

Lecture #10 Instructor Notes (Parabolic Equation or PE)

A. Derivation of basic PE

Let's look at a very clever and popular way to solve the acoustic wave equation, called the "parabolic equation (PE)" method. This is an "old method" now (transcribed for underwater acoustics from Russian electromagnetics work by Fred Tappert in 1977), but it is still being improved upon today, and is the workhorse for most lower frequency acoustics calculations.

We start out with "standard forms" for a second order differential equation in (x,y) or (r,z) as we will get to soon.

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G$$

Depending on whether the discriminant $B^2 - 4AC$ is greater than, less than, or equal to zero, the equation is Hyperbolic, Elliptic, or Parabolic.

The acoustic wave equation is elliptic by nature, which in a practical vein means it is laborious to solve. However, a parabolic equation can be solved quickly and easily by Fourier Transform methods, and so we would like to see if we can reasonably approximate our elliptic equation with a parabolic one. (Obviously we can, or I wouldn't be writing this lecture!) Let's look at the details!

We start with our (elliptic) Helmholtz equation (for the velocity potential here)

$$\nabla^2 \varphi + k^2 \varphi = 0$$

Using an index of refraction form, we write this as

$$\nabla^2 \varphi + k_0^2 n^2 \varphi = 0$$

where

$$k = \omega / c, \quad k_0 = \omega / c_0, \quad \text{and} \quad n = c_0 / c.$$

Let's again look at a cylindrically symmetric geometry, and put the wave equation in cylindrical coordinates.

$$\frac{\partial^2 \varphi}{\partial r^2} + \frac{1}{r} \frac{\partial \varphi}{\partial r} + \frac{\partial^2 \varphi}{\partial z^2} + k_0^2 n^2 = 0$$

As we did with range dependent normal modes, we assume a “partially separable” solution, i.e.

$$\varphi = \psi(r, z)S(r)$$

where the $\psi(r, z)$ term varies slowly, but $S(r)$ varies quickly. Inserting this solution into our DE, we get

$$\psi \left[\frac{\partial^2 S}{\partial r^2} + \frac{1}{r} \frac{\partial S}{\partial r} \right] + S \left[\frac{\partial^2 \psi}{\partial r^2} + \frac{\partial^2 \psi}{\partial z^2} + \left(\frac{1}{r} + \frac{2}{S} \frac{\partial S}{\partial r} \right) \frac{\partial \psi}{\partial r} + k_0^2 n^2 \psi \right] = 0$$

We use the k_0^2 as a separation constant to get two equations, one purely is S and the other mixed (but not for long!):

$$\frac{\partial^2 S}{\partial r^2} + \frac{1}{r} \frac{\partial S}{\partial r} + k_0^2 S = 0$$

$$\frac{\partial^2 \psi}{\partial r^2} + \frac{\partial^2 \psi}{\partial z^2} + \left(\frac{1}{r} + \frac{2}{S} \frac{\partial S}{\partial r} \right) \frac{\partial \psi}{\partial r} + k_0^2 n^2 \psi - k_0^2 \psi = 0$$

The first equation is just the familiar Bessel’s equation, with solution $J_0(k_0 r) = S(r)$. We can use the asymptotic form for this, i.e.

$$S(r) = \sqrt{\frac{2}{\pi k_0 r}} \exp(i[k_0 r - \pi/4])$$

We now explicitly plunk this solution for $S(r)$ into the second wave equation to get rid of the “mixed” form. We get

$$\frac{\partial^2 \psi}{\partial r^2} + \frac{\partial^2 \psi}{\partial z^2} + 2ik_0 \frac{\partial \psi}{\partial r} + k_0^2 (n^2 - 1) \psi = 0$$

We are almost there. We now make the paraxial (small angle) approximation, which mathematically is

$$\frac{\partial^2 \psi}{\partial r^2} \ll 2k_0 \frac{\partial \psi}{\partial r}$$

to get the “old school” parabolic equation

$$\frac{\partial^2 \psi}{\partial z^2} + 2ik_0 \frac{\partial \psi}{\partial r} + k_0^2 (n^2 - 1)\psi = 0$$

This equation is amenable to FFT “marching solutions” that are fast and efficient. Let’s next look at that. BTW, setting up this marching solution is one of the term projects, and is also well described in COA.

