Lecture # 7 Course Notes

Dealing with the Continuum – Virtual Modes:

When sound energy reflects off the seabed, a portion of it past critical angle is lost by transmission to infinity. The same is true to a lesser extent for the sea surface (which is not a perfect pressure release surface in reality.) In analogy with other physical systems (e.g. atoms, molecules, nuclei), the energy that is not bound (trapped) by the ocean's boundaries is called continuum energy. This is actually very important energy to consider, as many ocean processes scatter energy continually to angles above the critical grazing angle, and it constitutes a considerable loss mechanism for sound in the ocean.

The formal mathematical way to deal with the ocean acoustic continuum, which is described in a number of standard texts, is the "EJP Branch Line Integral." (EJP are Ewing, Jardetsky and Press). This is a rather involved integral to evaluate, and while one gets an exact answer at the far end, there is little physical insight gained.

However, there is a way to approximately evaluate the EJP integral that gives a lot of physical insight, as well as a reasonably correct answer. This is the "method of virtual modes" which appeared in a 1976 paper by Tindle, Stamp and Guthrie (J.Sound and Vib. 49, 231-240) entitled "Virtual modes and the surface boundary condition in underwater acoustics." I will follow this paper in the notes, as well as ask people to read the original.

TSG start out with the point source wave equation for pressure (so the normalization is $1/\rho$ and not ρ, remember!), which in cylindrical coordinates is:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\partial p}{\partial r}\right)+\frac{\partial^2 p}{\partial z^2}+\frac{\omega^2}{c^2}p=\frac{-2}{r}Q\delta(r)\delta(z-z_0)$$

Here, the source is at (r=0, z_0) and the receiver is at (r,z). As before, we separate the homogeneous version of this wave equation to get an $H_0^1(kr)$ radial pressure dependence and also a Helmholtz equation for the vertical modes, i.e.

$$\frac{d^2}{dz^2} Z(z) + \left[\frac{\omega^2}{c^2(z)} - k_r^2 \right] Z(z) = 0$$

We now assume a Pekeris-like waveguide, i.e one with a non-trivial water column soundspeed profile overlying an isovelocity half space. The boundary conditions across the interfaces are the usual ones, i.e.

$$Z(0)=0 \quad ; \quad Z(H^-) = Z(H^+) \quad ; \quad \frac{Z'(H^-)}{\rho_1} = \frac{Z'(H^+)}{\rho_2}$$

This system will have both discrete normal modes and continuous ones. The normalized modes of the system may be written as:

$$U_n(z) = N_n \rho^{-1/2}(z) Z_n(z)$$

where

$$N_n = \left[\int_0^\infty \frac{1}{\rho(z)} Z_n^2(z) dz \right]^{-1/2}$$

For discrete modes, this is business as usual. However, for the continuous modes where the mode function is nonzero at infinity, this normalization diverges. We'll need to deal with this.

Let's now look at the modes in the water and in the bottom.

Since c_2 =const. , the solution in the bottom is either sinusoidal or a decaying exponential. Thus we write:

$$Z(z) = A \sin(\gamma_2 z + \delta) \qquad \text{for} \quad z>H \text{ and } k_r < \frac{\omega}{c_2}$$

$$Z(z) = B \exp(-|\gamma_2| z) \qquad \text{for} \quad z>H \text{ and } k_r > \frac{\omega}{c_2}$$

where γ_2 is the vertical wavenumber in the bottom given by:

$$\gamma_2 = \left[\frac{\omega^2}{c_2^2} - k_r^2\right]^{-1/2}$$

The constants A,B and the phase δ are to be determined by the normalization and the boundary conditions. (A is important for the continuum, whereas B is just our old trapped mode result).

Looking at the modes in the water (z<H), we shift our notation for the mode to Z(z) → ξ(z). We also assume that ξ(z) is either known, or that we can obtain it easily (e.g. for a Pekeris waveguide, $\xi(z) = \sin(\gamma_1 z)$ where γ_1 is the vertical wavenumber in the water column.)

With forms for the modes in hand, we can proceed to matching boundary conditions and obtaining the constants we mentioned before.

$$\xi(H) = A\sin(\gamma_2 H + \delta)$$

$$\xi'(H)/\rho_1 = \gamma_2 A\cos(\gamma_2 H + \delta)/\rho_2$$

Squaring both equations and adding, we get the result:

$$\xi^2(H) + \left[\frac{\xi'(H)\rho_2}{\gamma_2\rho_1}\right]^2 = A^2(\sin^2 x + \cos^2 x) = A^2$$

where

$$x \equiv \gamma_2 H + \delta .$$

So we get A in terms of known quantities! Now let's get the normalization. To do this, we introduce an artificial boundary condition at a large depth L which we later allow to approach infinity, i.e. Z(L)=0 for L large. This in turn gives us the equation:

$\gamma_2 L + \delta = m\pi$ where m=0,1,2,3….

