

## Lecture #8: Ray-mode picture connections and Perturbation Theory

### A. Rays as interfering modes

We will follow a simple, physical argument paper by Tindle and Guthrie entitled “Rays as interfering modes in underwater acoustics”, J. Sound and Vib 34(2) 291-295 (1974).

The traditional picture of modes is that of constructively interfering plane waves at angles  $\pm \theta$ . We will show here that rays can be represented as constructively interfering groups of neighboring normal modes centered at certain angles. We will use simple WKB arguments for this.

We have seen that the WKB solution for the  $n$ th normal mode is:

$$U_n(z) = \frac{1}{\gamma_n^{1/2}} \sin \left[ \int_{z_n}^z \gamma_n dz \right]$$

where  $z$  is the depth downward,  $z_n$  is the upper turning point of the solution and

$$\gamma_n^2(z) = \frac{\omega^2}{c^2(z)} - k_n^2$$

The WKB solution is good for amplitude and phase away from the turning points. Phase is also good near the turning points (whereas amplitude diverges). Since phase is our biggest concern here, we can take the WKB solution to be good everywhere.

As usual, we can write the pressure as:

$$p(r, z) = A \sum_n U_n(z) U_n(z_s) H_0^1(k_n r)$$

Using the WKB mode expression and the asymptotic form of the Hankel function, we get the phase each mode accumulates in its travel from source to receiver, i.e.

$$\phi = \pm \int_{z_s}^z \left( \frac{\omega^2}{c^2(z)} - k_n^2 \right)^{1/2} dz + k_n r$$

Where the +/- comes from the square root, the first term on the RHS is the vertical phase accumulation, and the second term on the RHS is the horizontal phase accumulation. We have four forms of phase possible, since besides the +/- sign, the integral can be taken directly (as above) or via the upper turning point, i.e.

$$\int_{z_s}^z \text{ or } -\int_{z_s}^{z_n} + \int_{z_n}^z$$

Now we examine the condition for the successive terms in the mode sum to be in phase. Successive modes interfere constructively provided the change in phase  $\Delta\phi$  corresponding to a change in mode number  $\Delta n$  is equal to some multiple of  $2\pi$ .

Thus:

$$\frac{\Delta\phi}{\Delta n} = 2m\pi$$

or

$$\pm \int_{z_s}^z \frac{k_n (\Delta k_n / \Delta n) dz}{\left(\frac{\omega^2}{c^2(z)} - k_n^2\right)^{1/2}} + (\Delta k_n / \Delta n)r = 2\pi m$$

Strong reinforcement occurs only over a short range of mode numbers  $n$ , since  $\phi$  varies rapidly with  $n$ .

Groups of neighboring modes thus interfere constructively at depth, range  $(z, r)$  given by

$$r = \pm \int_{z_s}^z \left[ \frac{(\omega/k_n)^2}{c^2(z)} - 1 \right]^{-1/2} dz + m \frac{2\pi}{(\Delta k_n / \Delta n)}$$

This is interesting, since in ray theory the equation of a ray path which leaves a source of depth  $z_s$  and crosses the channel axis (where  $c(z) = c_0 = \text{minimum}$ ) at angle  $\theta_0$  is:

$$r = \pm \int_{z_s}^z \left[ \frac{(c_0 / \cos \theta_0)^2}{c^2(z)} - 1 \right]^{-1/2} dz + mD$$

The second term on the RHS is the full number of ray cycle distances, whereas the first term expresses the remaining distance (less than a full cycle).

We can “compare notes” between the two equations for  $r$  and see that the ray equation is the same as the constructive interference conditions for modes if:

$$\frac{c_0}{\cos \theta_0} = \frac{\omega}{k_n}$$

$$D = \frac{2\pi}{(\Delta k_n / \Delta n)}$$

In the first equation above, we can always pick an angle  $\theta_0$  that satisfies the equation. The second equation is just that the ray cycle distance equals the mode cycle distance, which it does in the absence of beam displacement. Thus, the mode interference equation is the same as the ray equation!

## B. Perturbation Theory

One of the most useful mathematical techniques available is “perturbation theory”, where one finds the mathematical description of an unknown system in terms of a known system that is nearby (“within a perturbation”). This technique is used in all branches of science and engineering, and there are many books available that discuss aspects of this technique. In this course, we will just look at a variant called “time independent perturbation theory” which comes from the quantum mechanical Schroedinger equation, which is of course an analogue to the Helmholtz equation of acoustics. In fact, it is worth showing how the quantum system and our acoustic system correspond, in that many useful variants of perturbation theory are found in standard physics texts.

Let’s start out with our modal equation:

$$Z''(z) + \gamma_m^2 Z(z) = 0$$

where as usual  $k(z) = \omega/c(z)$  and  $k_m^2 + \gamma_m^2 = k^2(z)$ . We can thus write the mode equation as

$$Z''(z) + k^2(z)Z(z) = k_m^2 Z(z)$$

or in 1-D as

$$\left( \frac{d^2}{dz^2} + k^2(z) \right) Z(z) = k_m^2 Z(z)$$

In quantum mechanics, the analogue of the above equation is

$$H^0 \psi_m^0 = E_m^0 \psi_m^0$$

The above two equations are the unperturbed Helmholtz and Schrodinger equations, respectively, and form the “known systems” that we have solutions for. Now we put the useful twist on the problem. Suppose we add a little extra term to  $H^0$  in quantum mechanics (or  $k(z)$  in acoustics.) This will give a small change to the eigenvalues and eigenvectors in each case. Let’s look at what happens in the Schrodinger equation form:

$$(H^0 + \lambda H')(\psi_m^0 + \Delta\psi_m) = (E_m^0 + \Delta E_m)(\psi_m^0 + \Delta\psi_m)$$

