**Lecture #12 Instructor Notes (Rough surface scattering theory)**

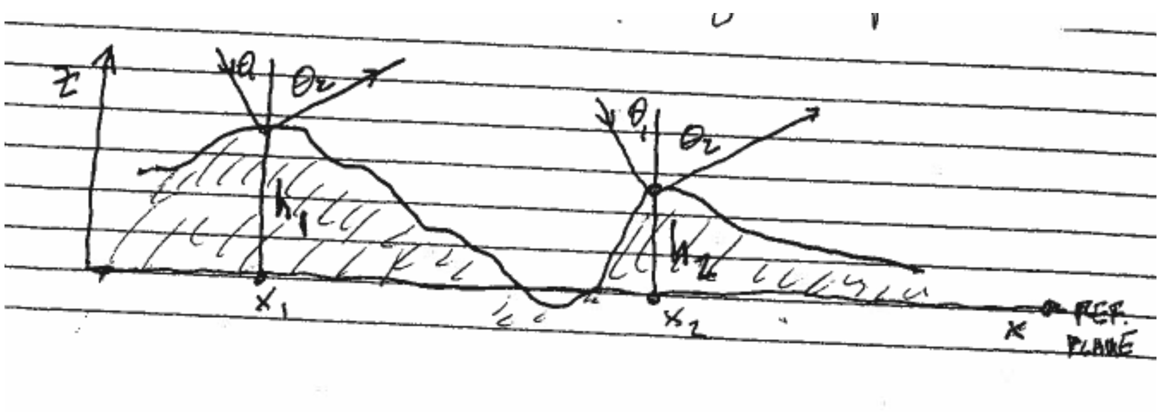
In these notes on roughness, I'll be paraphrasing two major references, along with including some other material. The two references are:

1) J.A. Ogilvie, "theory of wave scattering from random rough surfaces", IOP Publishing, 1992. (This is an expensive and hard to obtain reference these days, but still a very good one.)

2) L.M.Brekhovskikh and Y. Lysanov, "Fundamentals of ocean acoustics", either 2nd edition, Springer 1991 or 3rd edition, Springer 2003. This is a more common reference, at a usual price.

A. Characterization of roughness – Rayleigh prameter

Rayleigh (1877) provided the first really useful characterization of roughness for scattering. Let's look at this, the so called "Rayleigh Criterion" for scattering. Consider a plane monochromatic wave incident at an angle $\theta_1$ onto the rough surface shown below.




Then the phase difference between the energy scattered at points $x_1$ and $x_2$ on the rough surface, and scattered to the same angle $\theta_2$ (the angle at which we observe the scattered energy) is given by

$$\Delta\varphi = k\left[(h_1 - h_2)(\cos\theta_1 + \cos\theta_2) + (x_2 - x_1)(\sin\theta_1 - \sin\theta_2)\right]$$

where as usual $k = \omega/c = 2\pi/\lambda$. For specular scattering ($\theta_1 = \theta_2$), we get the simple phase difference

$$\Delta\varphi = 2k\Delta h\cos\theta_1$$

where $\Delta h = h_1 - h_2$.

For $\Delta\varphi << \pi$, the scattered waves are almost in phase and interfere constructively, i.e. there is little scattering.

For $\Delta\varphi \sim \pi$,, there is destructive interference and strong scattering.

When we average the roughness over the surface, $\Delta h \rightarrow \sigma$, the RMS roughness. It is usual to define "Rayleigh parameter" for roughness in terms of the RMS roughness, i.e.

$$R = 2k\sigma\cos\theta_1$$

Thus we see that the "roughness" we consider for a scattering calculation depends on the surface, the freq2uency and the incident angle, not just the surface.

## B. Huygens Principle

We are all familiar with Huygens Principle, which states that each point on a rough surface acts as a radiator of spherical waves (the scattered energy). The relative phase of these wavelets is given by our equation above. Let's look at how this works. For a smooth surface, $h_1 = h_2$ everywhere, so

$$\Delta\varphi = k(x_1 - x_2)(\sin\theta_1 - \sin\theta_2)$$

In the specular direction, ($\theta_1 = \theta_2$), so $\Delta\varphi = 0$. Thus all the waves from the surface interfere constructively. For $h_1 = h_2$, but ($\theta_1 \neq \theta_2$), we get destructive interference for the various $\Delta x$ values, i.e. the waves cancel.

We note that for both rough and smooth surfaces of finite extent, we get beampattern effects, i.e. a diffraction pattern, with a main lobe and sidelobes. These edge effects can be very important to the scattering description.

For a rough surface, looking at the specular direction, we have (as before)

$$\Delta\varphi = 2\pi\Delta k cos\theta_1$$

for the phase difference between the secondary (scattered) wavelets. When $\Delta\varphi \sim \pi$, the destructive interference reduces the amplitude of the specular field. The extent to which it is reduced depends on the average value of the equation above. We will show later, via Kirchoff theory, that the reduction in the coherent reflection coefficient is given by $R_{coh=}Rexp(-\frac{g}{2})$ where

$$g = 4k^2\sigma^2cos^2\theta_1$$

This equation works both for specular reflection and backscattering. (Note: You should be able to easily check the above equation numerically using Huygen's principle on a computer.)

