2.29 Numerical Fluid Mechanics
Spring 2015 – **Lecture 15**

REVIEW Lecture 14:

- **Elliptic PDEs, Continued**
  - Examples, Higher order finite differences
  - Irregular boundaries: Dirichlet and Von Neumann BCs
  - Internal boundaries
- **Parabolic PDEs and Stability**
  - Explicit schemes (1D-space)
    - Von Neumann
  - Implicit schemes (1D-space): simple and Crank-Nicholson, von Neumann
  - Examples
  - Extensions to 2D and 3D
    - Explicit and Implicit schemes
    - Alternating-Direction Implicit (ADI) schemes

TODAY (Lecture 15): FINITE VOLUME METHODS

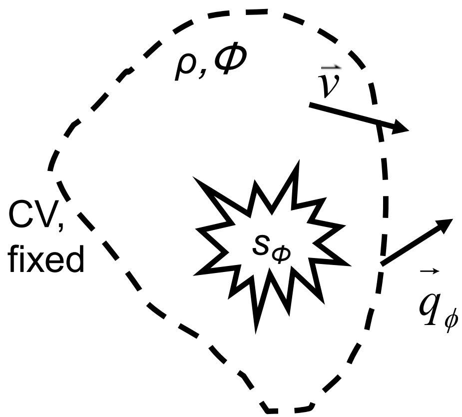
- Integral forms of the conservation laws
- Introduction to FV Methods
- Approximations needed and basic elements of a FV scheme
  - FV grids: Cell centered (Nodes or CV-faces) vs. Cell vertex; Structured vs. Unstructured
  - Approximation of surface integrals (leading to symbolic formulas)
  - Approximation of volume integrals (leading to symbolic formulas)
- Summary: Steps to step-up FV scheme
- Examples: one-dimensional examples
  - Generic equations
  - Linear Convection (Sommerfeld eqn.): convective fluxes
    - 2^{nd} order in space, 4^{th} order in space, links to CDS
  - Unsteady Diffusion equation: diffusive fluxes
    - Two approaches for 2^{nd} order in space, links to CDS

References and Reading Assignments

- Chapter 29.4 on "The control-volume approach for elliptic equations" of "Chapra and Canale, Numerical Methods for Engineers, 2014/2010/2006."
- Chapter 4 on "Finite Volume Methods" of "J. H. Ferziger and M. Peric, Computational Methods for Fluid Dynamics. Springer, NY, 3rd edition, 2002"
- Chapter 5 on "Finite Volume Methods" of "H. Lomax, T. H. Pulliam, D.W. Zingg, *Fundamentals of Computational Fluid Dynamics (Scientific Computation).* Springer, 2003"
- Chapter 5.6 on "Finite-Volume Methods" of T. Cebeci, J. P. Shao, F. Kafyeke and E. Laurendeau, Computational Fluid Dynamics for Engineers. Springer, 2005.

Integral Conservation Law for a scalar ϕ

$$\left\{ \frac{d}{dt} \int_{CM} \rho\phi dV = \right\} \frac{d}{dt} \int_{CV_{\text{fixed}}} \rho\phi dV + \underbrace{\int_{CS} \rho\phi\, (\vec{v}.\vec{n}) dA}_{\substack{\text{Advective fluxes} \\ \text{(Adv.\& diff. fluxes = "convective" fluxes)}}} = \underbrace{-\int_{CS} \vec{q}_\phi.\vec{n}\, dA}_{\text{Other transports (diffusion, etc)}} + \underbrace{\sum \int_{CV_{\text{fixed}}} s_\phi\, dV}_{\substack{\text{Sum of sources and} \\ \text{sinks terms (reactions, etc)}}}$$



Applying the Gauss Theorem, for any arbitrary CV gives:

$$\frac{\partial \rho\phi}{\partial t} + \nabla.(\rho\phi\vec{v}) = -\nabla.\vec{q}_\phi + s_\phi$$

For a common diffusive flux model (Fick's law, Fourier's law):

$$\vec{q}_\phi = -k\nabla\phi$$

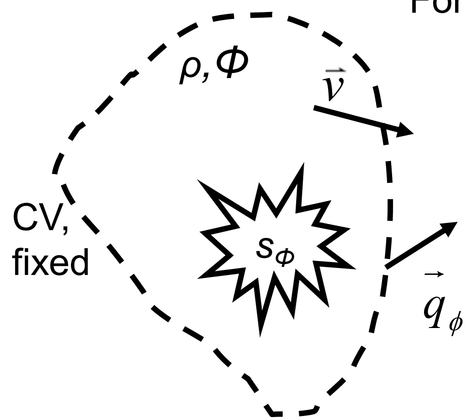
Conservative form of the PDE

$$\frac{\partial \rho\phi}{\partial t} + \nabla.(\rho\phi\vec{v}) = \nabla.(k\nabla\phi) + s_\phi$$

Strong-Conservative form of the Navier-Stokes Equations ($\phi \Rightarrow \boldsymbol{v}$)

Cons. of Momentum: $$\frac{d}{dt}\int_{CV} \rho\vec{v}dV + \int_{CS} \rho\vec{v}(\vec{v}.\vec{n})dA = \underbrace{\int_{CS} -p\,\vec{n}dA + \int_{CS} \vec{\vec{\tau}}.\vec{n}dA + \int_{CV} \rho\vec{g}dV}_{=\sum\vec{F}}$$

Applying the Gauss Theorem gives: $$= \int_{CV} \left(-\nabla p + \nabla.\vec{\vec{\tau}} + \rho\vec{g}\right)dV$$



For any arbitrary CV gives: $$\frac{\partial\rho\vec{v}}{\partial t} + \nabla.(\rho\vec{v}\;\vec{v}) = -\nabla p + \nabla.\vec{\vec{\tau}} + \rho\vec{g}$$ Cauchy Mom. Eqn.

