2008

