2.29 Numerical Fluid Mechanics
Spring 2015 – **Lecture 13**

REVIEW Lecture 12:

- **Grid-Refinement and Error estimation**
  - Estimation of the order of convergence and of the discretization error
  - Richardson's extrapolation and Iterative improvements using Roomberg's algorithm

- **Fourier Error Analysis**
  - Provide additional information to truncation error: indicates how well Fourier mode solution, i.e. wavenumber and phase speed, is represented
    - Effective wavenumber: $\left(\frac{\partial\, e^{ikx}}{\partial x}\right)_j = i\, k_{\text{eff}}\, e^{ikx_j}$ (for CDS, 2$^{\text{nd}}$ order, $k_{\text{eff}} = \frac{\sin(k\Delta x)}{\Delta x}$)
    - Effective wave speed (for linear convection eqn., $\frac{\partial f}{\partial t} + c\frac{\partial f}{\partial x} = 0$, integrating in time exactly):

$$\frac{d f_k^{num.}}{dt} = -f_k^{num.}(t)\, c\, i\, k_{\text{eff}} \Rightarrow f_{\text{numerical}}(x,t) = \sum_{k=-\infty}^{\infty} f_k(0)\, e^{ikx - ik_{\text{eff}}\, t} = \sum_{k=-\infty}^{\infty} f_k(0)\, e^{ik(x - c_{\text{eff}}\, t)} \quad \Rightarrow \frac{c_{eff}}{c} = \frac{\sigma_{eff}}{\sigma} = \frac{k_{eff}}{k}$$

$$(\text{with } \sigma_{\text{eff}} = -i\, k_{\text{eff}}\, c = -i\, k\, c_{\text{eff}})$$

2.29 Numerical Fluid Mechanics
Spring 2015 – **Lecture 13**

REVIEW Lecture 12, Cont'd:

- **Stability**
  - Heuristic Method: trial and error
  - Energy Method: Find a quantity, l_2 norm $\sum_j \left(\phi_j^n\right)^2$, and then aim to show that it remains bounded for all n.
    - Example: for $\frac{\partial \phi}{\partial t} + c\frac{\partial \phi}{\partial x} = 0$ we obtained $0 \leq \frac{c\,\Delta t}{\Delta x} \leq 1$
  - Von Neumann Method (Introduction), also called Fourier Analysis Method/Stability
- **Hyperbolic PDEs and Stability**
  - 2nd order wave equation and waves on a string
    - Characteristic finite-difference solution (review)
    - Stability of C – C (CDS in time/space, explicit): $C = \frac{c\,\Delta t}{\Delta x} < 1$
    - Example: Effective numerical wave numbers and dispersion

FINITE DIFFERENCES – Outline for Today

- Fourier Analysis and Error Analysis
  - Differentiation, definition and smoothness of solution for ≠ order of spatial operators
- Stability
  - Heuristic Method
  - Energy Method
  - Von Neumann Method (Introduction) : 1st order linear convection/wave eqn.
- Hyperbolic PDEs and Stability
  - Example: 2nd order wave equation and waves on a string
    - Effective numerical wave numbers and dispersion
  - CFL condition:
    - Definition
    - Examples: 1st order linear convection/wave eqn., 2nd order wave eqn., other FD schemes
  - Von Neumann examples: 1st order linear convection/wave eqn.
  - Tables of schemes for 1st order linear convection/wave eqn.
- Elliptic PDEs
  - FD schemes for 2D problems (Laplace, Poisson and Helmholtz eqns.)

References and Reading Assignments

- Lapidus and Pinder, 1982: Numerical solutions of PDEs in Science and Engineering. Section 4.5 on “Stability”.
- Chapter 3 on “Finite Difference Methods” of “J. H. Ferziger and M. Peric, Computational Methods for Fluid Dynamics. Springer, NY, 3rd edition, 2002”
- Chapter 3 on “Finite Difference Approximations” of “H. Lomax, T. H. Pulliam, D.W. Zingg, *Fundamentals of Computational Fluid Dynamics (Scientific Computation).* Springer, 2003”
- Chapter 29 and 30 on “Finite Difference: Elliptic and Parabolic equations” of “Chapra and Canale, Numerical Methods for Engineers, 2014/2010/2006.”

Von Neumann Stability

- Widely used procedure
- Assumes initial error can be represented as a Fourier Series and considers growth or decay of these errors
- In theoretical sense, applies only to periodic BC problems and to linear problems
  - Superposition of Fourier modes can then be used
- Again, use, $f(x,t) = \sum_{k=-\infty}^{\infty} f_k(t)\, e^{ikx}$ but for the error: $\varepsilon(x,t) = \sum_{\beta=-\infty}^{\infty} \varepsilon_\beta(t)\, e^{i\beta x}$
- Being interested in error growth/decay, consider only one mode:

$$\varepsilon_\beta(t)\, e^{i\beta x} \approx e^{\gamma t}\, e^{i\beta x} \quad \text{where } \gamma \text{ is in general complex and function of } \beta:\ \gamma = \gamma(\beta)$$

- Strict Stability: The error will not to grow in time if $\left|e^{\gamma t}\right| \le 1 \quad \forall \gamma$
  - in other words, for $t = n\Delta t$, the condition for strict stability can be written:

$$\left|e^{\gamma \Delta t}\right| \le 1 \quad \text{or for } \xi = e^{\gamma \Delta t}, \quad |\xi| \le 1$$

von Neumann condition

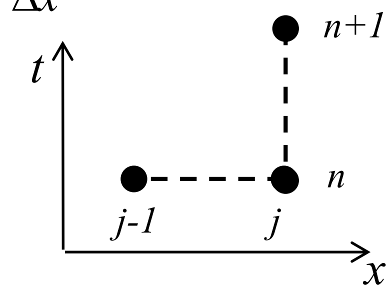
Norm of amplification factor ξ smaller or equal to1

Evaluation of the Stability of a FD Scheme
Von Neumann Example

- Consider again: $$\frac{\partial \phi}{\partial t} + c\frac{\partial \phi}{\partial x} = 0$$
- A possible FD formula ("upwind" scheme) $$\frac{\phi_j^{n+1} - \phi_j^n}{\Delta t} + c\frac{\phi_j^n - \phi_{j-1}^n}{\Delta x} = 0$$

  $(t = n\Delta t,\ x = j\Delta x)$ which can be rewritten:

$$\phi_j^{n+1} = (1-\mu)\,\phi_j^n + \mu\,\phi_{j-1}^n \quad \text{with} \quad \mu = \frac{c\,\Delta t}{\Delta x}$$



- Consider the Fourier error decomposition (one mode) and discretize it:

$$\varepsilon(x,t) = \varepsilon_\beta(t)\,e^{i\beta x} = e^{\gamma t}e^{i\beta x} \Rightarrow \underline{\varepsilon_j^n = e^{\gamma n\Delta t}e^{i\beta j\Delta x}}$$

- Insert it in the FD scheme, assuming the error mode satisfies the FD (strictly valid for linear eq. only):

$$\underline{\varepsilon_j^{n+1} = (1-\mu)\,\varepsilon_j^n + \mu\,\varepsilon_{j-1}^n} \quad \Rightarrow \quad e^{\gamma(n+1)\Delta t}e^{i\beta j\Delta x} = (1-\mu)\,e^{\gamma n\Delta t}e^{i\beta j\Delta x} + \mu\,e^{\gamma n\Delta t}e^{i\beta(j-1)\Delta x}$$

- Cancel the common term (which is $\varepsilon_j^n = e^{\gamma n\Delta t}e^{i\beta j\Delta x}$) in (linear) eq. and obtain:

$$\underline{e^{\gamma\Delta t} = (1-\mu) + \mu\,e^{-i\beta\Delta x}}$$

Evaluation of the Stability of a FD Scheme
von Neumann Example

- The magnitude of $\xi = e^{\gamma \Delta t}$ is then obtained by multiplying ξ with its complex conjugate:

$$|\xi|^2 = \left((1-\mu) + \mu\, e^{-i\beta\Delta x}\right)\left((1-\mu) + \mu\, e^{i\beta\Delta x}\right) = 1 - 2\mu(1-\mu)\left(1 - \frac{e^{i\beta\Delta x} + e^{-i\beta\Delta x}}{2}\right)$$

Since $\frac{e^{i\beta\Delta x} + e^{-i\beta\Delta x}}{2} = \cos(\beta\Delta x)$ and $1 - \cos(\beta\Delta x) = 2\sin^2(\frac{\beta\Delta x}{2}) \Rightarrow$

$$|\xi|^2 = 1 - 2\mu(1-\mu)\left(1 - \cos(\beta\Delta x)\right) = 1 - 4\mu(1-\mu)\sin^2(\frac{\beta\Delta x}{2})$$

- Thus, the strict von Neumann stability criterion gives

$$|\xi| \le 1 \quad \Leftrightarrow \quad \left|1 - 4\mu(1-\mu)\sin^2(\frac{\beta\Delta x}{2})\right| \le 1$$