With Newtonian fluid + incompressible + constant μ:

Momentum: $$\frac{\partial\rho\vec{v}}{\partial t} + \nabla.(\rho\vec{v}\;\vec{v}) = -\nabla p + \mu\nabla^2\vec{v} + \rho\vec{g}$$

Mass: $$\nabla.\vec{v} = 0$$

Equations are said to be in "strong conservative form" if all terms have the form of the divergence of a vector or a tensor. For the i^{th} Cartesian component, in the general Newtonian fluid case:

With Newtonian fluid only: $$\frac{\partial\rho v_i}{\partial t} + \nabla.(\rho v_i\;\vec{v}) = \nabla.\left(-p\,\vec{e}_i + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\vec{e}_j - \frac{2}{3}\mu\frac{\partial u_j}{\partial x_j}\vec{e}_i + \rho g_i x_i \vec{e}_i\right)$$

FINITE VOLUME METHODS: Introduction

- Finite Difference Methods are based on a discretization of the differential forms of the conservation equations
- Finite Volume Methods are based on a discretization of the integral forms of the conservation equations:

$$\frac{d}{dt}\int_{CV_{\text{fixed}}} \rho\phi dV + \underbrace{\int_{CS} \rho\phi\,(\vec{v}.\vec{n})dA}_{\substack{\text{Advective fluxes} \\ \text{(Adv.\& diff. fluxes = "convective" fluxes)}}} = \underbrace{-\int_{CS} \vec{q}_{\phi}.\vec{n}\; dA}_{\text{Other transports (diffusion, etc)}} + \underbrace{\sum\int_{CV_{\text{fixed}}} s_{\phi}\; dV}_{\substack{\text{Sum of sources and} \\ \text{sinks terms (reactions, etc)}}}$$

- Basic ideas/steps to set-up a FV scheme:
  - Grid generation (CVs):
    - Divide the simulation domain into a set of discrete control volumes (CVs)
    - For maintenance of conservation, usually important that CVs don’t overlap
  - Discretize the integral/conservation equation on CVs:
    - Satisfy the integral form of the conservation law to some degree of approximation for each of the many contiguous control volumes
  - Solve the resultant discrete integral/flux equations

FV METHODS: Introduction

- FV approach has two main advantages:
  - Ensures that the discretization is conservative, locally and globally
    - Mass, Momentum and often Energy are conserved in a discrete sense
    - In general, if discrete equations are summed over all CVs, the global conservation equation are retrieved (surface integrals cancel out)
    - These local/global conservations can be obtained from Finite Differences (FDs) (strong conservative form), but they are natural/direct for a FV formulation
  - Does not require a coordinate transformation to be applied to irregular meshes
    - Can be applied directly to unstructured meshes (arbitrary polyhedra in 3D or polygons in 2D)
- In our examples, we will work with

$$\frac{d}{dt}\int_{V(t)} \rho\phi dV + \int_{S(t)} \rho\phi\,(\vec{v}.\vec{n})dA = -\int_{S(t)} \vec{q}_{\phi}.\vec{n}\; dA + \int_{V(t)} s_{\phi}\; dV$$

where $V(t)$ is any discrete control volume. We will assume for now that they don't vary in time: $V(t)=V$

FV METHODS
Several Approximations Needed

- To integrate discrete CV equation:

$$\frac{d}{dt}\int_V \rho\phi dV + \int_S \rho\phi\,(\vec{v}.\vec{n})dA = -\int_S \vec{q}_\phi.\vec{n}\;dA + \int_V s_\phi\;dV$$

  – A "time-marching method" needs to be used to integrate $\Phi = \int_V \rho\phi dV$ to the next time step(s)

$$\frac{d}{dt}\int_V \rho\phi dV = \frac{d\Phi}{dt}$$

  – Total flux estimate F_ϕ is required at the boundary of each CV

$$\int_S \vec{F}_\phi.\vec{n}\;dA = \int_S \rho\phi\,(\vec{v}.\vec{n})dA + \int_S \vec{q}_\phi.\vec{n}\;dA$$

e.g. F_ϕ = advection + diffusion fluxes

  – Total source term (sum of sources) must be integrated over each CV

$$S_\phi = \int_V s_\phi\;dV$$

    - Hence cons. eqn. becomes: $\frac{d\Phi}{dt} + \int_S \vec{F}_\phi.\vec{n}\;dA = S_\phi$

- These needs lead to basic elements of a FV scheme, but we also need to relate Φ and ϕ

FV METHODS
Several Approximations Needed, Cont'd

- "Time-marching method" for CV equation: $\frac{d\Phi}{dt} + \int_S \vec{F}_\phi . \vec{n}\; dA = S_\phi$
  - The average of ϕ over a CV cell, $\bar{\Phi} = \frac{1}{V}\int_V \rho\phi dV$, satisfies

$$V\frac{d\bar{\Phi}}{dt} + \int_S \vec{F}_\phi . \vec{n}\; dA = S_\phi$$

(since $\frac{d}{dt}\int_V \rho\phi dV = \frac{d}{dt}(V\frac{1}{V}\int_V \rho\phi dV)$)

for V fixed in time.