We can identify the four bracketed terms as new variables, i.e.

$$H_{new}\psi'_m = E'_m \psi'_m$$

We now expand the new eigenvalues and eigenvectors in a power series using the small parameter  $\lambda$ , i.e.

$$\psi'_m = \phi_m^0 + \lambda\phi_m^1 + \lambda^2\phi_m^2 + \dots$$

$$E'_m = A_m^0 + \lambda A_m^1 + \lambda^2 A_m^2 + \dots$$

Now we substitute these back into our “new” eigenvalue equation to get (to 2<sup>nd</sup> order):

$$H^0 \phi_m^0 + \lambda[H^0 \phi_m^1 + H' \phi_m^0] + \lambda^2[H^0 \phi_m^2 + H' \phi_m^1] + \dots = A_m^0 \phi_m^0 + \lambda[A_m^0 \phi_m^1 + A_m^1 \phi_m^0] + \lambda^2[A_m^0 \phi_m^2 + A_m^1 \phi_m^1 + A_m^2 \phi_m^0] + \dots$$

We now equate powers of  $\lambda$  on each side, and get a hierarchy of equations. At zeroth order we get:

$$H^0 \phi_m^0 = A_m^0 \phi_m^0 \text{ which implies } \phi_m^0 = \psi_m^0 \text{ and } A_m^0 = E_m^0$$

At first order, we obtain:

$$[H^0 \phi_m^1 + H' \phi_m^0] = [A_m^0 \phi_m^1 + A_m^1 \phi_m^0] \text{ which we operate on with } \int \phi_m^0 \rho(z) dz .$$

Doing this, the first terms on the LHS and RHS cancel each other out, and we are left with:

$$\int_0^\infty dz \rho(z) \phi_m^0(z) \phi_m^0(z) H' = A_m^1$$

Setting  $\rho = 1$  for convenience (we don't need to, but...), we re-identify the variables as above to get:

$$\Delta E_m^1 = \int_0 dz Z_m^2(z) H'$$

This is our first order energy/wavenumber correction, and the perturbation part of the problem is done. Now we just want to identify  $H'$  with some sort of useful perturbation. Let's start out with attenuation, which is normally treated by adding a small imaginary part to the wavenumber. If we let  $k \rightarrow k' + i\alpha$  and note that for small  $\alpha$   $k \approx k' = \omega/c$ , we can write

$$k^2 = \left(\frac{\omega}{c} + i\alpha\right)^2 = \frac{\omega^2}{c^2} + 2i\alpha \frac{\omega}{c} + O(\alpha^2)$$

Thus we see that

$$H' = 2i\alpha \frac{\omega}{c}$$

This is still a bit off from the horizontal wavenumber form we need for ocean acoustics, since what is used is in ocean acoustics  $k_m$  and not  $k_m^2$  (the latter of which is proportional to energy in quantum mechanics). So we write:

$$E_m + \Delta E_m^1 = (k_m + \Delta k_m)^2 = (k_m^2 + 2k_m \Delta k_m + \Delta k_m^2)$$

Since we know the zeroth order equation, and we can ignore the higher order terms, we get

$$\Delta E_m^1 = 2k_m \Delta k_m$$

Thus we can write, using the attenuation as a vertical profile (function of  $z$ ), that

$$\Delta k_m \equiv i\beta_m = \frac{i\omega}{k_m} \int_0^\infty dz \frac{\alpha(z)}{c(z)} Z^2(z) \quad \text{Attenuation Perturbation}$$

This is the perturbation form that is commonly used to insert attenuation effects into modal eigenvalue calculations (rather than using a complex root finder, which is far harder computationally).

One can also perturb the wavenumber by perturbing the soundspeed, which is a very useful thing to do. Thus we write a “soundspeed perturbed” wavenumber as

$$k^2(z) = \left( \frac{\omega}{c_0(z) + \Delta c(z)} \right)^2$$

Using our previous notation, this quickly gives

$$H' = \Delta k^2 = 2k_0^2(z) \frac{\Delta c(z)}{c_0(z)}$$

Thus

$$\Delta E_m^1 = 2k_m \Delta k_m = \int_0^\infty dz \rho(z) k_0^2(z) \frac{\Delta c(z)}{c_0(z)} Z^2(z)$$

which in terms of the wavenumber perturbation is

$$\Delta k_m^1 = \frac{1}{k_m} \int_0^\infty dz \rho(z) k_0^2(z) \frac{\Delta c(z)}{c_0(z)} Z^2(z) \quad \text{Soundspeed Perturbation}$$

This is very useful in that one can take a simple system (say a hard bottom waveguide) and add some realistic  $\Delta c(z)$  profiles to it to mimic rather complicated ocean waveguides. This type of calculation can give a lot of physical insight into how the modes of complicated waveguides work, in fact often in an analytic form!! This has turned out to be a very useful approach for 3-D acoustics. The perturbation integrals are also “Fredholm Equations of the first kind” which can be inverted for  $\alpha(z)$  and  $\Delta c(z)$  given that one has experimentally measured the  $\beta_m$  and/or the  $\Delta k_m$  experimentally. This will be discussed in the section of the course treating inverse theory.

In finishing this section on wavenumber perturbation, it should be mentioned that one can perturb the frequency  $\omega$  as well. This leads to an integral expression for the group velocity, which is useful in that the usual derivative based expression for modal group velocity can easily suffer from numerical noise problems, whereas integrals do not. This derivation can be found in COA.

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