Since $\sin^2(\frac{\beta\Delta x}{2}) \ge 0 \quad \forall\beta \qquad \left(\left(1 - \cos(\beta\Delta x)\right) \ge 0 \quad \forall\beta\right)$

we obtain the same result as for the energy method:

$$|\xi| \le 1 \quad \Leftrightarrow \quad \mu(1-\mu) \ge 0 \quad \Leftrightarrow \quad 0 \le \frac{c\,\Delta t}{\Delta x} \le 1 \qquad (\mu = \frac{c\,\Delta t}{\Delta x})$$

Equivalent to the CFL condition

Partial Differential Equations

Hyperbolic PDE: $B^2 - 4AC > 0$

Examples:

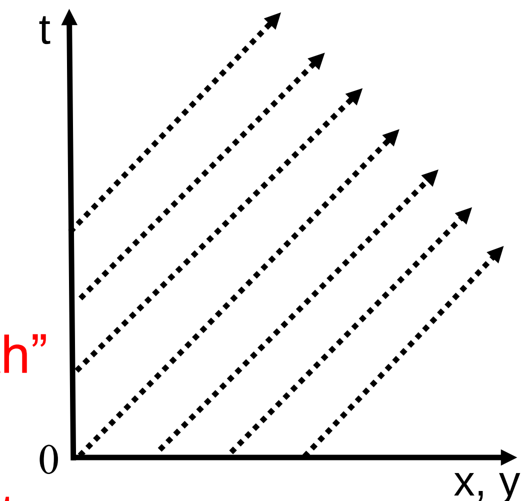
(1) $$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$ ← Wave equation, 2nd order

(2) $$\frac{\partial u}{\partial t} \pm c \frac{\partial u}{\partial x} = 0$$ ← Sommerfeld Wave/radiation equation, 1st order

(3) $$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{U} \cdot \nabla)\, \mathbf{u} = \mathbf{g}$$ ← Unsteady (linearized) inviscid convection (Wave equation first order)

(4) $$(\mathbf{U} \cdot \nabla)\, \mathbf{u} = \mathbf{g}$$ ← Steady (linearized) inviscid convection

- Allows non-smooth solutions
- Information travels along characteristics, e.g.:
  - For (3) above: $\frac{d\,\mathbf{x_c}}{dt} = \mathbf{U}(\mathbf{x_c}(t))$
  - For (4), along streamlines: $\frac{d\,\mathbf{x_c}}{ds} = \mathbf{U}$
- Domain of dependence of $\mathbf{u}(\mathbf{x},T)$ = "characteristic path"
  - e.g., for (3), it is: $\mathbf{x}_c(\mathrm{t})$ for $0 < \mathrm{t} < \mathrm{T}$
- Finite Differences, Finite Volumes and Finite Elements
  - Upwind schemes



Partial Differential Equations
Hyperbolic PDE - Example

Waves on a String

$$\frac{\partial^2 u(x,t)}{\partial t^2} = c^2 \frac{\partial^2 u(x,t)}{\partial x^2} \qquad 0 < x < L, \quad 0 < t < \infty$$

Initial Conditions

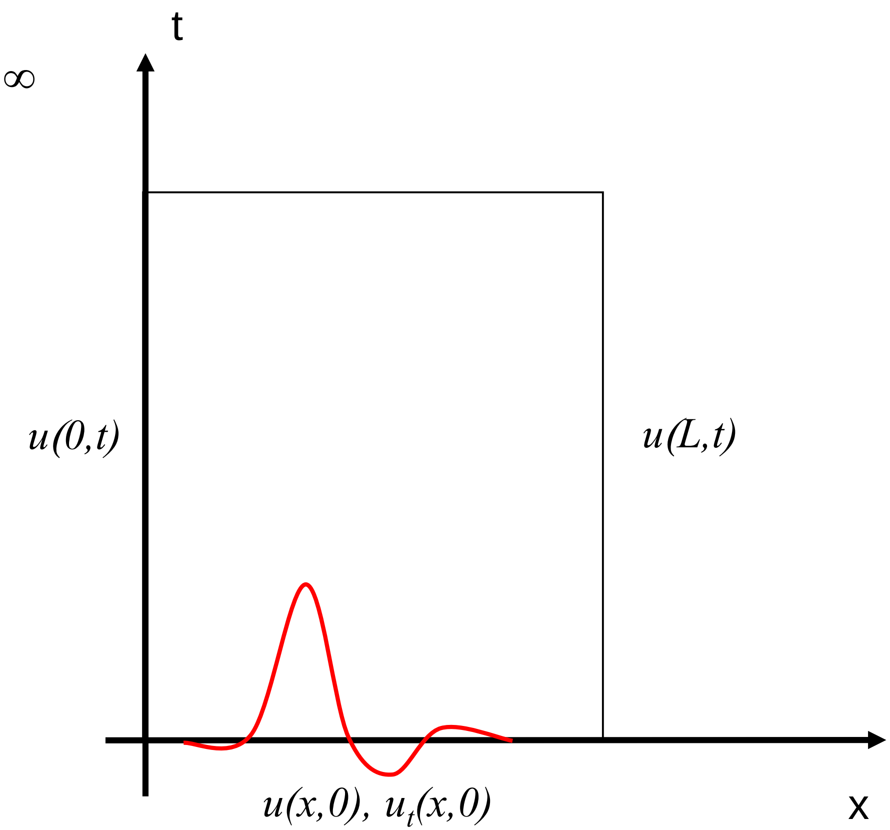
$$\begin{aligned} u(x,0) &= f(x), \;\; 0 \le x \le L \\ u_t(x,0) &= g(x), \;\; 0 < x < L \end{aligned}$$

Boundary Conditions

$$\begin{aligned} u(0,t) &= 0, \;, \; 0 < t < \infty \\ u(L,t) &= 0, \;, \; 0 < t < \infty \end{aligned}$$

Wave Solutions

$$u = \begin{cases} F(x-ct) & \text{Forward propagating wave} \\ G(x+ct) & \text{Backward propagating wave} \end{cases}$$



Typically Initial Value Problems in Time, Boundary Value Problems in Space

Time-Marching Solutions:

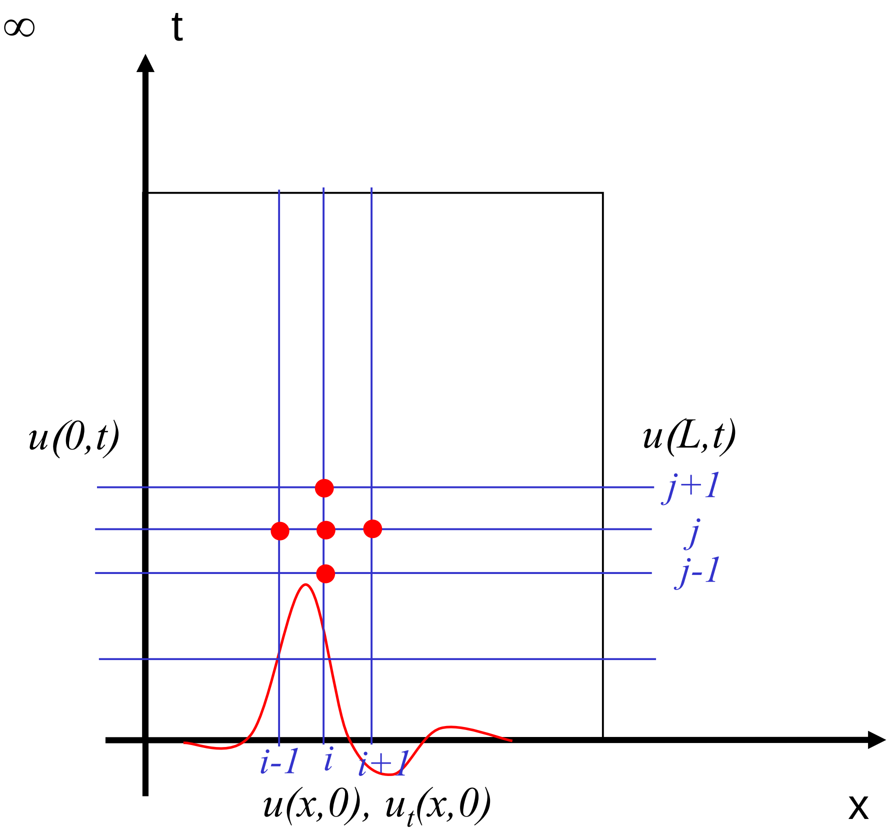
Implicit schemes generally stable

Explicit sometimes stable under certain conditions

Partial Differential Equations
Hyperbolic PDE - Example

Wave Equation

$$\frac{\partial^2 u(x,t)}{\partial t^2} = c^2 \frac{\partial^2 u(x,t)}{\partial x^2} \qquad 0 < x < L, \quad 0 < t < \infty$$



Discretization:

$$h = L/n$$
$$k = T/m$$
$$x_i = (i-1)h, \; i = 2, \ldots, n-1$$
$$t_j = (j-1)k, \; j = 1, \ldots, m$$