  - Hence, after discrete time-integration, we would have updated the cell-averaged quantities $\bar{\Phi}$
- For the total flux estimate F_ϕ at CV boundary: "Reconstruction" of ϕ from $\bar{\Phi}$
  - Fluxes are functions of $\phi \Rightarrow$ to evaluate them, we need to represent ϕ within the cell
  - This can be done by a piece-wise approximation which, when averaged over the CV, gives back $\bar{\Phi}$
  - But, each cell has a different piece-wise approximation => fluxes at boundaries can be discontinuous. Two example of remedies:
    - Take the average of these fluxes (this is a non-dissipative scheme, analogous to central differences)
    - Flux-difference splitting

FV METHODS
Basic Elements of FV Scheme

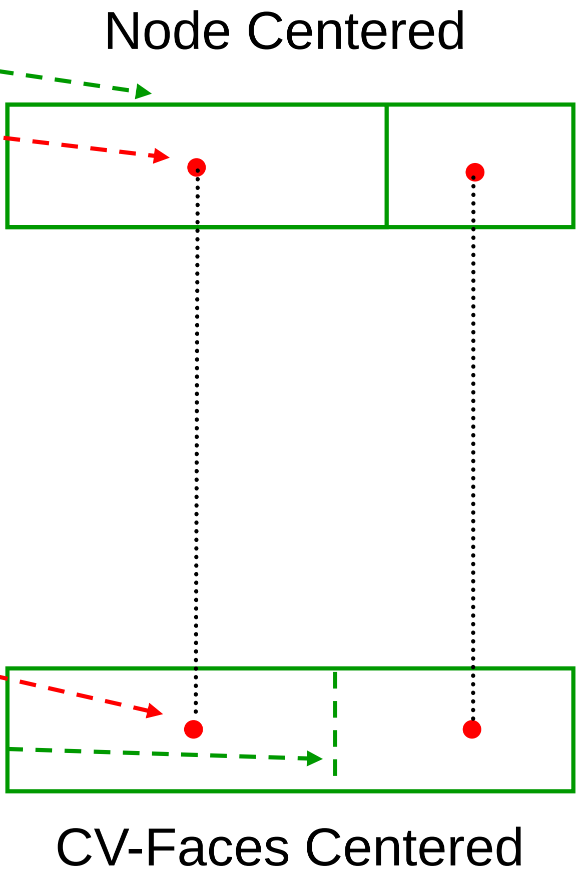
1. Given $\bar{\Phi}$ for each CV, construct an approximation to $\phi(x, y, z)$ in each CV and evaluate fluxes $F_\phi(x,y,z)$
   – Find ϕ at the boundary using this approximation, evaluate fluxes F_ϕ
   – This generally leads to two distinct values of the flux for each side of the boundary
2. Apply some strategy to resolve the flux discontinuity at the CV boundary to produce a single F_ϕ over the whole boundary
3. Integrate the fluxes F_ϕ to obtain $\int_S \vec{F}_\phi . \vec{n}\, dA$: Surface Integrals
4. Compute S_ϕ by integration over each CV: Volume Integrals
5. Advance the solution in time to obtain the new values of $\bar{\Phi}$

$$V \frac{d\bar{\Phi}}{dt} + \int_S \vec{F}_\phi . \vec{n}\, dA = S_\phi$$

Time-Marching

Different Types of FV Grids

- Usual approach (used here):
  - Define CVs by a suitable grid
  - Assign computational node to CV center
  - Advantages: nodal values will represent the mean over the CV at high(er) accuracy (second order) since node is centroid of CV
- Other approach:
  - Define nodal locations first
  - Construct CVs around them (so that CV faces lie midway between nodes)
  - Advantage: CDS approximations of derivatives (fluxes) at boundaries are more accurate (faces are midway between two nodes)



Different Types of FV Grids, Cont'd

- Other specialized variants
  - Cell centered vs. Cell vertex
  - Structured:
    - All mesh points lie on intersection two/three lines
  - vs. Unstructured:
    - Meshes formed of triangular or quadrilateral cells in 2D, or tetrahedra or pyramids in 3D
    - Cells are identified by their numbers (can not be indentified by coordinate lines, e.g. i,j)

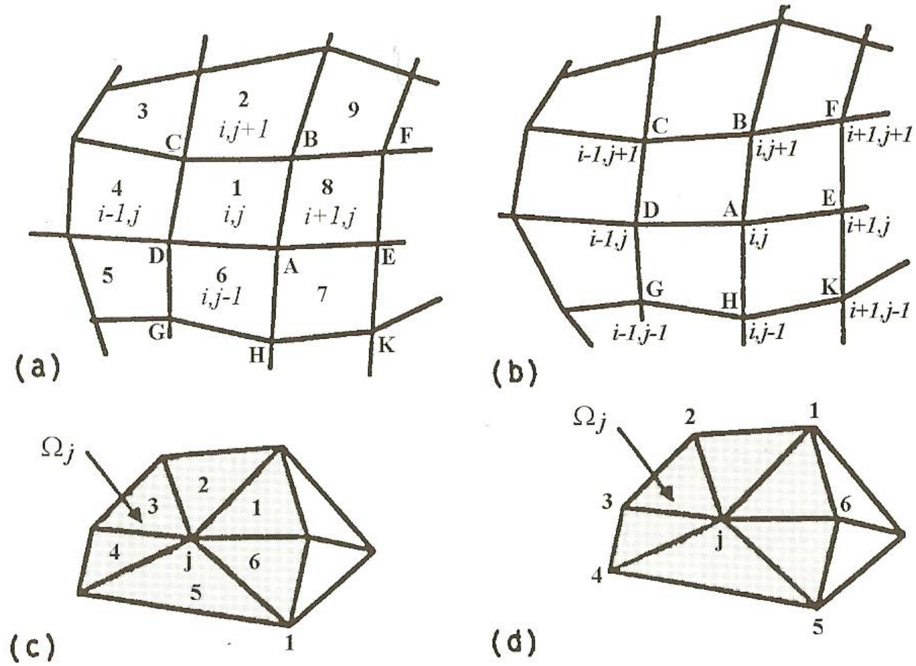


Fig. 5.2. Two-dimensional finite-volume mesh systems. (a) Cell centered structured finite-volume mesh; (b) cell vertex structured finite-volume mesh; (c) cell centered unstructured finite-volume mesh; (d) cell vertex unstructured finite-volume mesh.

