

MASSACHUSETTS INSTITUTE OF TECHNOLOGY
DEPARTMENT OF MECHANICAL ENGINEERING
CAMBRIDGE, MASSACHUSETTS 02139
2.29 NUMERICAL FLUID MECHANICS — SPRING 2015

Problem Set 2

Issued: Monday, February 23, 2015

Due: Wednesday, March 4, 2015

Grading Note: Please provide your solutions either as hand-written/hard-copy solutions or by submitting via course website. MATLAB[®] codes should be submitted via course website. The bulk of the grades will be given to detailed explanations and to algorithms and numerical schemes that capture the essence of the numerical problems. We know that successful coding of numerical schemes can be time consuming and prone to small errors. Such small errors or omissions in a code will not be heavily penalized.

The goals of this Problem Set are to: (i) review vector calculus and conservations laws; (ii) learn and utilize methods for solving linear algebraic systems; and (iii) apply these methods to simple differential equations in fluid mechanics and heat transfer.

Problem 1 (Modified from Chapra and Canale, Problems 9.10 and 9.11)

For each of the following two systems of equations

$$\begin{array}{rcl} -3x_2 + 7x_3 = 2 & & -2x_1 - 6x_2 - x_3 = -38 \\ x_1 + 2x_2 - x_3 = 3 & & -3x_1 - x_2 + 7x_3 = -34 \\ 5x_1 - 2x_2 = 2 & & -8x_1 + x_2 - 2x_3 = -20 \end{array}$$

- Compute the determinants (use the function $det()$ in MATLAB).
- Use Cramer's rule to solve for the x 's (using a code, e.g. in MATLAB).
- Use Gauss elimination with (i) partial pivoting, then (ii) full pivoting, to solve for the x 's. Show the steps of the recursive computations. To do so, write a small MATLAB script (i.e. similar to the script given in class for 2-by-2 matrices, but extended to 3-by-3).
- Substitute your results back into the original equations to check your solution.

Problem 2

Do problems 10.12 and 10.15 of Chapra and Canale.

Problem 3 (Reference: *Matrix Computations* by Gene Golub and Charles Van Loan)

Suppose we partition an n -by- n matrix A as follows

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

where A_{11} is r -by- r , A_{22} is $(n-r)$ -by- $(n-r)$ and A_{21} and A_{12} of compatible dimensions. Assume that A_{11} is nonsingular and has a LU factorization. Suppose that A_{21} is eliminated row by row by applying r steps of Gauss elimination, the result of this process being:

$$A^* = \begin{bmatrix} A_{11}^* & A_{12}^* \\ 0 & A_{22}^* \end{bmatrix}$$

Show that the bottom-right $(n-r)$ -by- $(n-r)$ block of the result, A_{22}^* , is the matrix S defined by $S = A_{22} - A_{21}A_{11}^{-1}A_{12}$, which is called the *Schur complement* of A_{11} in A .

Problem 4 (Reference: *Numerical Linear Algebra* by Lloyd N. Trefethen and David Bau, III, and Chapra and Canale)

Gauss elimination can be used to compute the inverse A^{-1} of a nonsingular matrix $A \in \mathbb{R}^{m \times m}$, though it is not often necessary to do so.

- Describe a 3-steps algorithm (pseudo-code) to compute A^{-1} by solving a system of m equations using LU decomposition. Show that its asymptotic operation count is $8m^3/3$ flops.
- Taking advantage of the fact that L and U have zeros above and below the diagonal, respectively, and that I is diagonal (L , U and I are sparse matrices), describe a variant of your algorithm that would take advantage of sparsity in some of the steps of the algorithm and so reduce the asymptotic operation count to $2m^3$ flops.
- Suppose one wishes to solve n systems of equations $Ax_j = b_j$. As seen in lecture, a block system $AX = B$ with $B \in \mathbb{R}^{m \times n}$ is equivalent and can be efficiently solved. What is the asymptotic operation count (a function of m and n) for doing this (i) directly from the LU factorization applied to $AX = B$ and (ii) with a preliminary computation of A^{-1} ?
- The following system of equations is designed to determine concentrations (the c 's in g/m^3) in a series of coupled reactors as a function of the amount of mass input to each reactor (the right hand side in g/day),

$$\begin{aligned} 15c_1 - 3c_2 - c_3 &= 3800 \\ -3c_1 + 18c_2 - 6c_3 &= 1200 \\ -4c_1 - c_2 + 12c_3 &= 2350 \end{aligned}$$

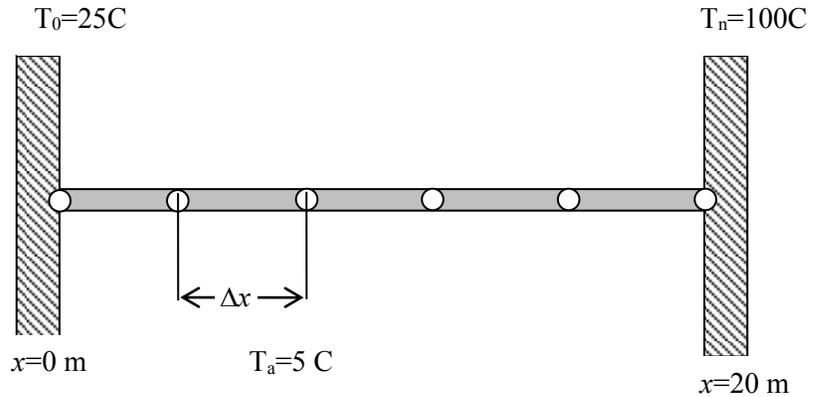
- Solve this system of equations by LU decomposition without pivoting.
- Determine the matrix inverse of this system and use it to determine the solution.
- How much will the concentration in reactor 3 be reduced if the rate of mass input to reactors 1 and 2 is reduced by 500 and 250 g/day , respectively?
- If you had to compute the solution of part iii. by only using either the LU decomposition or the inverse of A (computed in part i. and part ii. above), will one of these approaches lead to a faster solution?