Finite Difference Representations (centered)

$$u_{tt}(x,t) = \frac{u(x_i,t_{j-1}) - 2u(x_i,t_j) + u(x_i,t_{j+1})}{k^2} + O(k^2)$$

$$u_{xx}(x,t) = \frac{u(x_{i-1},t_j) - 2u(x_i,t_j) + u(x_{i+1},t_j)}{h^2} + O(h^2)$$

$$u_{i,j} = u(x_i, t_j)$$

Finite Difference Representations

$$\frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{k^2} = c^2 \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2}$$

Partial Differential Equations
Hyperbolic PDE - Example

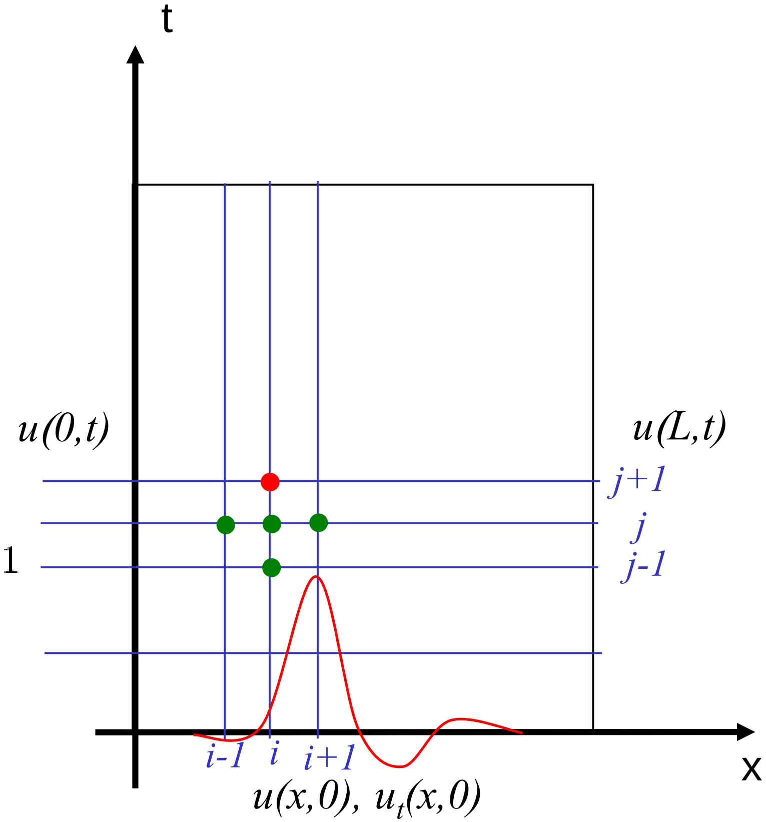
(from Lecture 10)

Introduce Dimensionless Wave Speed $C = \frac{ck}{h}$

Explicit Finite Difference Scheme

$$u_{i,j-1} - 2u_{i,j} + u_{i,j+1} = C^2(u_{i-1,j} - 2u_{i,j} + u_{i+1,j})$$

$$\boxed{u_{i,j+1}} = \boxed{(2 - 2C^2)u_{i,j} + C^2(u_{i+1,j} + u_{i-1,j}) - u_{i,j-1}} \quad i = 2, \dots n-1$$



Stability Requirement: $C = \frac{ck}{h} < 1$

$$C = \frac{c\,\Delta t}{\Delta x} < 1 \quad \text{Courant-Friedrichs-Lewy condition (CFL condition)}$$

Physical wave speed must be smaller than the largest numerical wave speed, or,
Time-step must be less than the time for the wave to travel to adjacent grid points:

$$c < \frac{\Delta x}{\Delta t} \quad \text{or} \quad \Delta t < \frac{\Delta x}{c}$$

Partial Differential Equations
Hyperbolic PDE - Example
Start of Integration: Euler and Higher Order Starters

Given ICs:
$$\begin{aligned} u(x,0) &= f(x), \;\; 0 \le x \le L \\ u_t(x,0) &= g(x), \;\; 0 < x < L \end{aligned}$$

1st order Euler Starter

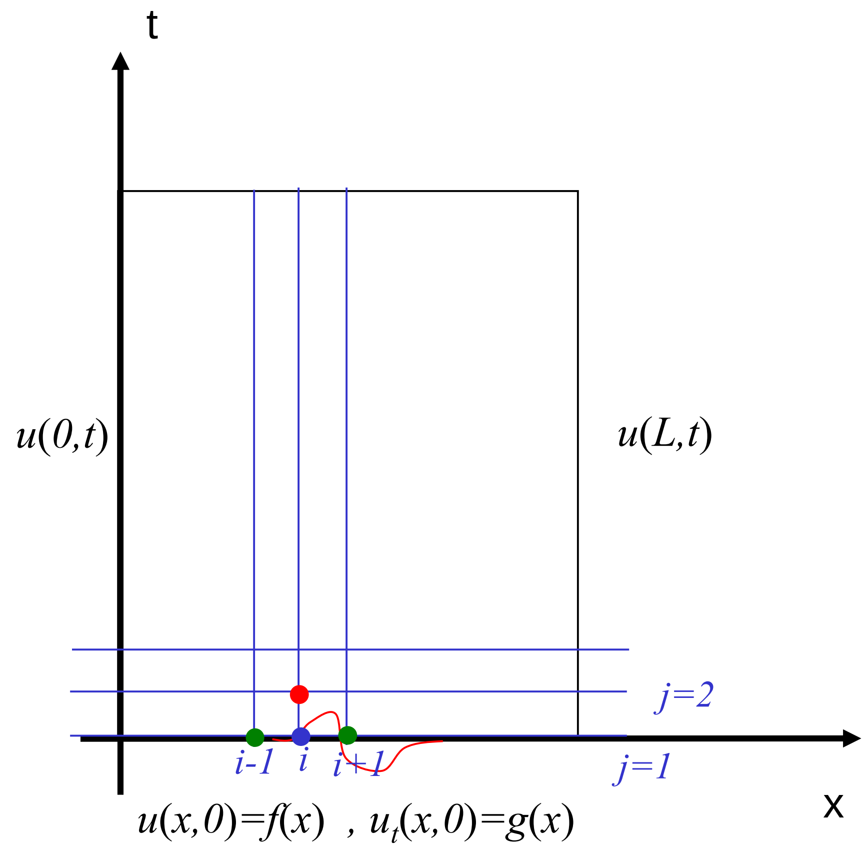
$$u_{i,2} = u(x_i,k) \simeq u(x_i,0) + k u_t(x,0) = f(x_i) + k g(x_i)$$

But, second derivative in x at $t = 0$ is known from IC:
$$u_{xx}(x,0) = f''$$

From Wave Equation
$$u_{tt}(x_i,0) = c^2 u_{xx}(x_i,0) = c^2 f_{xx}(x_i) = c^2 \frac{f_{i-1} - 2f_i + f_{i+1}}{h^2} + O(h^2)$$

Higher order Taylor Expansion
$$u(x,k) = u(x,0) + k u_t(x,0) + \frac{u_{tt}(x,0)k^2}{2} + O(k^3)$$

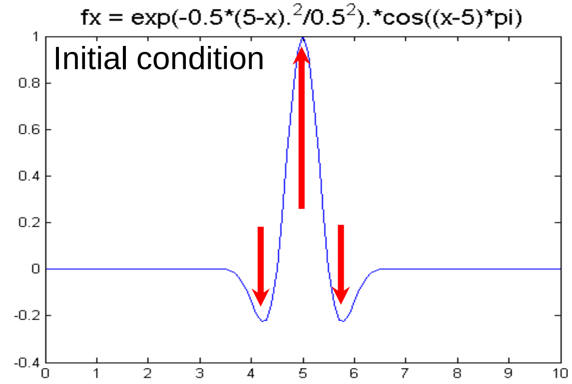
Higher Order Self Starter
$$u_{i,2} = u(x_i,k) = f_i + k g_i + \frac{c^2k^2}{2h^2}(f_{i-1} - 2f_i + f_{i+1}) + O(h^2k^2) + O(k^3)$$
$$= \left(1 - C^2\right) f_i + k g_i + \frac{C^2}{2}(f_{i+1} + f_{i-1})$$



General idea: use the PDE itself to get higher order integration

Waves on a String

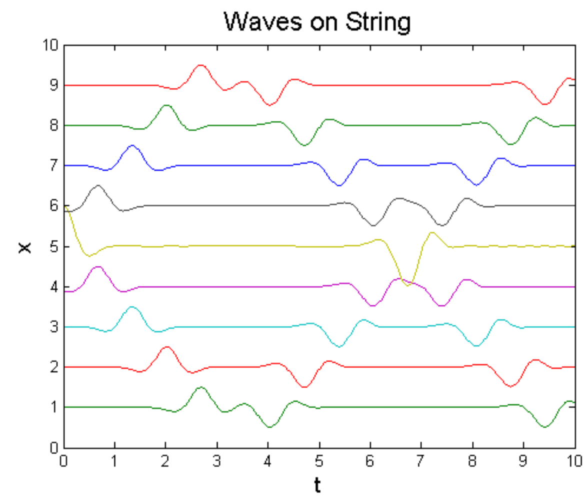
$$\frac{\partial^2 u(x,t)}{\partial t^2} = c^2 \frac{\partial^2 u(x,t)}{\partial x^2} \qquad 0 < x < L, \quad 0 < t < \infty$$