For a rough surface, now considering the non-specular direction $\theta_2$, the first term of our "long form" $\Delta\varphi$ equation comes into play and gives the so-called "diffuse field." The phase of the energy scattered in the off-specular direction varies randomly over 0 to 2π.

Note that is we look at the average amplitude of the scattered field,

$$\langle A_{scats} \rangle = \langle A_{coh} + A_{incoh} \rangle = \langle A_0 e^{i\varphi_{coh}} \rangle + \langle A_0 e^{i\varphi_{incoh}} \rangle$$

The last term on the RHS goes to zero due to the phase randomness of $\varphi_{incoh}$. Thus our amplitude average only reflects the coherent field! To look at the incoherent field, we must go to intensity statistics, ala $\langle AA^* \rangle_{scats}$, which we treat next.

C. Statistics of rough surfaces

There are two essential aspects to the nature of a random rough surface: 1) its spread of heights about some reference height and 2) the variation of these heights along the surface. Thus, the two quantities we will consider are the surface height distribution and the surface correlation function. These are the most common rough surface descriptors.

C.1. Height Distribution

We look at $h(\overrightarrow{r})$ where h is the surface height at some point $\vec{r} = (x, y)$of the surface. The distribution of heights is $p(h)$. We usually reference h to a mean surface, and moreover make the distribution zero mean, i.e.

$$\langle h \rangle_S = \int_{-\infty}^{\infty} h p(h) dh = 0$$

where the angle bracket denotes averaging over the surface.

The root-mean-square (RMS) height of the surface is simply given by

$$\sigma = \sqrt{\langle h^2 \rangle_S} = \left[ \int_{-\infty}^{\infty} h^2 p(h) dh \right]^{1/2}$$

Another useful descriptor of roughness is the arithmetic mean, given by

$$R_{arith} = \int_{-\infty}^{\infty} |h| p(h) dh$$

R (arith) can be easily related to σ. For a Gaussian distribution, $R_{arith} = \sigma\sqrt{2/\pi} = 0.8\sigma$.

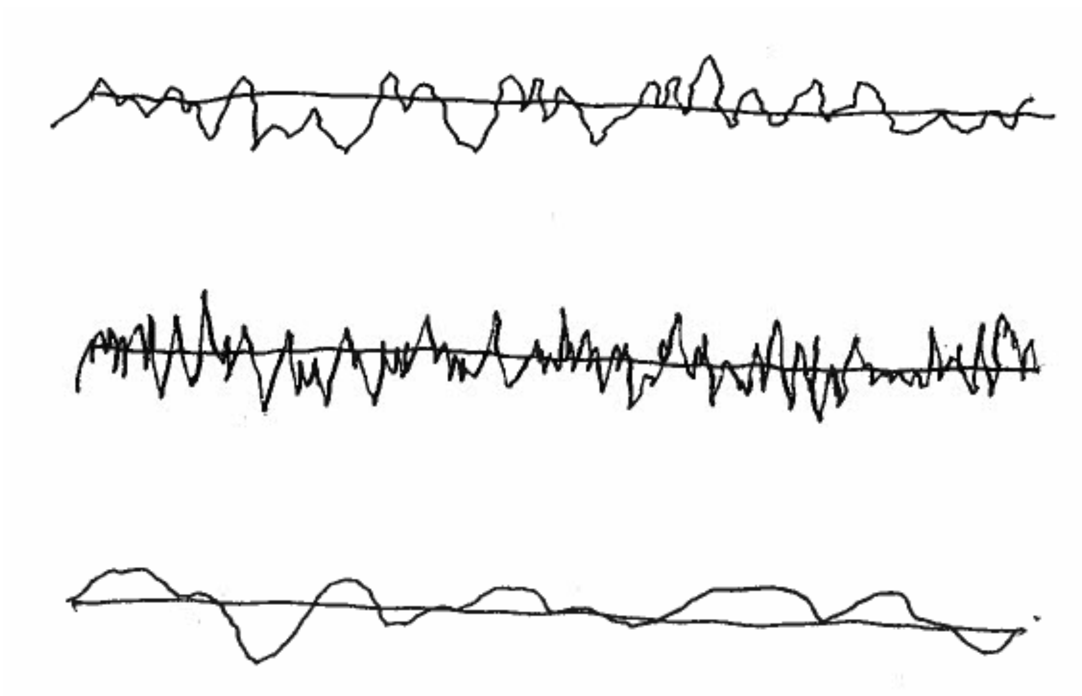
Much of the work on rough surfaces uses (either assumed or measured, with the latter being preferable!) Gaussian height distributions, i.e.

$$p(h) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{h^2}{2\sigma^2}\right)$$

Gaussian surfaces arise (often) when every point on the surface is the result of a large number of local events, the results of which are cumulative. This is a result of the Central Limit Theorem, i.e." a random variable x which is the sum of random variables $p_i$, where the $p_i$ are independent, is (under most conditions) Gaussian. (Note: the sea surface height distribution is a Gaussian – why?)