This "extra" boundary condition gives us discrete k_m eigenvalues which become continuous in the limit $L \to \infty$. We can now (at the expense of having an arbitrary parameter L floating around) do the normalization integral that was previously undefined for continuum modes.

$$N_m = \left[\int_0^{L \to \infty} \frac{1}{\rho(z)} Z_m^2(z) dz \right]^{-1/2}$$

If we take the difference between the shapes of the trapped and continuous modes to be small (e.g. both are sinusoids), and the water layer as being very thin compared to the depth L, we can write:

$$N_m \approx \left[\int_0^{L} (continuum\ \mathrm{mod}\, es) dz \right]^{-1/2}$$

Thus $Z^2(z) = A^2 \sin^2(\gamma_2 z + \delta)$

We now can take δ=0 to do the integrals, since: 1) this now gets the surface boundary condition right and 2) there is no boundary H in the integral that specifies any δ. Thus we get:

$$I = \int_0^L \frac{A^2}{\rho_2} \sin^2(\gamma_2 z) dz$$

Setting $\beta = \gamma_2 z$, we get the standard integral

$$I = \frac{A^2}{\gamma_2 \rho_2} \int_0^{\gamma_2 L} \sin^2 \beta d\beta$$

Using $\gamma_2 L = m\pi$ and evaluating the integral, one gets;

$$I = \frac{1}{N_m^2} = \frac{A^2 L}{2\rho_2}$$

Thus we finally get for the normalization factor:

$$N_m = \left(\frac{2\rho_2}{L} \right)^{1/2} (\xi^2(H) + \left[\frac{\xi'(H)\rho_2}{\gamma_2 \rho_1} \right]^2)^{-1/2}$$

The above normalization is called "box normalization" in that it depends on the size L of the box we created. Now we have a large amount of discrete eigenvalues and normalized eigenfunctions, so we can write the field as a sum of discrete and closely spaced "continuum" modes.

$$p(r,z,t) = i\pi Q\left[\sum_{n=discrete} U_n(z_0)U_n(z)H_0^1(k_n r) + \sum_{m=continuum} U_m(z_0)U_m(z)H_0^1(k_m r)\right]\exp(-i\omega t)$$

We can use this large sum directly in some calculations, but the usual way to do things is to convert the continuum sum into an integral. We do that as follows. Using

$\gamma_2 L = m\pi$ implies that $d\gamma_2 L = \pi dm$, and letting $\sum_m \to \int dm$, we get

$$\sum_{m=continuum} U_m(z_0)U_m(z)H_0^1(k_m r) \to \int U(z_0)U(z)H_0^1(k_r)\frac{L}{\pi}d\gamma_2$$

This factor of L neatly cancels the 1/L factor in the box normalization, so that our final answer is independent of the box size L we initially chose! We have a normalizable integral for the continuum! If we put together all the factors we've assembled, we get that the continuum contribution to the pressure is given by the integral:

$$p^c(r,z,t) = 2iQ\frac{\rho_2}{\rho_1}\int_0^\infty \frac{\xi(z_0)\xi(z)}{\left(\xi^2(H) + \left[\frac{\xi'(H)\rho_2}{\gamma_2\rho_1}\right]^2\right)} H_0^1(k_r r)d\gamma_2$$

This is the aforementioned EJP branch line integral, but with γ_2 as the integration variable instead of k_r. (Using $\gamma_2 d\gamma_2 = -k_r dk_r$ gives the usual EJP form.)

We want to evaluate this integral approximately. We do this by noting that there are three regions for the behavior of the integral.

1) $\gamma_2 > \omega/c_2$ In this region, the Hankel function decays exponentially, and thus there is negligible contribution to the integral.

2) "Moderate" γ_2 In this region, the numerator oscillates, and again there is negligible contribution

3) γ_2 small In this case, the numerator varies slowly, and the integral is governed by the denominator.

The denominator in our integral above has a well defined minimum at the value of γ_2 for which $\xi'(H) = 0$. This is a resonant condition for the integral. For the Pekeris waveguide, $\xi(z) = \sin(\gamma_1 z)$,so $\xi'(H) = 0$ corresponds to zero slope for the mode functions at the bottom. This happens when

$\gamma_1 H = (l - 1/2)\pi$ where l is an integer.

But this is the hard bottom waveguide eigenvalue equation!? Who ordered that? This is the Pekeris waveguide, not the hard bottom one! (There is a very simple physical reason – see the homework.)

Note also that the hard bottom eigenvalues introduce an l index to the resonances of the integral ! Thus we have γ_1, γ_2, k_r → $\gamma_{1l}, \gamma_{2l}, k_l$.

Given the resonances, it is reasonable to evaluate the integral only in the region of the resonances- they should dominate. Let us evaluate one resonance as an example. The variable of integration is expanded around the resonance as $\gamma_2 \approx \gamma_{2l} + \varepsilon$. This quickly gives, using asymptotic forms and the hard bottom resonance condition,

$$p_l(r,z) = 2iQ\frac{\rho_2}{\rho_1}\sin(\gamma_{1l}z)\sin(\gamma_{1l}z_0)\sqrt{\frac{2}{\pi k_l r}}\exp(ik_l r - i\frac{\pi}{4})I_l$$

where

$$I_l = \int \frac{\exp(-i\gamma_{2l}r\varepsilon/k_l)}{1 + \left[(\rho_1/\rho_2)^2 + (\gamma_{2l}/\gamma_{1l})^2\right]\varepsilon^2 H^2} d\varepsilon$$

Strictly, this integral should be evaluated only in the near vicinity of the resonance. However, since the integrand vanishes rapidly for large ε, the limits may be extended to +/- ∞. Then the integral is in a standard tabular form. So,if we evaluate it and add the "virtual mode" (the l resonances) sum to the trapped modes, we get:

$$p(r,z,t) = \sum_n UsualPe\ker isTrappedModes + i\pi Q \sum_l \frac{2\exp(-\rho_1\gamma_{2l} r / H\rho_2 k_l \sqrt{1-(\rho_1\gamma_{2l}/\rho_2\gamma_{1l})^2}}{H\sqrt{1-(\rho_1\gamma_{2l}/\rho_2\gamma_{1l})^2}} F$$

where

$$F = \sin(\gamma_{1l} z)\sin(\gamma_{1l} z_0)\sqrt{\frac{2}{\pi k_l r}}\exp(ik_l r - i\frac{\pi}{4})$$

We see that we now have a small number of trapped modes (at low frequency, of course!) and a small number of resonant "virtual modes." What is more, the continuum virtual modes continue the trapped mode set! That is, if n=1,2, then l=3,4 etc. Also, the virtual modes decay exponentially in range, which is expected from a steady leakage of energy to the bottom. (To show this is a simple homework problem).