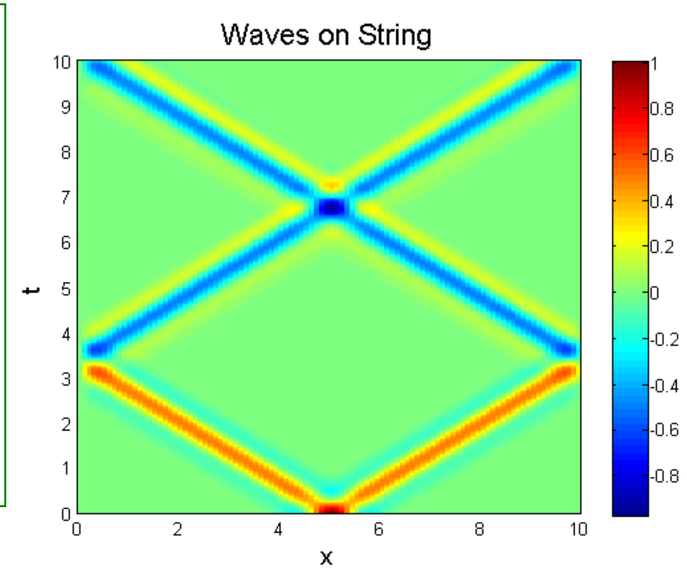


waveeq.m

```
L=10;
T=10;
c=1.5;
N=100;
h=L/N;
M=400;
k=T/M;
C=c*k/h
Lf=0.5;
x=[0:h:L]';
t=[0:k:T];
%fx=['exp(-0.5*(' num2str(L/2) '-x).^2/(' num2str(Lf) ').^2)'];
%gx='0';
fx='exp(-0.5*(5-x).^2/0.5^2).*cos((x-5)*pi)';
gx='0';  %Zero first time derivative at t=0
f=inline(fx,'x');
g=inline(gx,'x');

n=length(x);
m=length(t);
u=zeros(n,m);
%  Second order starter
u(2:n-1,1)=f(x(2:n-1));
for i=2:n-1
 u(i,2) = (1-C^2)*u(i,1) + k*g(x(i)) +C^2*(u(i-1,1)+u(i+1,1))/2;
end

%  CDS: Iteration in time (j) and space (i)
for j=2:m-1
   for i=2:n-1
u(i,j+1)=(2-2*C^2)*u(i,j) + C^2*(u(i+1,j)+u(i-1,j)) - u(i,j-1);
   end
end
```

```
figure(1)
plot(x,f(x));
a=title(['fx = ' fx]);
set(a,'FontSize',16);

figure(2)
wavei(u',x,t);
a=xlabel('x');
set(a,'Fontsize',14);
a=ylabel('t');
set(a,'Fontsize',14);
a=title('Waves on String');
set(a,'Fontsize',16);
colormap;
```

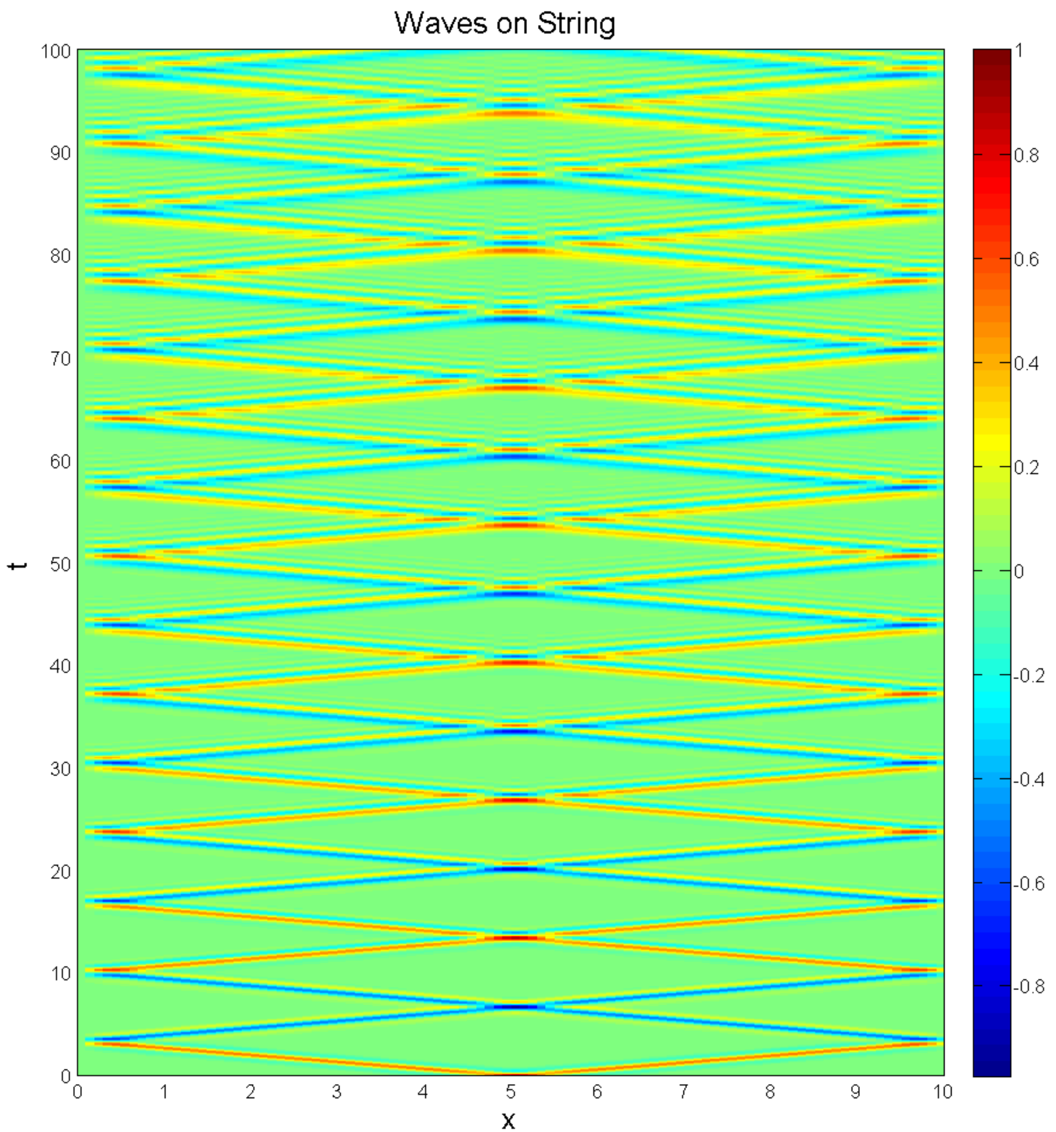




Waves on a String, Longer simulation:
Effects of dispersion and effective wavenumber/speed

waveeq.m

```
L=10;
T=10;
c=1.5;
N=100;
h=L/N;     % Horizontal resolution (Dx)
M=400;
%  Test: increase duration of simulation, to see effect of
%dispersion and  effective wavenumber/speed (due to 2nd order)
%T=100;M=4000;
k=T/M;     % Time resolution  (Dt)
C=c*k/h    % Try case C>1, e.g. decrease Dx or increase Dt
Lf=0.5;

x=[0:h:L]';
t=[0:k:T];
%fx=['exp(-0.5*(' num2str(L/2) '-x).^2/(' num2str(Lf) ').^2)'];
%gx='0';
fx='exp(-0.5*(5-x).^2/0.5^2).*cos((x-5)*pi)';
gx='0';
f=inline(fx,'x');
g=inline(gx,'x');

n=length(x);
m=length(t);
u=zeros(n,m);
%Second order starter
u(2:n-1,1)=f(x(2:n-1));
for i=2:n-1
 u(i,2) = (1-C^2)*u(i,1) + k*g(x(i)) +C^2*(u(i-1,1)+u(i+1,1))/2;
end

%CDS: Iteration in time (j) and space (i)
for j=2:m-1
   for i=2:n-1
u(i,j+1)=(2-2*C^2)*u(i,j) + C^2*(u(i+1,j)+u(i-1,j)) - u(i,j-1);
   end
end
```



Wave Equation
d'Alembert's Solution

Wave Equation

$$\frac{\partial^2 u(x,t)}{\partial t^2} = c^2 \frac{\partial^2 u(x,t)}{\partial x^2} \qquad 0 < x < L, \quad 0 < t < \infty$$