Source: Cebeci, T., J. Shao, et al. *Computational Fluid Dynamics for Engineers: From Panel to Navier-Stokes Methods with Computer Programs*. Springer, 2005.

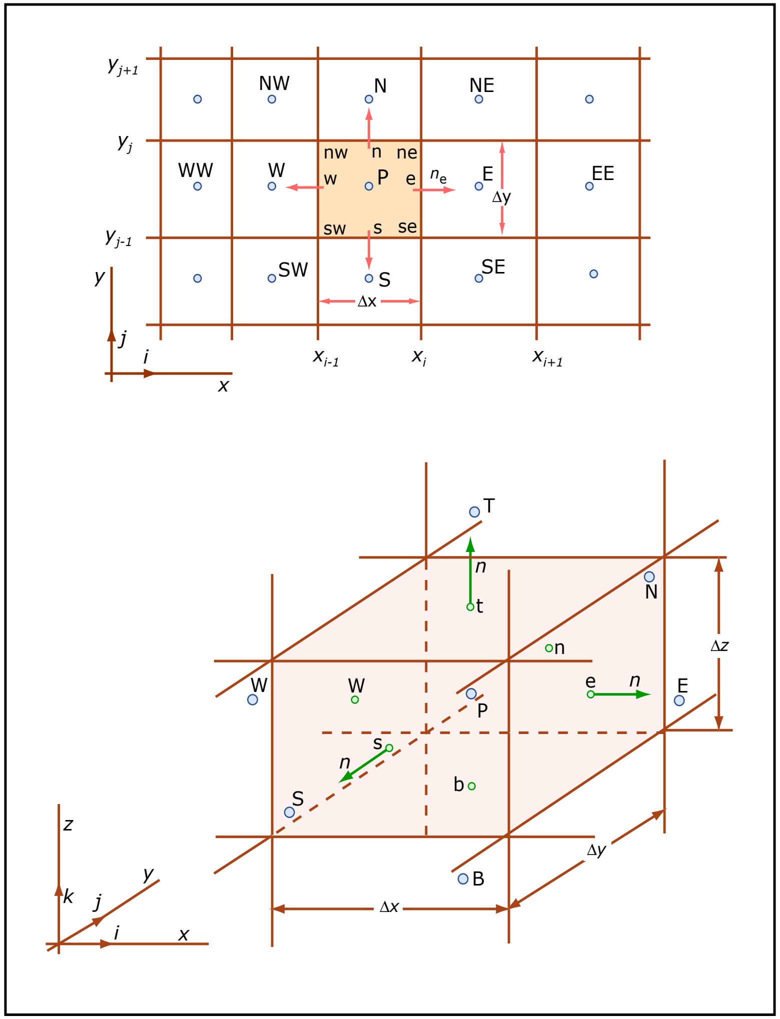
- Remarks
  - Discretization principles the same for all grid variants
    - => For now, we work with (a): Cell centered (i,j is the center of the cell, similar to FD)
    - In 3D, a cell has a finite volume (for extruded mesh, given distance ⊥ to plane is used ⇒ behaves as 2D)
  - What changes are the relations between various locations on the grid and accuracies

Approximation of Surface Integrals

- Typical (cell centered) 2D and 3D Cartesian CV (see conventions on 2 figs)
- Total/Net flux through CV boundary
  - is sum of integrals over four (2D) or six (3D) faces:

$$\int_S \vec{F}_\phi . \vec{n} \, dA = \sum_k \int_{S_k} f_\phi \, dA$$

  - for now, we will consider a single typical CV surface, the one labeled 'e'
- To compute surface integral, ϕ is needed everywhere on surface, but $\bar{\Phi}$ only known at nodal (CV center) values => two successive approximations needed:
  - Integral estimated based on values at one or more locations on the cell face
  - These cell faces values approximated in terms of nodal values



Notation used for a Cartesian 2D and 3D grid. Image by MIT OpenCourseWare.

Approximation of Surface Integrals, Cont'd

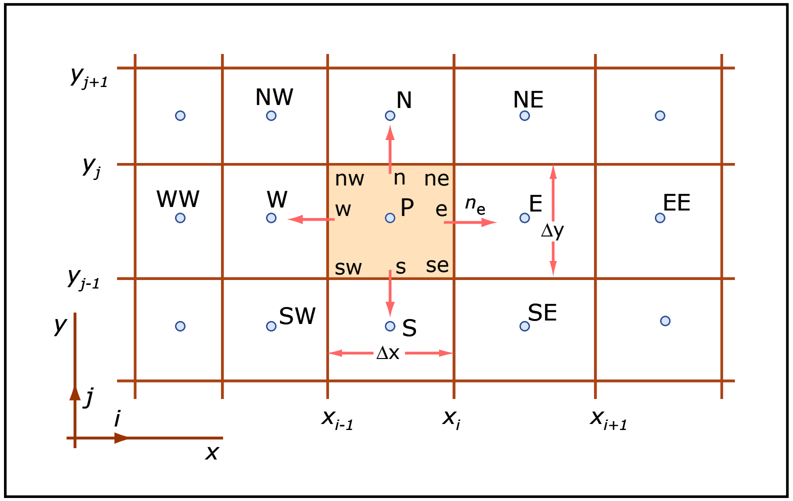
1D surfaces (2D CV)

- Goal: estimate $F_e = \int_{S_e} f_\phi \, dA$

- Simplest approximation:

  midpoint rule (2nd order)

  – F_e is approximated as a product of the integrand at cell-face center (itself approximation of mean value over surface) and the cell-face area



Notation used for a Cartesian 2D and 3D grid Image by MIT OpenCourseWare.