We will see some more of the interesting behavior of the virtual modes soon, after going through the Hankel transform and looking at the depth dependent Green's function.

The Hankel (Fourier-Bessel) Transform

For a cylindrically symmetric system such as the ocean is (approximately), the 2D Fourier transform collapses into the 1D Hankel transform. This transform is very useful for ocean acoustics, as we will see.

The Hankel transform pair is simply given as:

$$g(k_r) = \int_0^\infty r J_0(k_r r) p(r) dr$$

$$p(r) = \int_0^\infty k_r J_0(k_r r) g(k_r) dk_r$$

where $p(r)$ is the pressure field as a function of range and $g(k_r)$ is the "depth dependent Green's function." This transform is interesting in that: 1) one can clearly see the modal structure of the waveguide in it, 2) it is the basis for "wavenumber integration techniques" which we will discuss, and 3) it produces quantities that are very useful for inverses to determine ocean and seabed properties.

Let us look in some detail at how we go from the wave equation to the Hankel transform to the Green's function to normal modes, by way of explaining the above.

We start with the inhomogeneous Helmholtz equation:

$$\left[\nabla^2 + k^2(\vec{r})\right]p(\vec{r}) = -4\pi f(\vec{r})$$

We then put this in Green's function form by considering a delta function (point) source;

$$\left[\nabla^2 + k^2(\vec{r})\right]G(\vec{r},\vec{r}_0) = -4\pi\delta(\vec{r}-\vec{r}_0)$$

In cylindrical coordinates, we write this as:

$$\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{\partial^2}{\partial z^2} + k^2(z)\right]G(r,z,z_0) = \frac{-2}{r}\delta(r)\delta(z-z_0)$$

We can then Hankel Transform this equation using the identity

$$H.T.\left\{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)F(r)\right\} = -k_r^2 f(k_r)$$

This yields the result:

$$\left(\frac{d^2}{dz^2} + k^2(z) - k_r^2\right)g(k_r,z,z_0) = -2\delta(z-z_0)$$

where

$g(k_r, z, z_0) = H.T.[G(r, z, z_0)]$

Note that $k_z^2 = k^2 - k_r^2$ and that the 1-D homogeneous Helmholtz equation for modes that we saw before

$$\left(\frac{d^2}{dz^2} + k_z^2\right)\phi(z) = 0$$

We see the same structure in the two equations, which leads one to expect that $g(k_r, z, z_0)$ will show modal structure.

We can now use the endpoint method to write down the standard Green's function solution

$$g(k_r, z, z_0) = \frac{-2}{W(z_0)}\psi_s(k_r, z)\psi_B(k_r, z_0) \qquad 0 \le z \le z_0$$

$$g(k_r, z, z_0) = \frac{-2}{W(z_0)}\psi_s(k_r, z_0)\psi_B(k_r, z) \qquad z_0 \le z \le h$$

where the Wronskian is

$$W(z_0) = \psi_S(z_0)\psi_B'(z_0) - \psi'_S(z_0)\psi_B(z_0)$$

And ψ_s and ψ_B are linearly independent solutions of the homogeneous differential equation which satisfy homogeneous boundary conditions on the surface and bottom respectively.

We now put the boundary conditions in terms of the reflection coefficients.

$$\psi_s(z) = A[\exp(-i\gamma z) + R_S \exp(i\gamma z)] = A[\exp(-i\gamma z) - \exp(i\gamma z)]$$

since $R_s = -1$ for a pressure release surface.

$$\psi_B(z) = B[\exp(i\gamma z) + R_B(k_r)\exp(2i\gamma h)\exp(-i\gamma z)]$$

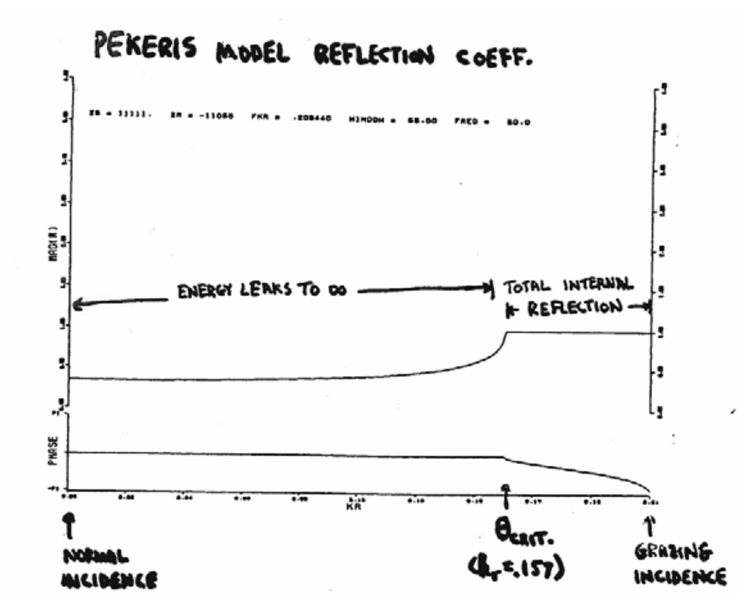
where $\gamma^2 = k^2 - k_r^2 \equiv k_z^2$.