Solution

$$u(x,t) = F(x-ct) + G(x+ct)\ ,\ 0 < x < L$$

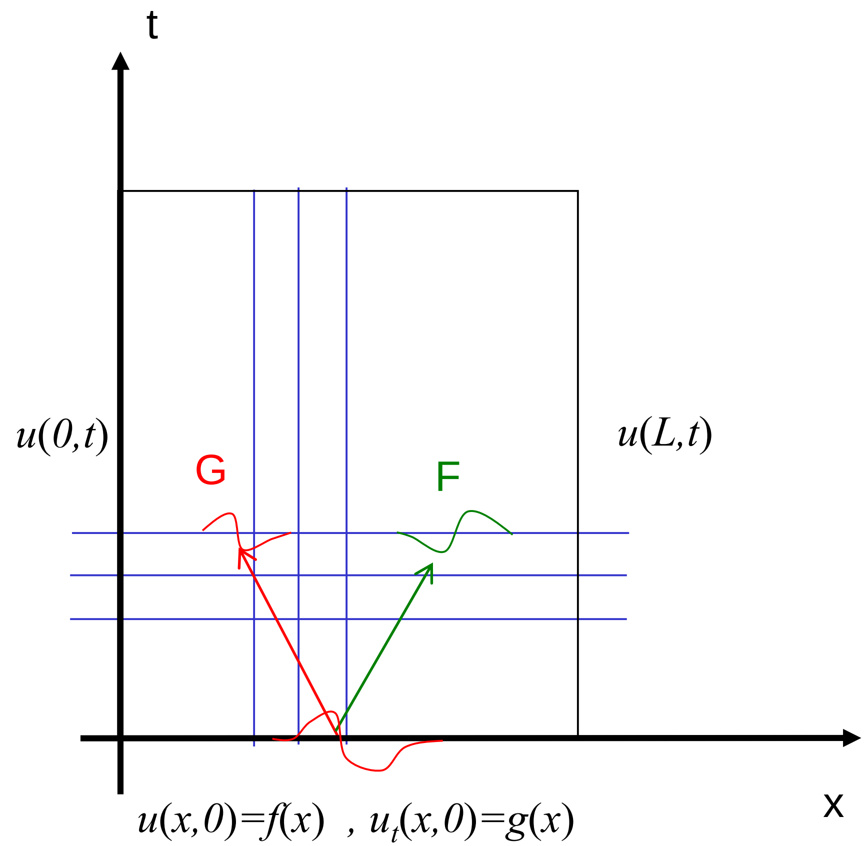
Periodicity Properties

$$\begin{aligned} F(-z) &= -F(z) \\ F(z+2L) &= F(z) \end{aligned}$$

$$\begin{aligned} G(-z) &= -G(z) \\ G(z+2L) &= G(z) \end{aligned}$$

Proof

$$\begin{aligned} u_{xx}(x,t) &= F''(x-ct) + G''(x+ct) \\ u_{tt}(x,t) &= c^2 F''(x-ct) + c^2 G''(x+ct) \\ &= c^2 u_{xx}(x,t) \end{aligned}$$

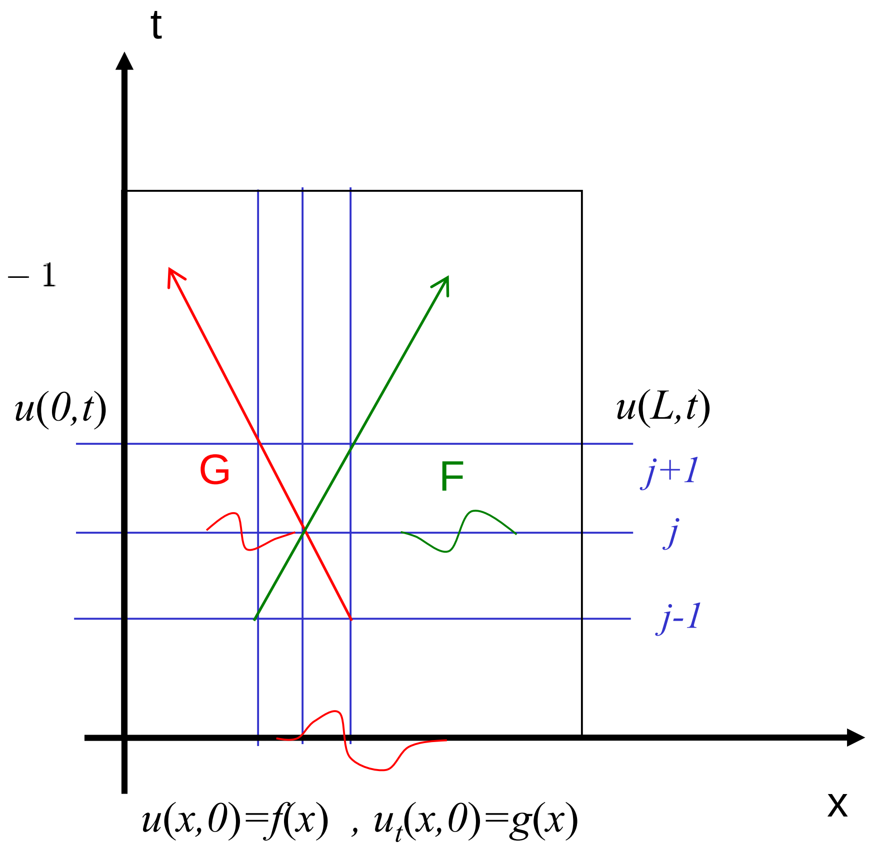


Hyperbolic PDE
Method of Characteristics

Explicit Finite Difference Scheme

$$u_{i,j-1} - 2u_{i,j} + u_{i,j+1} = C^2(u_{i-1,j} - 2u_{i,j} + u_{i+1,j})$$

$$u_{i,j+1} = (2 - 2C^2)u_{i,j} + C^2(u_{i+1,j} + u_{i-1,j}) - u_{i,j-1}, \; i = 2, \ldots n-1$$



First 2 Rows known

$$\begin{aligned} u_{i,1} &= u(x_i, 0) \\ u_{i,2} &= u(x_i, k) \end{aligned}$$

Characteristic Sampling

$$k = h/c \Rightarrow C = 1$$

Exact Discrete Solution

$$u_{i,j+1} = u_{i+1,j} + u_{i-1,j} - u_{i,j-1}$$

Hyperbolic PDE
Method of Characteristics

Let's proof the following FD scheme is an exact Discrete Solution

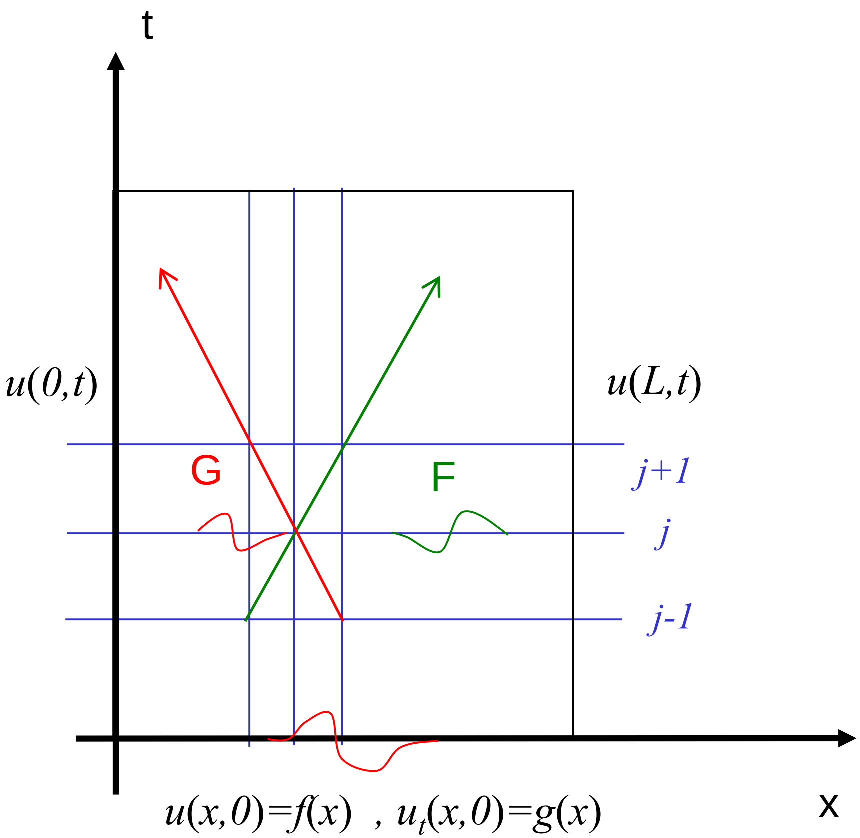
$$u_{i,j+1} = u_{i+1,j} + u_{i-1,j} - u_{i,j-1}$$