$$F_e = \int_{S_e} f_\phi \, dA = \bar{f}_e S_e = f_e S_e + O(\Delta y^2) \approx f_e S_e$$

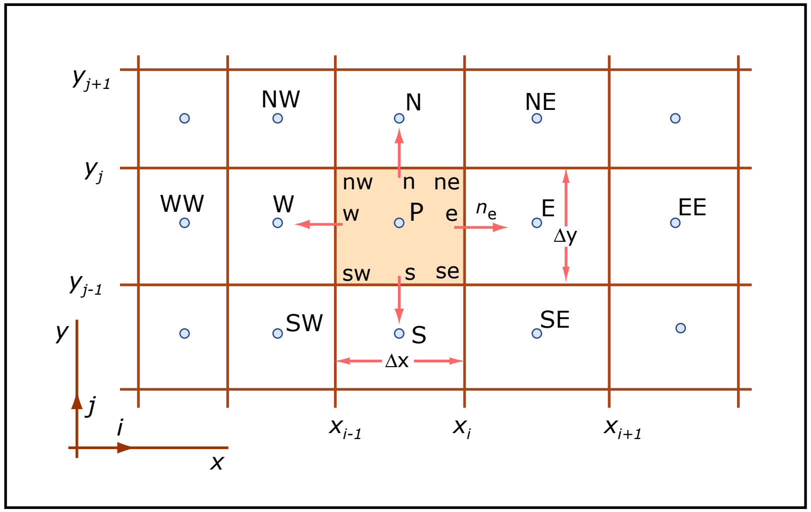
$$\left(f(y) = f(y_e) + \xi f'(y_e) + \frac{\xi^2}{2!} f''(y_e) + R_2 \right) \quad \xi = y - y_e$$

  – Since f_e is not available, it has to be obtained by interpolation

    - Has to be computed with 2nd order accuracy to preserve accuracy of midpoint rule

Approximation of Surface Integrals, Cont'd

- Goal: estimate $F_e = \int_{S_e} f_\phi \, dA$



Notation used for a Cartesian 2D and 3D grid. Image by MIT OpenCourseWare.

- Another 2nd order approximation: Trapezoid rule
  - F_e is approximated as:

$$F_e = \int_{S_e} f_\phi \, dA \approx S_e \frac{(f_{ne} + f_{se})}{2} + O(\Delta y^2)$$

  - In this case, it is the fluxes at the corners f_{ne} and f_{se} that need to be obtained by interpolation
    - Have to be computed with 2nd order accuracy to preserve accuracy
- Higher-order approximation of surface integrals require more than 2 points / locations on the cell-face
  - Simpson's rule (4th order approximation):

$$F_e = \int_{S_e} f_\phi \, dA \approx S_e \frac{(f_{ne} + 4 f_e + f_{se})}{6} + O(\Delta y^4)$$

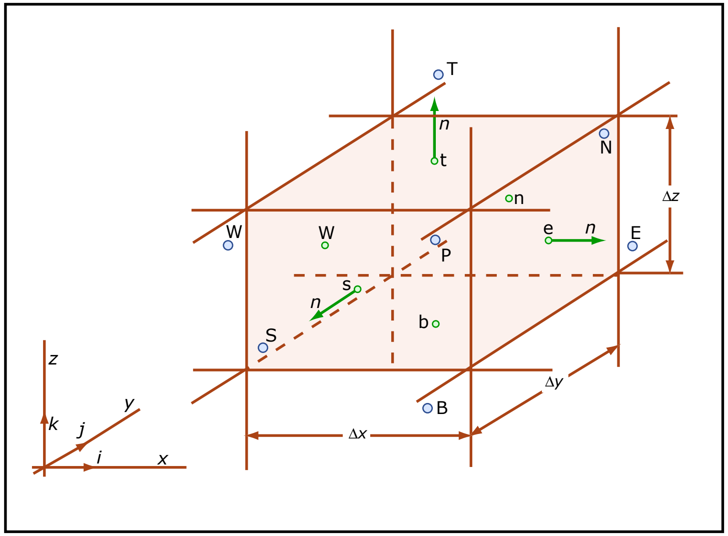
  - Values needed at 3 locations
  - To keep accuracy of integral: e.g. use cubic polynomials to estimate these values from $\bar{\Phi}_{\text{P}}$'s nearby

Approximation of Surface Integrals, Cont'd
2D surface (for **3D problems)**

- Goal: estimate $F_e = \int_{S_e} f_\phi \, dA$ for 3D CV

- Simplest approximation: still the midpoint rule (2nd order)
  - F_e is approximated as:

$$F_e = \int_{S_e} f_\phi \, dA \approx S_e f_e \quad + O(\Delta y^2, \Delta z^2)$$



Notation used for a Cartesian 2D and 3D grid. Image by MIT OpenCourseWare.

- Higher-order approximation (require values elsewhere e.g. at vertices) possible but more complicated to implement for 3D CV
- Integration easy if variation of f_e over 2D surface is assumed to have specific easy shape to integrate
  - e.g. assume 2D polynomial interpolation over surface, then complete (symbolic) integration