If we substitute the ψ_s and ψ_B into the Green's function solution, and do some algebra, we come up with the useful form:

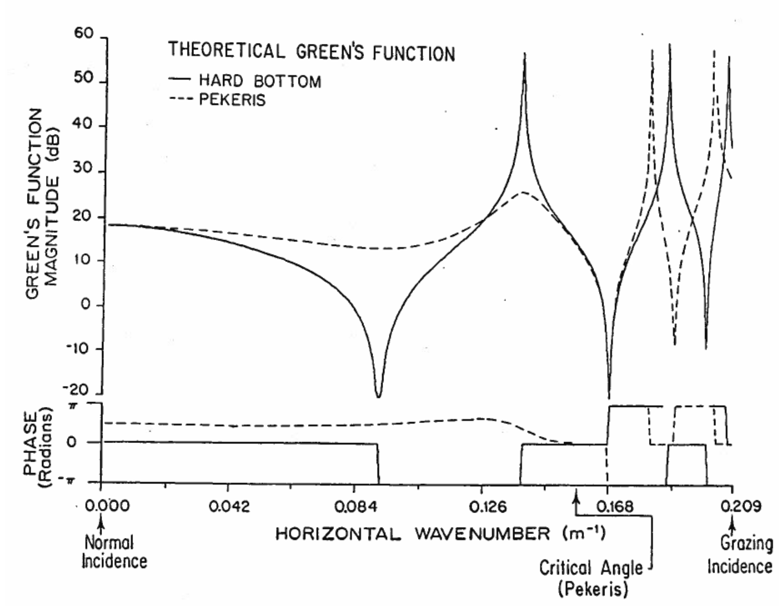
$$g(k_r, z, z_0) = -\frac{\exp(i\gamma|z - z_0|) - \exp(i\gamma(z + z_0) + R_B(k_r)\exp(2i\gamma h)[-\exp(-i\gamma|z - z_0|) + \exp(-i\gamma(z + z_0)]}{i\gamma[1 + R_B(k_r)\exp(2i\gamma h)]}$$

We immediately note that $[1 + R_B(k_r)\exp(2i\gamma h)]=0$ is the normal mode eigenvalue equation, so that $g(k_r, z, z_0)$ has poles at the mode eigenvalues! We have assumed nothing about the modal structure in doing this, but see that it occurs naturally in the formalism!

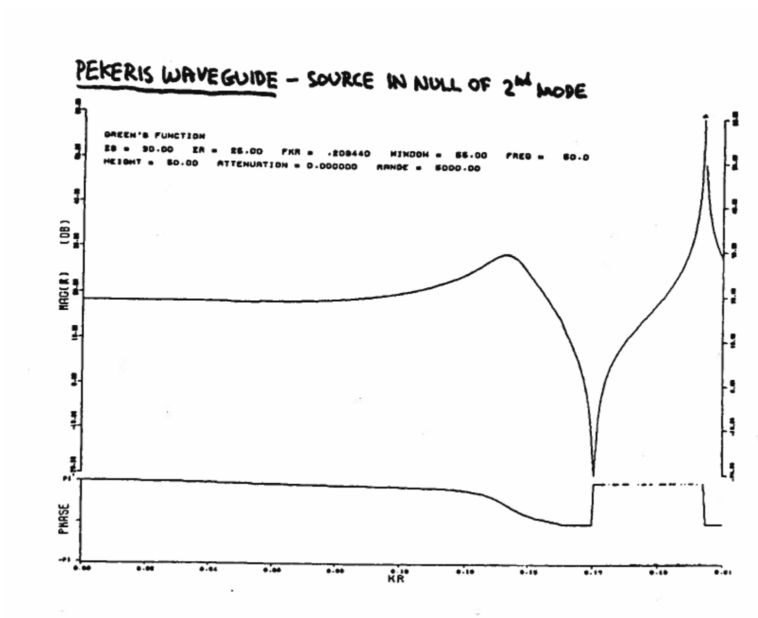
The physical input into the above scheme is the geometry of the waveguide and the plane wave reflection coefficients of the surface (trivial) and bottom (non-trivial). We can get the plane wave reflection coefficient for the bottom rather simply for the two cases of the "hard bottom" waveguide and the Pekeris waveguide (where the bottom is an infinite isovelocity halfspace), so let's look at $g(k_r, z, z_0)$ for these cases.



Typical Pekeris model bottom reflection coefficient (magnitude and phase) for a sandy bottom waveguide.