D'Alembert's Solution with $C=1$

$$\begin{aligned} x_i - ct_j &= (i-1)h - c(j-1)k \\ &= (i-1)h - (j-1)h \\ &= (i-j)h \end{aligned}$$

$$\begin{aligned} x_i + ct_j &= (i-1)h + c(j-1)k \\ &= (i-1)h + (j-1)h \\ &= (i+j-2)h \end{aligned}$$

$$u_{i,j} = F((i-j)h) + G((i+j-2)h)$$



Proof

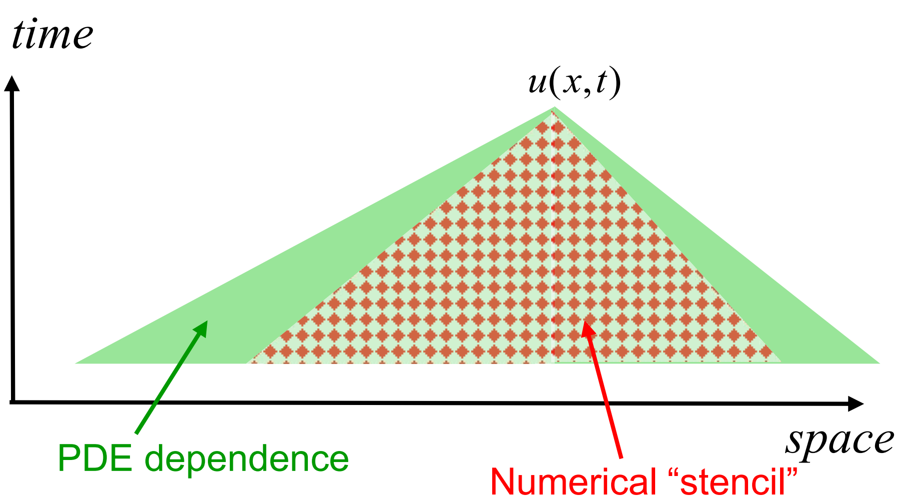
$$\begin{aligned} & u_{i+1,j} + u_{i-1,j} - u_{i,j-1} \\ = \; & F((i+1-j)h) + F((i-1-j)h) - F((i-(j-1))h) \\ & +G((i+1+j-2)h) + G((i-1+j-2)h) - G((i+j-1-2)h) \\ = \; & F((i-(j+1))h) + G((i+(j+1)-2)h) \\ = \; & u_{i,j+1} \end{aligned}$$

Courant-Fredrichs-Lewy Condition (1920's)

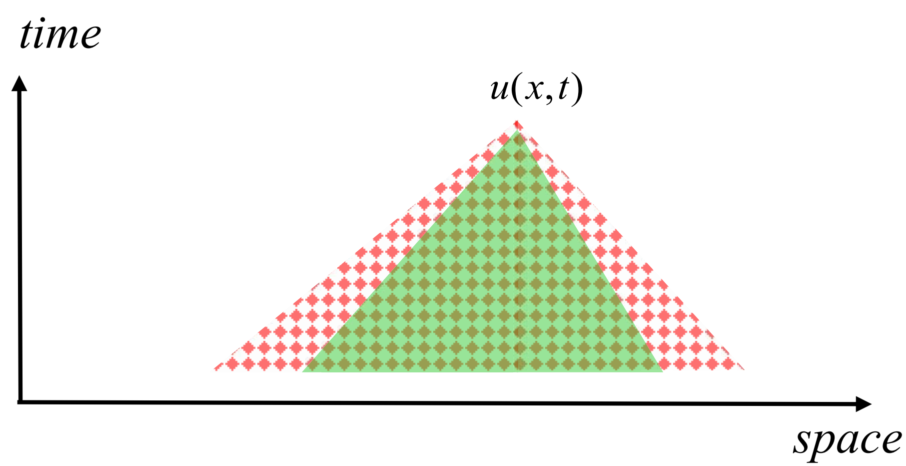
- Basic idea: the solution of the Finite-Difference (FD) equation can not be independent of the (past) information that determines the solution of the corresponding PDE
- In other words:

  The "Numerical domain of dependence of FD scheme must include the mathematical domain of dependence of the corresponding PDE"

CFL NOT satisfied



CFL satisfied



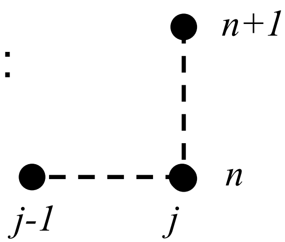
CFL: Linear convection (Sommerfeld Eqn) Example

Determine domain of dependence of PDE and of FD scheme

- PDE: $\dfrac{\partial u(x,t)}{\partial t} + c\,\dfrac{\partial u(x,t)}{\partial x} = 0$ Characteristics: If $\dfrac{dx}{dt} = c \;\Rightarrow\; x = c\,t + \zeta$ and $du = 0 \;\Rightarrow u = \text{cst}$

  Solution of the form: $u(x,t) = F(x - ct)$

- FD scheme. For our Upwind discretization, with $t = n\Delta t,\; x = j\Delta x$:



$$\frac{\phi_j^{n+1} - \phi_j^n}{\Delta t} + c\,\frac{\phi_j^n - \phi_{j-1}^n}{\Delta x} = 0$$

Slope of characteristic: $\dfrac{dt}{dx} = \dfrac{1}{c}$

Slope of Upwind scheme: $\dfrac{\Delta t}{\Delta x}$

=> CFL condition: $\dfrac{\Delta t}{\Delta x} \le \dfrac{1}{c}$

$$\boxed{\frac{c\,\Delta t}{\Delta x} \le 1}$$

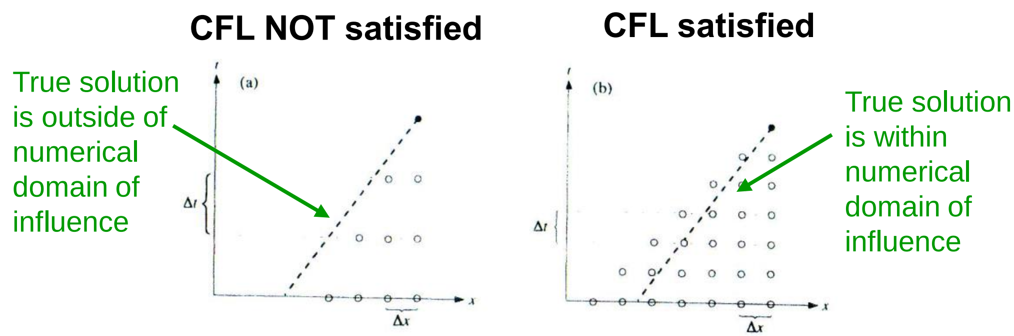


FIGURE 2.1. The influence of the time step on the relationship between the numerical domain of dependence of the upstream scheme (open circles) and the true domain of dependence of the advection equation (heavy dashed line): (a) unstable Δt, (b) stable Δt.

Source: Figure 2.1 from Durran, D. *Numerical Methods for Wave Equations in Geophysical Fluid Dynamics*. Springer, 1998.

CFL: 2nd order Wave equation Example

Determine domain of dependence of PDE and of FD scheme

- PDE, second order wave eqn example:

$$\frac{\partial^2 u(x,t)}{\partial t^2} = c^2 \frac{\partial^2 u(x,t)}{\partial x^2} \qquad 0 < x < L, \quad 0 < t < \infty$$

  – As seen before: $u(x,t) = F(x-ct) + G(x+ct) \Rightarrow$ slope of characteristics: $\frac{dt}{dx} = \pm \frac{1}{c}$

- FD scheme: discretize: $t = n\Delta t, \; x = j\Delta x$

  – CD scheme (CDS) in time and space (2nd order), explicit

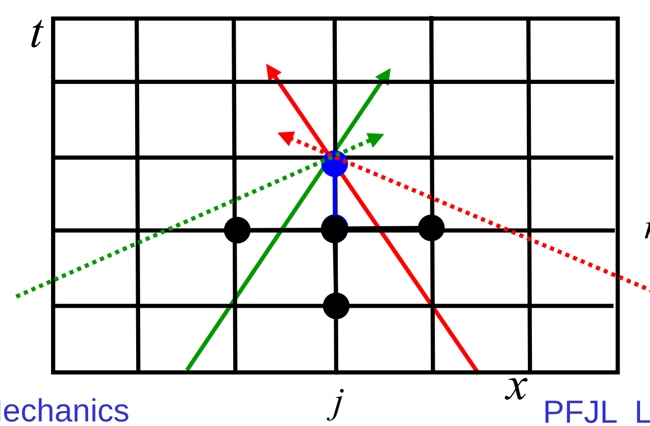
$$\frac{u_j^{n+1} - 2u_j^n + u_j^{n-1}}{\Delta t^2} = c^2 \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2} \Rightarrow u_j^{n+1} = (2-2C^2)u_j^n + C^2(u_{j+1}^n + u_{j-1}^n) - u_j^{n-1} \quad \text{where } C = \frac{c\Delta t}{\Delta x}$$

  – We obtain from the respective slopes:

$$\boxed{\frac{c\,\Delta t}{\Delta x} \le 1}$$

Full line case: CFL satisfied

Dotted lines case: c and Δt too big, Δx too small (CFL NOT satisfied)