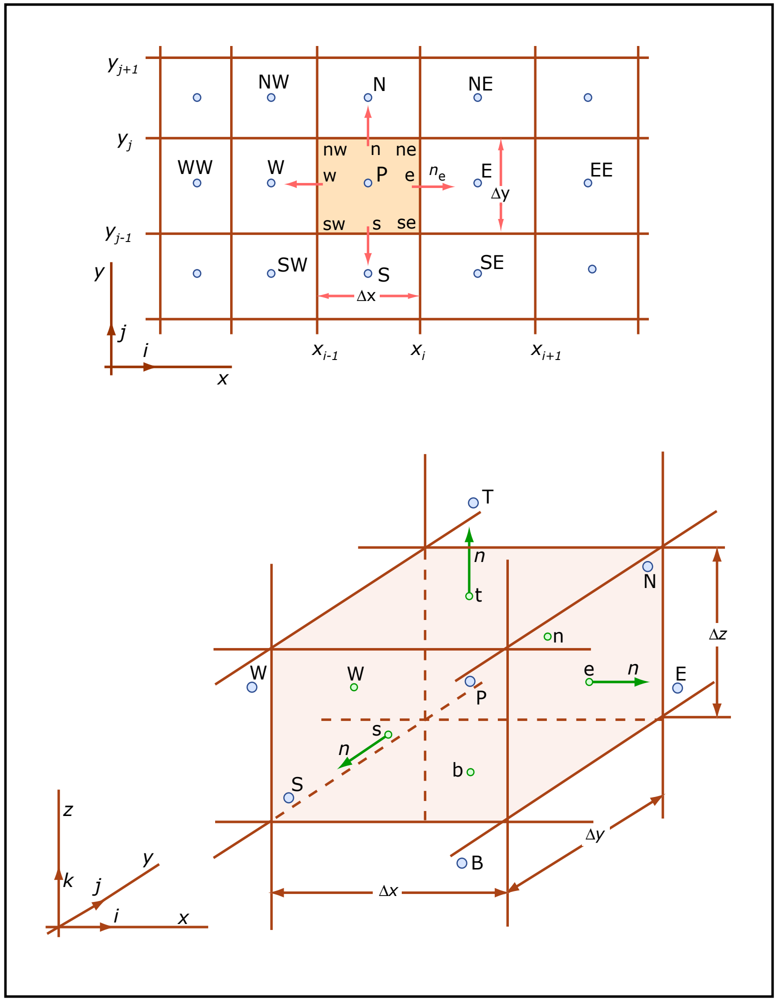
Approximation of VOLUME Integrals

- Goal: estimate $S_\phi = \int_V s_\phi \, dV$

$$\bar{\Phi} = \frac{1}{V} \int_V \rho \phi dV$$

- Simplest approximation: product of CV's volume with the mean value of the integrand (approximated by the value at the center of the node P)
  - S_P approximated as:

$$S_P = \int_V s_\phi \, dV = \bar{s}_P \, V \approx s_P \, V$$



Notation used for a Cartesian 2D and 3D grid. Image by MIT OpenCourseWare.

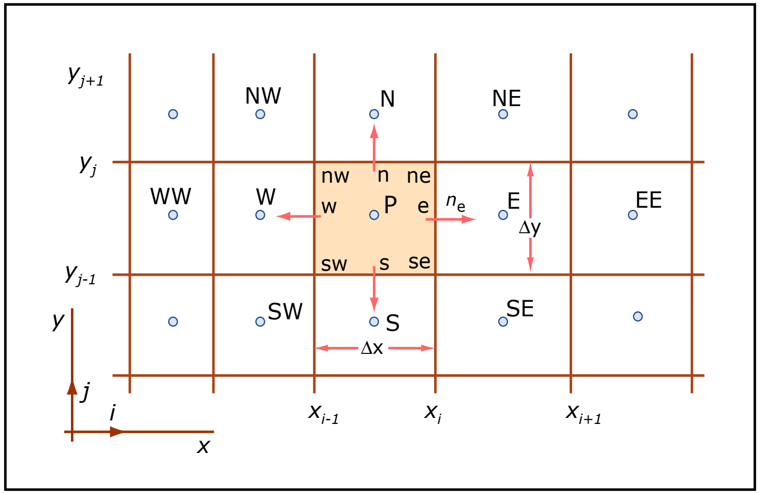
- Exact if s_p is constant or linear within CV
- 2nd order accurate otherwise
- Higher order approximation require more locations than just the center

Approximation of VOLUME Integrals

- Goal: estimate $S_\phi = \int_V s_\phi \, dV$

$$\bar{\Phi} = \frac{1}{V} \int_V \rho \phi dV$$

- Higher order approximations:
  – Requires $\bar{\Phi}$ values at other locations than P
  – Obtained either by interpolating neighbor nodal values or by using shape functions/ polynomials



Notation used for a Cartesian 2D and 3D grid. Image by MIT OpenCourseWare.

- Consider 2D case (volume integral is a surface integral) using shape functions
  – Bi-quadratic shape function leads to a 4th order approximation (9 coefficients)

$$s(x,y) = a_0 + a_1 x + a_2 y + a_3 x^2 + a_4 y^2 + a_5 xy + a_6 x^2 y + a_7 xy^2 + a_8 x^2 y^2$$

  – 9 coefficients obtained by fitting $s(x,y)$ to 9 node locations (center, corners, middles)
  – For Cartesian grid, this gives:

$$S_P = \int_V s_\phi \, dV = \Delta x \, \Delta y \left[a_0 + \frac{a_3}{12} \Delta x^2 + \frac{a_4}{12} \Delta y^2 + \frac{a_8}{144} \Delta x^2 \, \Delta y^2 \right]$$

Only 4 coefficients a_i (linear dependences cancel), but the a_i still depend on the 9 nodal values

Approximation of **VOLUME** Integrals, Cont'd 2D and 3D

- 2D case example, Cont'd
  - For a uniform Cartesian grid, one obtains the 2D integral as a function of the 9 nodal values:

$$S_P = \int_V s_\phi \, dV = \frac{\Delta x \, \Delta y}{36}\left[16 s_P + 4 s_s + 4 s_n + 4 s_w + 4 s_e + s_{se} + s_{sw} + s_{ne} + s_{nw}\right]$$

  - Since only value at node P is available, one must interpolate to obtain values at the nodal locations on the surface
  - Has to be at least 4th order accurate interpolation to retain order of integral approximation
- 3D case:
  - Techniques are similar to 2D case: above 4th order approx directly extended
  - For Higher Order
    - Integral approximation formulas are more complex
    - Interpolation of node values are more complex