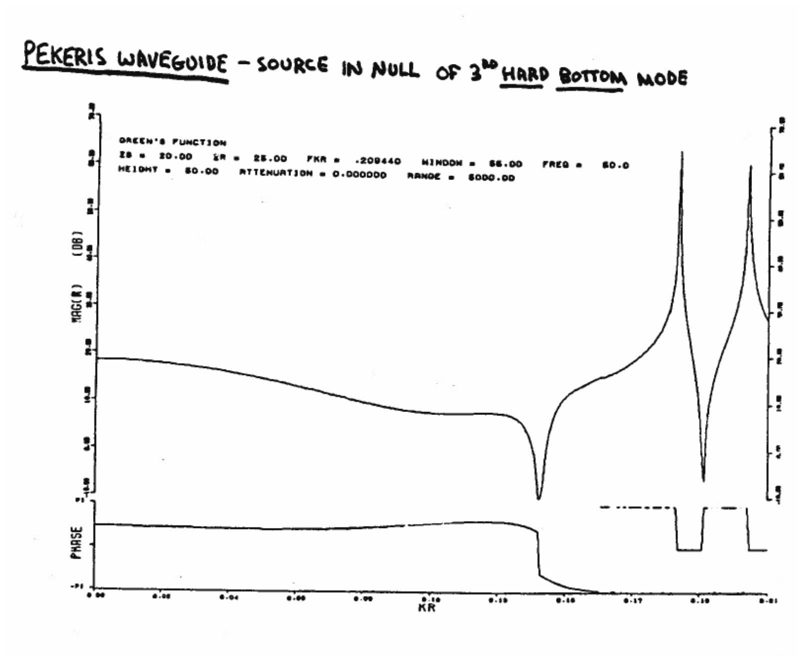


Theoretical Green's functions for both the hard bottom and Pekeris waveguides for a 50 Hz source in a 50m deep ocean waveguide. There are three hard bottom modes, and two Pekeris trapped modes and a virtual mode (right at the location of the third hard bottom mode!) Notice also the pi phase changes at the poles and nulls.



Frisk, G. V. and J. F. Lynch. "Shallow water waveguide characterization using the Hankel transform." *JASA* 79, no. 1 (1984): 205-16.

If you put the source or receiver in the null of a normal mode, then no energy is either transmitted or received in that mode. Here, we've nulled out the second Pekeris trapped mode, leaving the first mode and the virtual mode intact



Though it seems odd, the "virtual mode" acts just like a normal mode as regards being nulled out by putting the source/receiver in the null a the appropriate hard bottom waveguide mode (here mode #3). This is seen in the equations derived for the virtual modes in TSG.

We have done the previous calculations/examples going from the geoacoustic parameters to the reflection coefficient to $g(k_r, z, z_0)$. In the "practical universe", we would like to reverse this sequence, i.e. measure $g(k_r, z, z_0)$ (or something that provides it) and then obtain the geoacoustic parameters that describe the seabed. This is a very important inverse problem in ocean acoustics. To do this, we will go in the sequence: $p(r)$ → $g(k_r, z, z_0)$ →reflection coefficient → geoacoustic parameters. Thus we go:

Measure $p(r)$ using a "synthetic aperture" type of technique (see picture below)

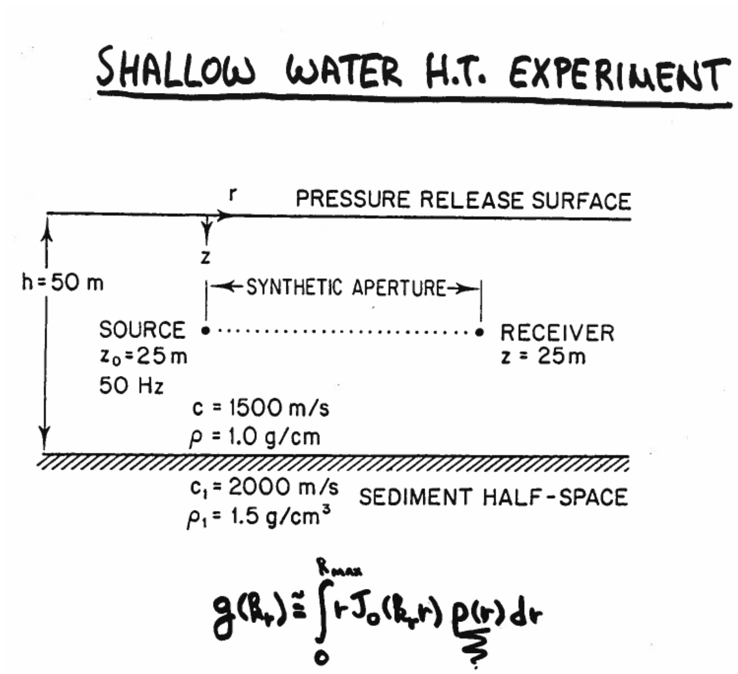
Use the inverse Hankel transform on $p(r)$ to get $g(k_r, z, z_0)$

Use a simple algebra connection to get $R_B(k_r)$ from $g(k_r, z, z_0)$, i.e.

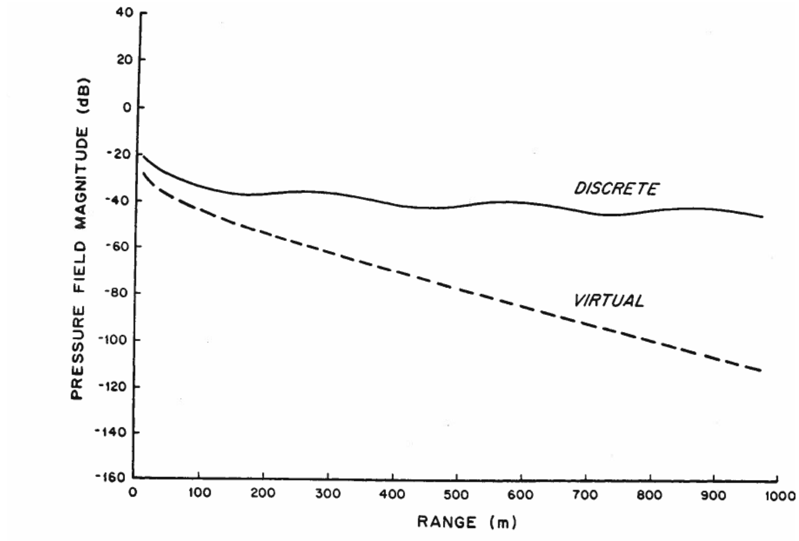
$$R_B(k_r) = \frac{[-i\gamma g - \exp(i\gamma|z - z_0|) + \exp(i\gamma(z + z_0)] \exp(-2i\gamma h)}{[i\gamma g - \exp(-i\gamma|z - z_0|) + \exp(-i\gamma(z + z_0)]}$$

Use iterated forward models or inverse modeling techniques to extract the geoacoustic parameters from either $g(k_r, z, z_0)$ or $R_B(k_r)$.