B. “Marching Solution” to the PE

Let’s start by Fourier transforming the PE

$$\int_{-\infty}^{\infty} \left\{ \frac{\partial^2 \psi}{\partial z^2} + 2ik_0 \frac{\partial \psi}{\partial r} + k_0^2 (n^2 - 1)\psi \right\} \exp(-isz) dz = 0$$

Let’s define the transform pair as:

$$\Psi(r, s) = \frac{1}{2\pi} \int \psi(r, z) \exp(-isz) dz$$

$$\psi(r, z) = \int \Psi(r, s) ds$$

where the limits of the integral are understood to be minus infinity to infinity unless otherwise stated.

We can rewrite this equation as

$$\int \frac{\partial^2 \psi}{\partial z^2} \exp(-isz) dz + 2ik_0 \frac{\partial}{\partial r} \left\{ \int \psi \exp(-isz) dz \right\} + k_0^2 (n^2 - 1) \int \psi \exp(-isz) dz = 0$$

To get the first integral in usable form, we integrate by parts:

$$\int \frac{\partial^2 \psi}{\partial z^2} \exp(-isz) dz = \frac{\partial \psi}{\partial z} \exp(-isz) \Big|_{-\infty}^{\infty} + is \int \psi \exp(-isz) dz - s^2 \int \psi \exp(-isz) dz = -s^2 \Psi$$

since ψ and its derivative vanish at infinity. (Apologies for the limits symbol! Old version of Word!)

This lets us write the transformed DE as

$$-s^2\Psi + 2ik_0 \frac{\partial\Psi}{\partial r} + k_0^2(n^2 - 1)\Psi = 0$$

We can rearrange this as a linear, first order homogeneous DE for $\Psi(r, s)$, i.e.

$$\frac{\partial\Psi}{\partial r} + \frac{k_0^2(n^2 - 1) - s^2}{2ik_0} \Psi = 0$$

This has the simple solution

$$\Psi(r, s) = \Psi(r_0, s) \exp\left(-\frac{[k_0^2(n^2 - 1) - s^2]}{2ik_0}(r - r_0)\right)$$

Now, all we have to do is inverse transform to get $\Psi(r, s) \rightarrow \psi(r, z)$

Defining $\Delta r = (r - r_0)$ and taking appropriate terms outside the integral, we get

$$\psi(r, z) = \exp(ik_0(n^2 - 1)\Delta r / 2) \int \Psi(r_0, s) \exp\left(-\frac{is^2\Delta r}{2k_0}\right) \exp(isz) ds$$

We can put this in a compact form using F as the symbol for the Fourier transform:

$$\psi(r + \Delta r, z) = \exp(ik_0(n^2 - 1)\Delta r / 2) F^{-1} \left\{ \exp\left(-\frac{is^2\Delta r}{2k_0}\right) F\{\psi(r, z)\} \right\}$$

This is the marching solution. If we know the solution at r_0 , we can quickly get it via FFT's at $r_0 + \Delta r$. You then repeat the process going out in range. The "starter field" for the PE is an interesting problem that is discussed in COA.

One little "gotcha" that you probably noticed is the limits on the FFT. We usually go from 0 to minus infinity if the ocean half-space problem. But we can fix this. If you remember that the BC at the sea surface is $\psi(r, z) = 0$ at $z=0$, then it makes sense to make $\psi(r, z)$ antisymmetric in z , i.e. the original solution is reflected about $z=0$ and phase shifted by 180 degrees. This gives the picture we see below. We also reflect the soundspeed profile (with no phase shift) about the origin. To insure we get $\psi(r, z) \rightarrow 0$ at $\pm\infty$, we add a small imaginary component to the soundspeed at very large depths, which attenuates the signal and prevents spurious reflections.

C. Discussion of error in the PE

In order to use the index of refraction $n(r,z)$ in the PE marching scheme, we have to use an appropriately small range step Δr . The larger the index of refraction gradient, the smaller the range step one can take. Let's see how this works.