Approx. of Surface/Volume Integrals: Classic symbolic formulas

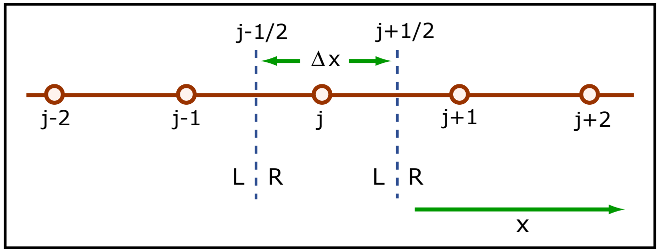
- Surface Integrals $F_e = \int_{S_e} f_\phi \, dA$
  - 2D problems (1D surface integrals)
    - Midpoint rule (2nd order): $F_e = \int_{S_e} f_\phi \, dA = \overline{f}_e S_e = f_e S_e + O(\Delta y^2) \approx f_e S_e$
    - Trapezoid rule (2nd order): $F_e = \int_{S_e} f_\phi \, dA \approx S_e \frac{(f_{ne} + f_{se})}{2} + O(\Delta y^2)$
    - Simpson's rule (4th order): $F_e = \int_{S_e} f_\phi \, dA \approx S_e \frac{(f_{ne} + 4 f_e + f_{se})}{6} + O(\Delta y^4)$
  - 3D problems (2D surface integrals)
    - Midpoint rule (2nd order): $F_e = \int_{S_e} f_\phi \, dA \approx S_e f_e \quad + O(\Delta y^2, \Delta z^2)$
    - Higher order more complicated to implement in 3D
- Volume Integrals: $S_\phi = \int_V s_\phi \, dV \,, \quad \overline{\Phi} = \frac{1}{V} \int_V \rho \phi dV$
  - 2D/3D problems, Midpoint rule (2nd order): $S_P = \int_V s_\phi \, dV = \overline{s}_P \, V \approx s_P \, V$
  - 2D, bi-quadratic (4th order, Cartesian): $S_P = \frac{\Delta x \, \Delta y}{36} [16 s_P + 4 s_s + 4 s_n + 4 s_w + 4 s_e + s_{se} + s_{sw} + s_{ne} + s_{nw}]$

Summary: 3 basic steps to set-up a FV scheme

- Grid generation ("create CVs")
- Discretize integral/conservation equation on CVs
  - This integral eqn. is: $\frac{d\Phi}{dt} + \int_S \vec{F}_\phi . \vec{n} \, dA = S_\phi$
  - Which becomes for V fixed in time: $V \frac{d\bar{\Phi}}{dt} + \int_S \vec{F}_\phi . \vec{n} \, dA = S_\phi$

    where $\bar{\Phi} = \frac{1}{V} \int_V \rho \phi dV$ and $S_\phi = \int_V s_\phi \, dV$
  - This implies:
    - The discrete state variables are the averaged values over each cell (CV): $\bar{\Phi}_P$'s
    - Need rules to compute surface/volume integrals as a function of ϕ within CV
      - Evaluate integrals as a function of ϕ_e values at points on and near CS/CV.
      - Need to interpolate to obtain these ϕ_e values from averaged values $\bar{\Phi}_P$'s of nearby CVs
    - Other approach: impose piece-wise function ϕ within CV, ensures that it satisfies $\bar{\Phi}_P$'s constraints, then evaluate integrals (surface and volume). We use this in the examples next.
    - Select scheme to resolve/address discontinuities
- Solve resultant discrete integral/flux eqns: (Linear) algebraic system for $\bar{\Phi}_P$'s

One-Dimensional Examples: Generic 1D FV



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- Grid generation (fixed CVs)
  - Consider equispaced grid: $x_j = j\Delta x$
  - Control volume j extends from $x_j - \Delta x/2$ to $x_j + \Delta x/2$
  - Boundary (surface) values are: $\phi_{j\pm 1/2} = \phi(x_{j\pm 1/2})$
  - Boundary total fluxes (convective+diffusive) are: $f_{j\pm 1/2} = f(\phi_{j\pm 1/2})$
  - Average cell and source values:

$$\bar{\Phi}_j(t) = \frac{1}{V}\int_V \rho\phi \, dV = \frac{1}{\Delta x}\int_{x_{j-1/2}}^{x_{j+1/2}} \phi(x,t)\, dx$$

$$S_j(t) = \int_V s_{\phi_j} \, dV = \int_{x_{j-1/2}}^{x_{j+1/2}} s_\phi(x,t)\, dx$$

- Discretize generic integral/conservation equation on CVs
  - The integral form $V\frac{d\bar{\Phi}}{dt} + \int_S \vec{F}_\phi . \vec{n} \, dA = S_\phi$ becomes:

$$\frac{d\left(\Delta x \, \bar{\Phi}_j\right)}{dt} + f_{j+1/2} - f_{j-1/2} = \int_{x_{j-1/2}}^{x_{j+1/2}} s_\phi(x,t)\, dx$$

One-Dimensional Examples, Cont'd

Note: Cell-average vs. Center value

- With $\xi = x - x_j$ and a Taylor series expansion

$$\bar{\Phi}_j(t) = \frac{1}{\Delta x}\int_{x_{j-1/2}}^{x_{j+1/2}} \phi(x,t)\,dx$$

$$= \frac{1}{\Delta x}\int_{-\Delta x/2}^{\Delta x/2}\left[\phi_j + \xi \left.\frac{\partial \phi}{\partial x}\right|_j + \frac{\xi^2}{2}\left.\frac{\partial^2 \phi}{\partial x^2}\right|_j + R_2\right] d\xi$$

$$= \phi_j + \frac{\Delta x^2}{24}\left.\frac{\partial^2 \phi}{\partial x^2}\right|_j + O(\Delta x^4)$$