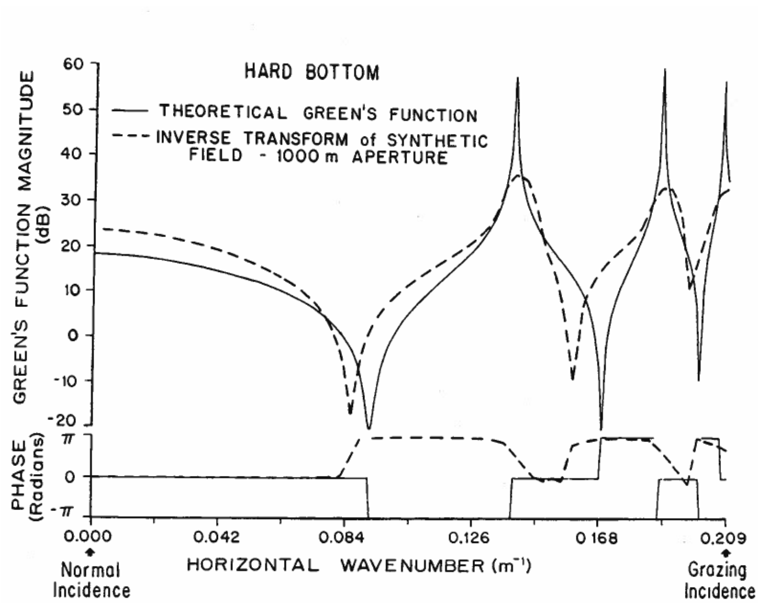
A very straightforward experiment one can conduct in the ocean to get $p(r)$ is to tow a source away from a receiver (or vice-versa by reciprocity). This gives one a finite aperture $p(r)$ to work with, as shown in the picture below, which has an R_{max} securely attached as the upper limit of the inverse HT integral.



Synthetic aperture array experiment used to get $p(r)$. Note we have a finite aperture approximation to the HT.

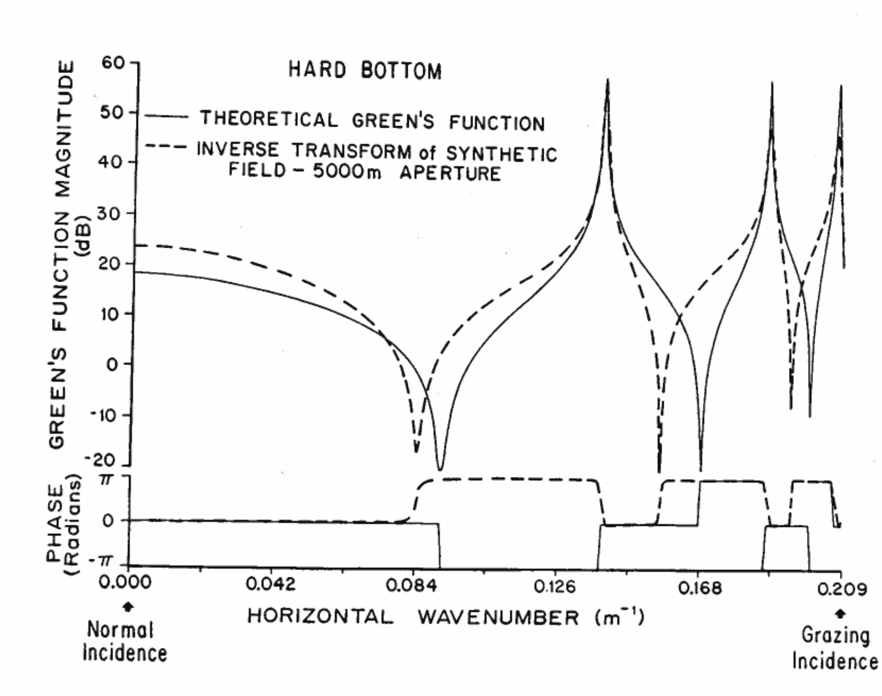


The pressure field $p(r)$ that one would see from a Pekeris waveguide with two trapped modes and one virtual mode. Notice the two mode interference pattern and the exponential decay of the

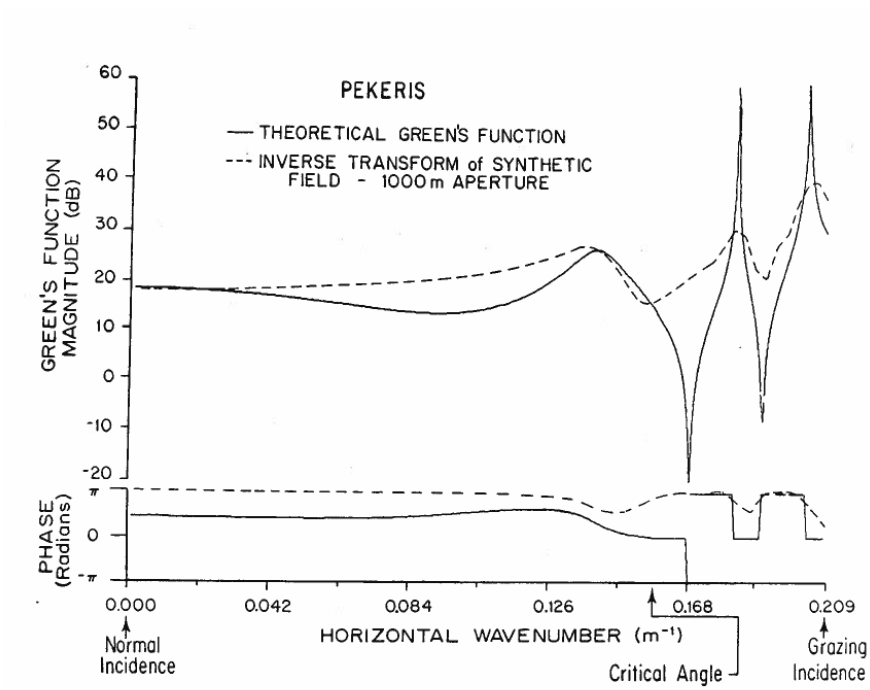


virtual mode!