CFL Condition: Some comments

- CFL is only a necessary condition for stability
- Other (sufficient) stability conditions are often more restrictive
  - For example: if $\frac{\partial u(x,t)}{\partial t} + c\,\frac{\partial u(x,t)}{\partial x} = 0$ is discretized as

$$\left(\frac{\partial u(x,t)}{\partial t}\right)_{\text{CD},2^{\text{nd}}\text{ order in }t} + c\left(\frac{\partial u(x,t)}{\partial x}\right)_{\text{CD},4^{\text{th}}\text{ order in x}} \approx 0$$

Five grid-points stencil: (-1,8,0,-8,1) / 12
See Taylor tables in previous lecture and eqn. sheet

  - One obtains from the CFL: $\frac{c\,\Delta t}{\Delta x} \leq 2$
  - While a Von Neumann analysis leads: $\frac{c\,\Delta t}{\Delta x} \leq 0.728$

- For equations that are not purely hyperbolic or that can change of type (e.g. as diffusion term increases), CFL condition can at times be violated locally for a short time, without leading to global instability further in time

von Neumann Examples

- Forward in time (Euler), centered in space, Explicit

$$\frac{\phi_j^{n+1}-\phi_j^n}{\Delta t}+c\frac{\phi_{j+1}^n-\phi_{j-1}^n}{2\Delta x}=0 \quad\Rightarrow\quad \phi_j^{n+1}=\phi_j^n-\frac{C}{2}(\phi_{j+1}^n-\phi_{j-1}^n)$$

  – Von Neumann: insert $\varepsilon(x,t)=\varepsilon_\beta(t)\,e^{i\beta x}=e^{\gamma t}e^{i\beta x}\Rightarrow\quad \varepsilon_j^n=e^{\gamma n\Delta t}e^{i\beta j\Delta x}$

$$\Rightarrow\quad \varepsilon_j^{n+1}=\varepsilon_j^n-\frac{C}{2}(\varepsilon_{j+1}^n-\varepsilon_{j-1}^n)\quad\Rightarrow\quad e^{\gamma\Delta t}=1-\frac{C}{2}\left(e^{i\beta\Delta x}-e^{-i\beta\Delta x}\right)=1-Ci\sin(\beta\Delta x)$$

    - Taking the norm:

$$\left|e^{\gamma t}\right|^2=\left|\xi\right|^2=\left(1-Ci\sin(\beta\Delta x)\right)\left(1+Ci\sin(\beta\Delta x)\right)=1+C^2\sin^2(\beta\Delta x)\quad\geq 1\ \text{ for }\ C\neq 0\ !$$

    - Unconditionally Unstable

- Implicit scheme (backward in time, centered in space)

$$\frac{\phi_j^{n+1}-\phi_j^n}{\Delta t}+c\frac{\phi_{j+1}^{n+1}-\phi_{j-1}^{n+1}}{2\Delta x}=0\quad\Rightarrow\quad \varepsilon_j^{n+1}=\varepsilon_j^n-\frac{C}{2}(\varepsilon_{j+1}^{n+1}-\varepsilon_{j-1}^{n+1})\quad\Rightarrow\quad e^{\gamma\Delta t}=1-e^{\gamma\Delta t}Ci\sin(\beta\Delta x)$$

$$\Rightarrow\quad \left|e^{\gamma t}\right|^2=\left|\xi\right|^2=\frac{1}{\left(1+Ci\sin(\beta\Delta x)\right)\left(1-Ci\sin(\beta\Delta x)\right)}=\frac{1}{1+C^2\sin^2(\beta\Delta x)}\quad\leq 1\ \text{ for }\ C\neq 0\ !$$

    - Unconditionally Stable

Stability of FD schemes for $u_t + b\, u_y = 0$ (<u>t denoted x below</u>)

Table showing various finite difference forms removed due to copyright restrictions. Please see Table 6.1 in Lapidus, L., and G. Pinder. *Numerical Solution of Partial Differential Equations in Science and Engineering*. Wiley-Interscience, 1982.

Stability of FD schemes for $u_t + b\, u_y = 0$, Cont.

Table showing various finite difference forms removed due to copyright restrictions.
Please see Table 6.1 in Lapidus, L., and G. Pinder. *Numerical Solution of Partial Differential Equations in Science and Engineering*. Wiley-Interscience, 1982.

Partial Differential Equations
Elliptic PDE

Laplace Operator $\nabla^2 \equiv u_{xx} + u_{yy}$

Examples:

$$\nabla^2 \phi = 0$$

Laplace Equation – Potential Flow

$$\nabla^2 \phi = g(x, y)$$

Poisson Equation
- Potential Flow with sources
- Steady heat conduction in plate + source

$$\nabla^2 u + f(x, y) u = 0$$

Helmholtz equation – Vibration of plates

$$\mathbf{U} \cdot \nabla \mathbf{u} = \nu \nabla^2 \mathbf{u}$$

Steady Convection-Diffusion

- Smooth solutions ("diffusion effect")
- Very often, steady state problems
- Domain of dependence of u is the full domain **D**(x,y) => "global" solutions
- Finite differ./volumes/elements, boundary integral methods (Panel methods)

Partial Differential Equations
Elliptic PDE - Example

$$0 \le x \le a \;,\;\; 0 \le y \le b;$$

Equidistant Sampling

$$h = a/(n-1)$$
$$h = b/(m-1)$$

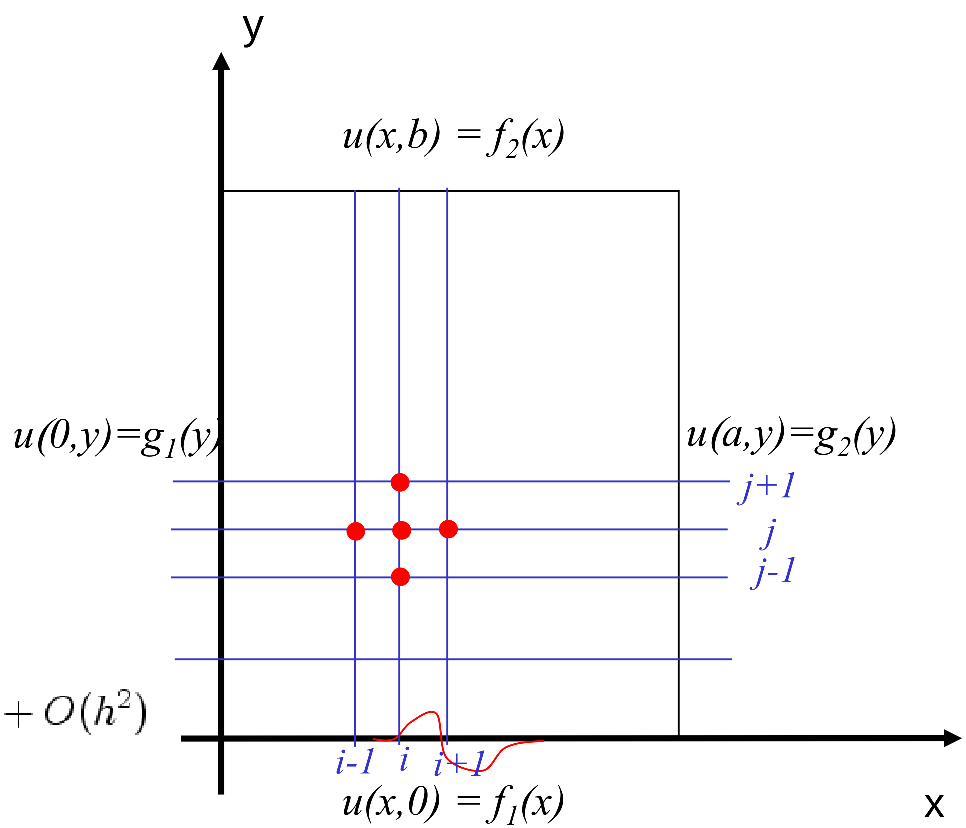
Discretization

$$x_i = (i-1)h, \; i = 1, \ldots, n$$
$$y_j = (j-1)h, \; j = 1, \ldots, m$$

Finite Differences

$$u_{xx}(x,t) = \frac{u(x_{i-1},y_j) - 2u(x_i,y_j) + u(x_{i+1},y_j)}{h^2} + O(h^2)$$

$$u_{yy}(x,t) = \frac{u(x_i,y_{j-1}) - 2u(x_i,y_j) + u(x_i,y_{j+1})}{h^2} + O(h^2)$$



Dirichlet BC

Partial Differential Equations
Elliptic PDE - Example

Discretized Laplace Equation

$$\nabla^2 u = \frac{u(x_{i-1},y_j) + u(x_i,y_{j-1}) - 4u(x_i,y_j) + u(x_{i+1},y_j) + u(x_i,y_{j+1})}{h^2} = 0$$