Problem 5: (Modified from Chapra and Canale, Problem 12.38)

Linear algebraic equations often arise in the numerical solution of differential equations. Consider for example the following differential equation which derives from the energy balance along a thin fin (see figure),

$$\frac{d^2T}{dx^2} + h(T_a - T) = 0$$

where: T is temperature ($^{\circ}\text{C}$), x is distance along the rod (m), h is a heat transfer coefficient between the fin and the ambient air (m^{-2}), and T_a is the temperature of the surrounding air ($^{\circ}\text{C}$). This is the ‘fin equation’ for convective cooling of an extended surface.



a) Develop an analytical solution for this equation. Solve for coefficients symbolically (you do not need to simplify your coefficients).

b) Substitute the centered finite difference $\frac{\partial^2 T}{\partial x^2} = \frac{T(x + \Delta x) - 2T(x) + T(x - \Delta x)}{(\Delta x)^2}$ in the above

differential fin equation. Re-arrange terms to have all known values on the right hand side (RHS) and all unknowns on the LHS. What are the coefficients in front of each unknown? You can provide your answer in terms of a ‘computational cell,’ e.g. in a table similar to the following:

	$x - \Delta x$	x	$x + \Delta x$
Coefficient	a_1	a_2	a_3

This gives the coefficients in the ‘ \mathbf{A} ’ matrix for the interior nodes, where $\mathbf{AT}=\mathbf{b}$, with ‘ \mathbf{A} ’ as the coefficient matrix, ‘ \mathbf{T} ’ as the vector of temperatures, and ‘ \mathbf{b} ’ as the RHS vector.

c) Implement part b) in MATLAB using the linear system solver (*lin_sys_solver.m*) that we provide you (with appropriate tolerance). What is the name of the algorithm used in this *lin_sys_solver.m*?

d) Plot your solution for a 20-m rod with $T_a=5$, $T(x=0)=25$, $T(x=20)=100$, and $h=[0.4, 0.04, 0.02, 0.00001]$ using 6 nodes. For your plots, graph both the analytical solution (using a solid line) and the numerical solution [using markers, e.g. `plot(Solution,'*')`]. You can either plot all the solutions on a single graph, or on four separate graphs.

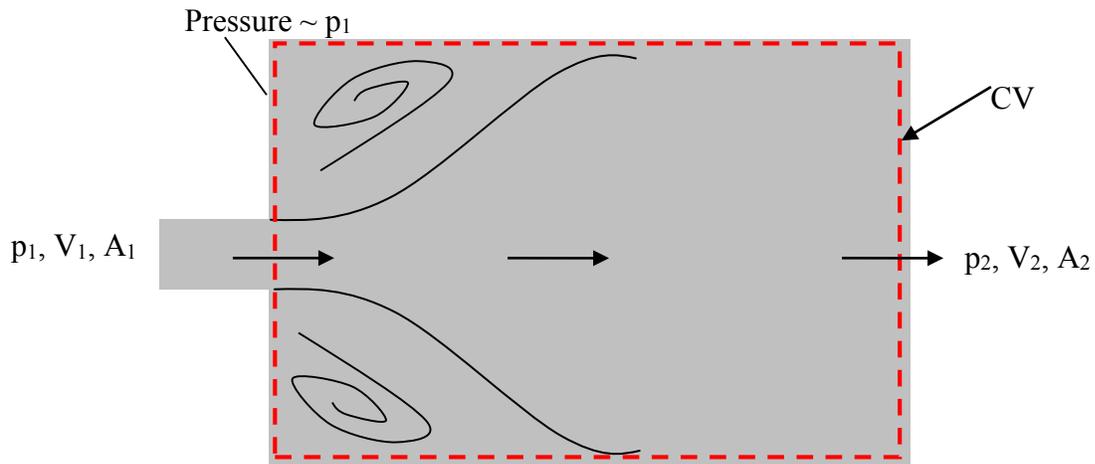
e) Briefly give a physical interpretation of the effect of the different values of h on the temperature profile? (A sentence or two will be fine, referring to your plots in part d).

f) Examine the ‘ \mathbf{A} ’ matrix you created and comment on its structure (MATLAB’s ‘spy’ command may be useful here). How does it affect computations?

g) Could you have used the Thomas algorithm for solving this problem? If yes, briefly discuss its cost in comparison to that of the algorithm used in *lin_sys_solver.m*.

Problem 8 (Modified from *Fluid Mechanics (4th)* by Frank White, Problem 3.59)

When a pipe flow suddenly expands from A_1 to A_2 , low speed, low-friction eddies appear in the corners and the flow gradually expands to A_2 downstream (see figure).



i) Using the suggested control volume for incompressible steady flow and assuming that $p \approx p_1$ on the corner annular ring as shown, show that the downstream pressure is given by $p_2 = p_1 + \rho V_1^2 \frac{A_1}{A_2} \left(1 - \frac{A_1}{A_2}\right)$.

Neglect wall friction.

ii) Compare your above result with the result obtained using the Bernoulli equation. Which one is more reasonable and why?

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