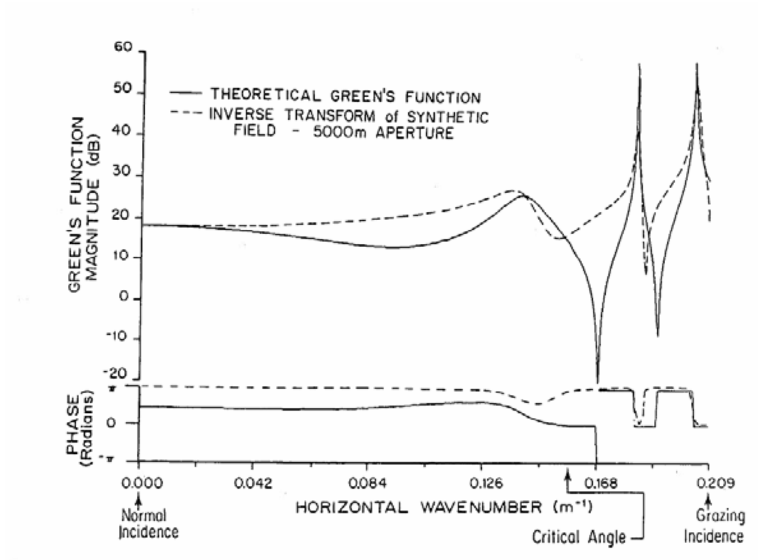
The result of doing the IHT on a 1000 m long “synthetic aperture array” versus the answer for an infinite array (the theoretical Green’s function) for a hard bottom waveguide. Notice the smearing of the finite array peaks.



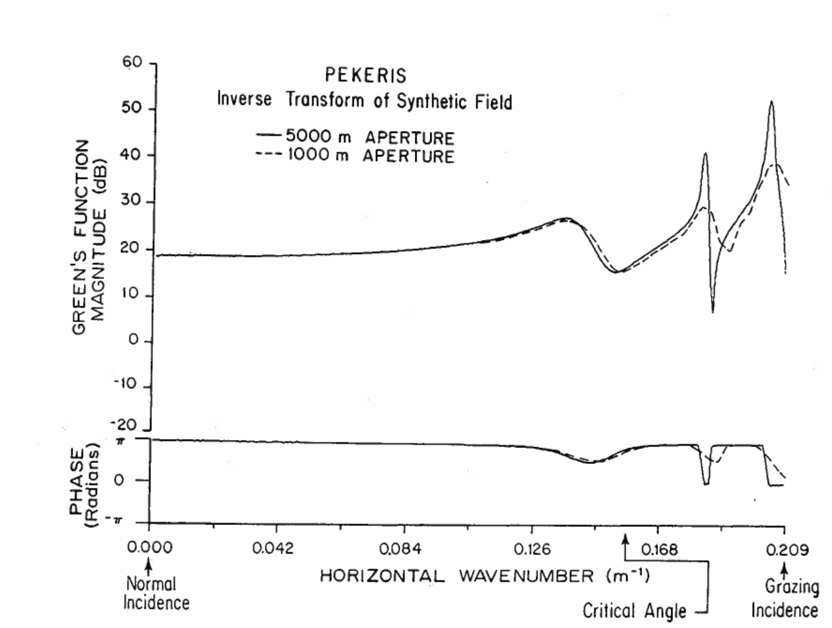
The same hard bottom case as above, only with a 5000m synthetic aperture. Notice that the positions of the poles are well estimated in this case!



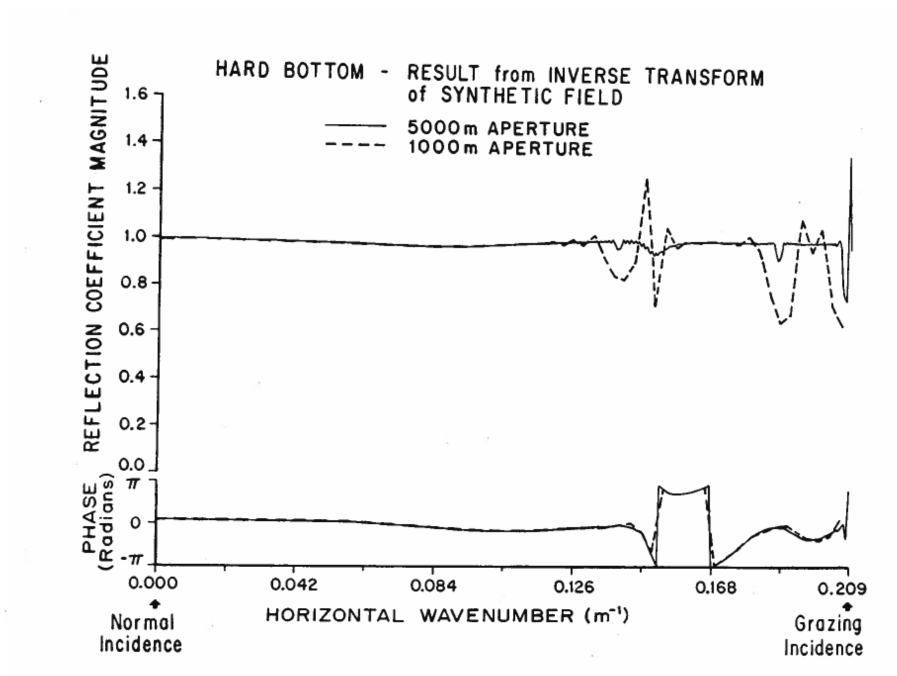
The 1000m aperture case for the Pekeris waveguide. Again the trapped peaks are poorly estimated, but interestingly the virtual mode peak is not done so badly. This is because most of the energy in the virtual mode is seen within the first kilometer range due to its exponential decay.