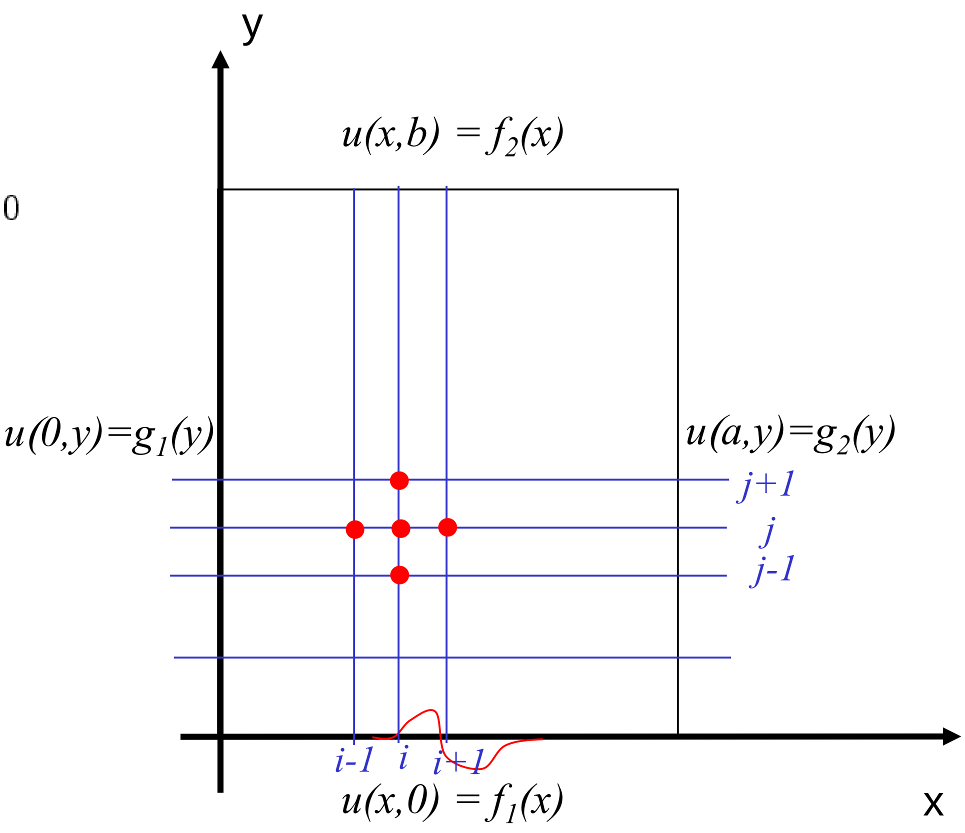
$$u_{i,j} = u(x_i, t_j)$$

Finite Difference Scheme

$$u_{i+1,j} + u_{i-1,j} + u_{i,j-1} + u_{i,j+1} - 4u_{i,j} = 0$$

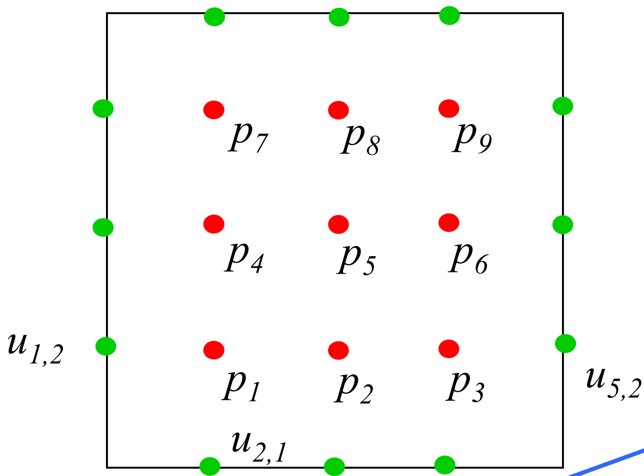
Boundary Conditions

$$\begin{aligned} u(x_1,y_j) &= u_{1,j}\ , \quad 2 \le j \le m-1 \\ u(x_n,y_j) &= u_{n,j}\ , \quad 2 \le j \le m-1 \\ u(x_i,y_1) &= u_{i,1}\ , \quad 2 \le i \le n-1 \\ u(x_i,y_n) &= u_{i,n}\ , \quad 2 \le i \le n-1 \end{aligned}$$



Global Solution Required

Elliptic PDEs
Laplace Equation, Global Solvers



Dirichlet BC

Leads to $\mathbf{Ax} = \mathbf{b}$, with $\mathbf{A}$ block-tridiagonal:
$\mathbf{A} = \text{tri}\,\{\,\mathbf{I}\,,\,\mathbf{T}\,,\,\mathbf{I}\,\}$

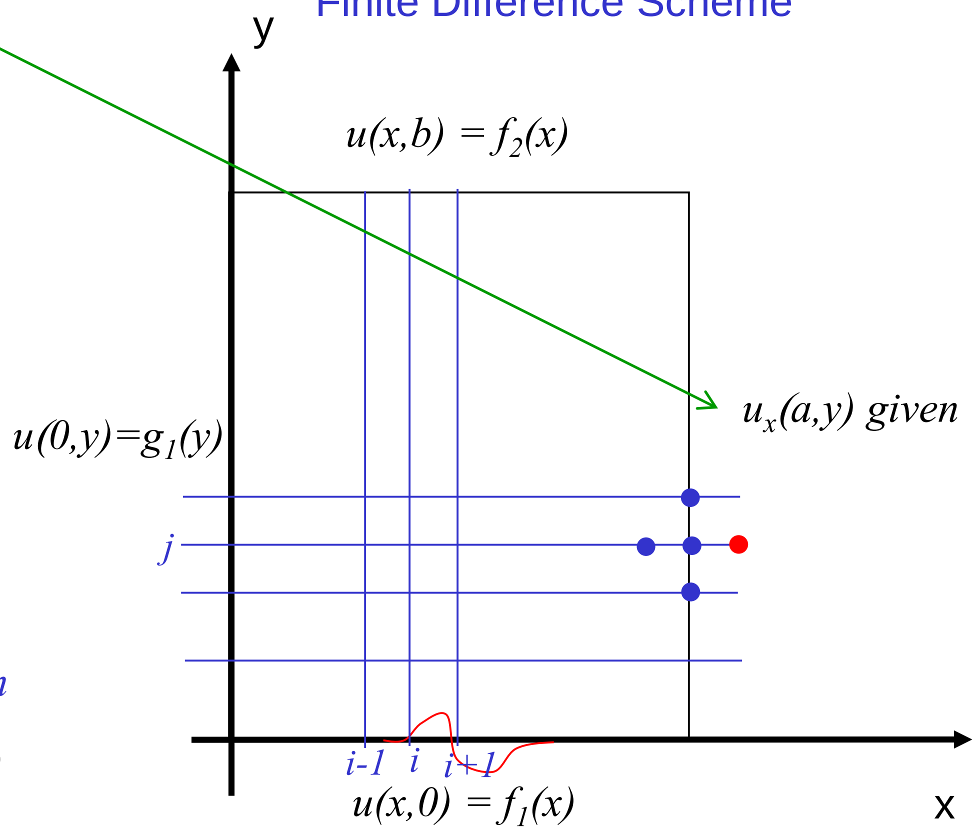
$$
\begin{array}{rcl}
-4p_1 + p_2 + p_4 & = & -u_{2,1} - u_{1.2} \\
p_1 - 4p_2 + p_3 + p_5 & = & -u_{3,1} \\
p_2 - 4p_3 + p_6 & = & -u_{4,1} - u_{5,2} \\
p_1 - 4p_4 + p_5 + p_7 & = & -u_{1,3} \\
p_2 + p_4 - 4p_5 + p_6 + p_8 & = & 0 \\
p_3 + p_5 - 4p_6 + p_9 & = & -u_{5,3} \\
p_4 - 4p_7 + p_8 & = & -u_{2,5} - u_{1,4} \\
p_5 + p_7 - 4p_8 + p_9 & = & -u_{3,5} \\
p_6 + p_8 - 4p_9 & = & -u_{4,5} - u_{5,4}
\end{array}
$$

Ellipticic PDEs
Neumann Boundary Conditions

Neumann (Derivative) Boundary Condition

$$\frac{\partial}{\partial N} u(x,y) \quad \text{given}$$

Finite Difference Scheme



Finite Difference Scheme at $i = n$

$$u_{n+1,j} + u_{n-1,j} + u_{n,j+1} + u_{n,j-1} - 4u_{n,j} = 0$$

Derivative BC: Finite Difference

$$\frac{u_{n+1,j} - u_{n-1,j}}{2h} \simeq u_x(x_n, y_j)$$

$$\boxed{u_{n+1,j}} = u_{n-1,j} + 2h u_x(x_n, y_j)$$

Boundary Finite Difference Scheme at $i = n$

$$u_{n-1,j} + 2\Delta x \left.\frac{\partial u}{\partial x}\right|_n + u_{n-1,j} + u_{n,j+1} + u_{n,j-1} - 4u_{n,j} = 0$$

Leads to a factor 2 (a matrix $2\,\mathbf{I}$ in $\mathbf{A}$) for points along boundary