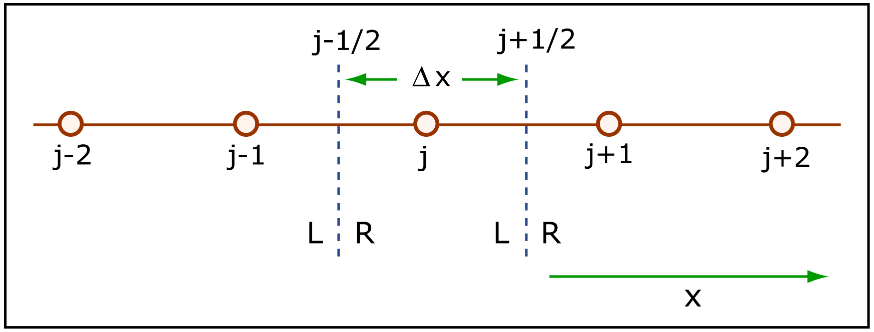

Image by MIT OpenCourseWare.

$$\Rightarrow \quad \bar{\Phi}_j(t) = \phi_j + O(\Delta x^2)$$

- Thus: cell-average value and center value differ only by second order term

One-Dimensional Example I
Linear Convection (Sommerfeld) Eqn: $\frac{\partial \phi(x,t)}{\partial t} + \frac{\partial c\,\phi(x,t)}{\partial x} = 0$

- With convection only, our generic 1D eqn.

$$\frac{d\left(\Delta x\, \bar{\Phi}_j\right)}{dt} + f_{j+1/2} - f_{j-1/2} = \int_{x_{j-1/2}}^{x_{j+1/2}} s_\phi(x,t)\, dx$$

becomes:

$$\frac{d\left(\Delta x\, \bar{\Phi}_j\right)}{dt} + f_{j+1/2} - f_{j-1/2} = 0$$

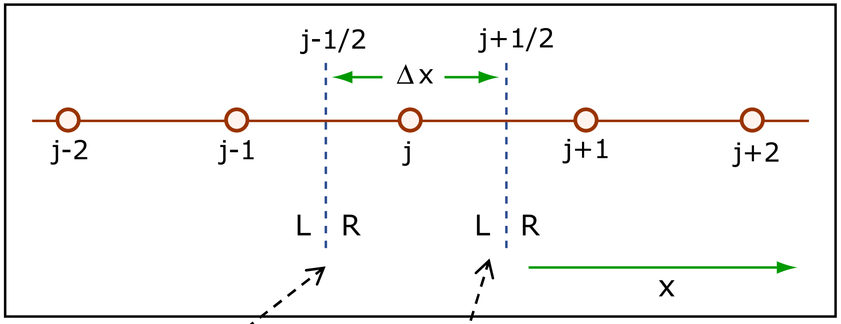

Image by MIT OpenCourseWare.

- Compute surface/volume integrals as a function of ϕ within CV
  – Here impose/choose first piecewise-constant approximation to $\phi(x)$:

$$\phi(x) = \bar{\phi}_j \quad \forall\, x_{j-1/2} \le x \le x_{j+1/2}$$

  – This gives simple flux terms. The only issue is that they differ depending on the cell from which the flux is computed:

$$f^L_{j+1/2} = f(\phi^L_{j+1/2}) = c\bar{\phi}_j \qquad f^R_{j+1/2} = f(\phi^R_{j+1/2}) = c\bar{\phi}_{j+1}$$

$$f^L_{j-1/2} = f(\phi^L_{j-1/2}) = c\bar{\phi}_{j-1} \qquad f^R_{j-1/2} = f(\phi^R_{j-1/2}) = c\bar{\phi}_j$$

One-Dimensional Example I
Linear Convection (Sommerfeld) Eqn, Cont'd

- Now, we have obtained the fluxes at the CV boundaries in terms of the CV-averaged values

- We need to resolve the flux discontinuity => average values of the fluxes on either side, leading the (2nd order) estimates:

$$\hat{f}_{j-1/2} = \frac{f^L_{j-1/2} + f^R_{j-1/2}}{2} = \frac{c\bar{\phi}_{j-1} + c\bar{\phi}_j}{2} \qquad \hat{f}_{j+1/2} = \frac{f^L_{j+1/2} + f^R_{j+1/2}}{2} = \frac{c\bar{\phi}_j + c\bar{\phi}_{j+1}}{2}$$

- Substitute into integral equation

$$\frac{d\left(\Delta x\, \bar{\Phi}_j\right)}{dt} + f_{j+1/2} - f_{j-1/2} \approx \frac{d\left(\Delta x\, \bar{\phi}_j\right)}{dt} + \hat{f}_{j+1/2} - \hat{f}_{j-1/2} = \frac{d\left(\Delta x\, \bar{\phi}_j\right)}{dt} + \frac{c\bar{\phi}_j + c\bar{\phi}_{j+1}}{2} - \frac{c\bar{\phi}_{j-1} + c\bar{\phi}_j}{2}$$

$$\Rightarrow \Delta x \frac{d\bar{\phi}_j}{dt} + \frac{c\bar{\phi}_{j+1} - c\bar{\phi}_{j-1}}{2} = 0$$

- With periodic BCs, storing all cell-averaged values into a vector $\bar{\mathbf{\Phi}}$

$$\frac{d\,\bar{\mathbf{\Phi}}}{dt} + \frac{c}{2\Delta x}\mathbf{B}_P(-1,0,1)\bar{\mathbf{\Phi}} = 0$$

(where $\mathbf{B}_P$ is a circulant tri-diagonal matrix, P for periodic)

2.29 Numerical Fluid Mechanics
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