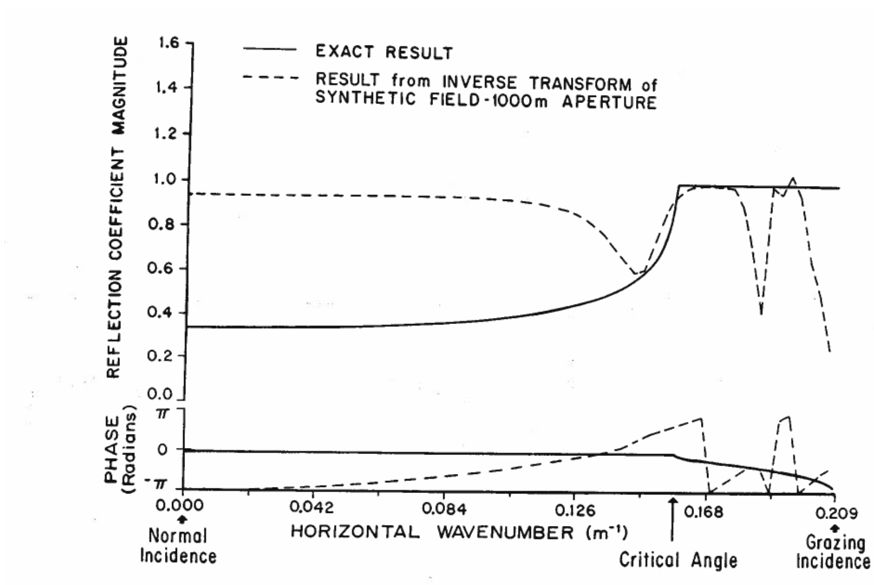
The same as above, only for a 5000m aperture. The trapped mode peaks are now well estimated, but there is no change in the virtual mode estimation!



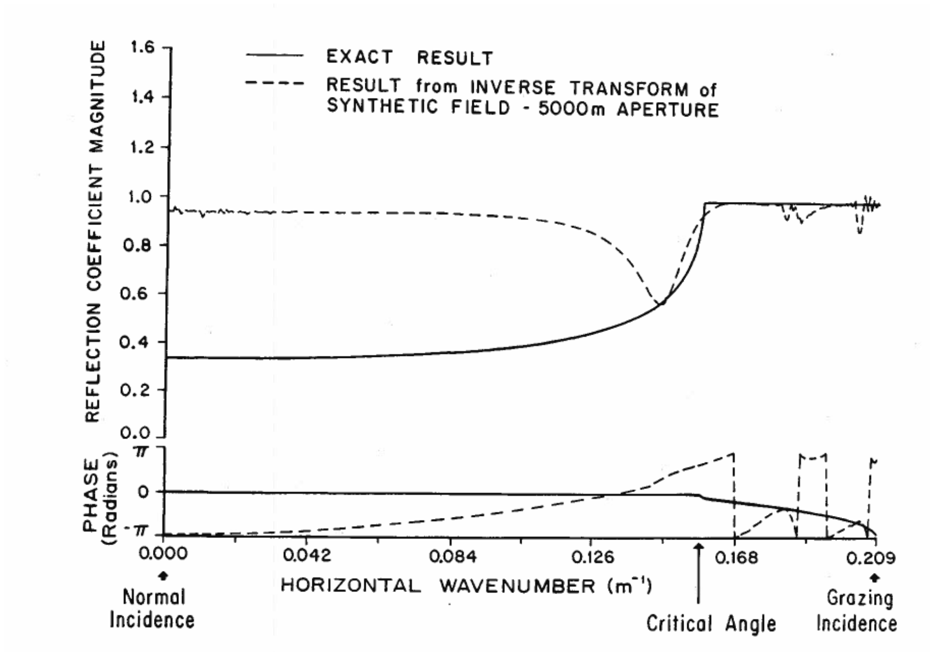
Comparison of the Green's functions for the 1000m and 5000m IHT's for the Pekeris waveguide. The only real change is in the trapped mode region.



Estimation of the plane wave reflection coefficient for the hard bottom case using 1km and 5km apertures. The 5km magnitude result seems pretty good, but the phase has some interesting jumps (which turn out to be due to the source and receiver positions.)



Estimation of the plane wave reflection coefficient for the Pekeris case using 1 km aperture. “Terrible” would be a nice way of describing this estimate!



Estimation of the plane wave reflection coefficient for the Pekeris case using 5 km aperture. “Terrible” would be a nice way of describing most of this estimate! However, it seems to give a good description of the magnitude of the reflection coefficient in the trapped mode region. Is this useful? Hmmmm.