If we put our PE marching solution back into the original PE with $n=n(r,z)$, we will no longer have an exact solution. Terms NOT satisfying the PE have derivatives of n with respect to r or z . They are:

$$\left(\frac{\partial \psi}{\partial r}\right)_{error} = ik_0 n \Delta r \frac{\partial n}{\partial r} \psi(r, z)$$

$$\left(\frac{\partial^2 \psi}{\partial z^2}\right)_{error} = 2ik\Delta r \frac{\partial n}{\partial z} \frac{\partial \psi}{\partial z} + (k\Delta r \frac{\partial n}{\partial z})^2 \psi + ik\Delta r \left(\frac{\partial^2 n}{\partial z^2} + \left(\frac{\partial n}{\partial z}\right)^2\right) \psi$$

where we have assumed $n \sim 1$. The total error per range step is:

$$E = \left(\frac{\partial^2 \psi}{\partial z^2}\right)_{error} + 2ik_0 \left(\frac{\partial \psi}{\partial r}\right)_{error}$$

or

$$E = 2ik\Delta r \frac{\partial n}{\partial z} \frac{\partial \psi}{\partial z} - 2k^2 \Delta r \frac{\partial n}{\partial r} \psi + \left[k\Delta r \left(\frac{\partial n}{\partial z}\right)^2 + i \left(\frac{\partial n}{\partial z}\right)^2 + i \frac{\partial^2 n}{\partial z^2} \right] k\Delta r \psi$$

The last two terms in parentheses are small, so we get:

$$E = 2ik\Delta r \frac{\partial n}{\partial z} \frac{\partial \psi}{\partial z} - \left(\frac{\partial n}{\partial r} - \frac{\Delta r}{2} \left(\frac{\partial n}{\partial z}\right)^2\right) 2k^2 \Delta r \psi$$

If we put in some numbers for the gradients in deep and shallow water, we can use the above for estimates of error and for step size determination

DEEP WATER

$$10^{-5} < \left(\frac{\partial n}{\partial z}\right)_{max} < 10^{-3} \quad \text{and} \quad \frac{\partial n}{\partial r} < 10^{-6}$$

SHALLOW WATER

$$10^{-2} < \left(\frac{\partial n}{\partial z}\right)_{max} < 1 \quad \text{and} \quad \frac{\partial n}{\partial r} < 10^{-4} \quad \text{where we used } m^{-1} \text{ units}$$

Both the vertical and horizontal gradients are much larger in shallow water, which is why the PE was originally used as a deep water computational tool, and much later (after a lot of development) became useful for shallow water.

D. Comparison of the PE solution with the normal mode solution

As this is right out of Brekhovskikh and Lysanov's book, just some comments are needed here.

The major point of the section is to compare a simple modal form with a simple PE solution. For modes, we get:

$$p_m = A_m r^{-\frac{1}{2}} u_m(z) \exp(i\zeta_m r)$$

whereas for PE we get

$$p_m = A_m r^{-\frac{1}{2}} u_m(z) \exp\left(i \frac{(k_0^2 + \zeta_m^2)}{2k_0} r\right)$$

If we define $\frac{(k_0^2 + \zeta_m^2)}{2k_0} = \zeta'_m$, we clearly see that the PE replaces the modal wavenumber with a perturbed form of it that is weighted toward the "water wavenumber" k_0 , which corresponds to the low, small grazing angle modes. Thus the PE is expected to work best at small grazing angles, i.e. it is paraxial.

The next simple, but useful, thing that we can do is replace the modal wavenumber in the PE form above with that of an ideal hard bottom waveguide (a nice analytic form) and see what we can see. We get for the hard bottom eigenvalue

$$\zeta_m^2 = k_0^2 - \left(\frac{\pi}{h}\right)^2 \left(m + \frac{1}{2}\right)^2$$

If we now look at the error in the modal phase that the PE causes in this case, we see:

$$\Delta\phi = (\zeta'_m - \zeta_m)r = \frac{1}{8} k_0 r \sin^4 \chi$$

where χ is the horizontal grazing angle. Thus the error grows linearly with frequency and range, but as $\sin^4 \chi$ with grazing angle! That is a very strong dependence, and indeed the "old school" PE could only handle angles up to around 15° , which was fine for the deep ocean, but not for shallow water. The current generation of PE, however, is vastly more capable in terms of angle, frequency and range.

MIT OpenCourseWare
<http://ocw.mit.edu>

2.682 Acoustical Oceanography
Spring 2012

For information about citing these materials or our Terms of Use, visit: <http://ocw.mit.edu/terms>.