C.2. Surface correlations

The specification of a height distribution does not discriminate between the three surfaces in the following figure. Each is a Gaussian with σ=SAME, but they obviously differ in the length scale of the roughness.



These samples CAN be distinguished by their correlation functions, however. We define the correlation function as:

$$C(\vec{R}) = \langle h(\vec{r})h(\vec{r}+\vec{R})\rangle/\sigma^2$$

There is also the well known "autocovariance function" $C_0(\vec{R}) = \sigma^2 C(\vec{R})$, which is just the unnormalized correlation function. Obviously, C(0)=1. Also, for a sinusoidal surface, C(R)= $\cos(ar)$. Only for truly uncorrelated states does → 0 as R → large. (And we often see a large scale correlation that is nonzero).

For simplicity (and sadly, often for oversimplicity), one can assume that the surface correlation functions are Gaussian, i.e.

$$C(R) = \exp\,(-\frac{R^2}{\lambda_0^2})$$

In the above, $\lambda_0$ is the correlation length, i.e. the e-folding length of $C(R)$. {NOTE: We have discussed a scalar R in the above, but it really is a vector – as we will examine soon. The above assumes angular isotropy, which is often a bad assumption. }

Data are often fit by an exponential correlation function, i.e. $C(R) = \exp\,(-\frac{|R|}{\lambda_0})$. This function runs into difficulty in the gradient of $C(R)$ at the origin. [We will see this in the Method of Small Perturbations for the Neumann BC case.]

We can get the best of both worlds (Gaussian being smooth at the origin and exponential having a good decay law) from two functional forms:

$$C(R) = \text{sech}\,(R/\lambda_0$$

$$C(R) = \exp\,\{\left(-\frac{R^2}{\lambda_1}\right)\left[1-\left(-\frac{R^2}{\lambda_2}\right)\right]\}$$

These functions both decay as an exponential at large R, but are smooth at the origin. $\lambda_1/\lambda_2$ in the latter form determines where the transition in behavior occurs.

Coutionary notes: 1) in measuring surface roughness, we need to use a discretization of $I \leq (\frac{1}{10})\lambda_0$, or we will not see the exponential nature of the surface. 2) For a finite sized rough surface, the surface extent provides a long wavelength cutoff to the power spectrum which can introduce misleading oscillations in $C(r)$. 3) It is often noticed that more than one correlation length appears, i.e. two-scale roughness. This is quite real. (As an example, one can look at capillary waves (e.g. "cats paws") on top of long surface waves.

C.3. The Structure Function

Another (occasionally) useful surface descriptor is the "structure function" which is defined as

$$S(\vec{R}) = \langle [h(\vec{r}) - h(\vec{r} + \vec{R})]^2 \rangle$$

It is related to the correlation function for stationary surfaces by

$$S(\vec{R}) = 2\sigma^2[1 - C(\vec{R})]$$

The structure function has the advantage that it is independent of the choice of surfaces to which $h(\vec{r})$ is referenced. BUT, it does not appear naturally in scattering theory.

C.4. The Characteristic Function

The characteristic function is the Fourier Transform of the PDF of the height distribution, i.e.

$$\chi(s) = \int_{-\infty}^{\infty} p(h) \exp(ish)\, dh$$

The 2-D characteristic function is also useful, i.e.

$$\chi_2(s_1, s_2, \vec{R}) = \int_{-\infty}^{\infty} p(h_1, h_2, \vec{R}) \exp[i(s_1 h_1 + s_2 h_2)]\, dh_1 dh_2$$

This function assumes stationarity, but one can relax the isotropy constraint that is implicitly in the 1D form.

C.5. The Power Spectrum

A very useful descriptor of surface roughness is the power spectrum, which is the Fourier Transform of the unnormalized correlation function, i.e.

$$P(\vec{k}) = \frac{\sigma^2}{(2\pi)^2} C(\vec{R}) \exp(i\vec{k} \cdot \vec{R})\, d\vec{R}$$

The power spectrum may be related to the FT of the surface by substituting for $C(\vec{R})$. We start by writing:

$$C(\vec{R}) = \lim_{A_m \to \infty} \frac{1}{A_m \sigma^2} \int_{-\infty}^{\infty} h(\vec{r}) h(\vec{r} + \vec{R}) d\vec{r}$$

We then substitute this to get

$$P(\vec{k}) = \lim_{A_m \to \infty} \frac{1}{A_m (2\pi)^2} \left| \int_{-\infty}^{\infty} h(\vec{r}) \exp(i\vec{k} \cdot \vec{r})\, d\vec{r} \right|^2$$

We see that the power spectrum, unlike previous descriptors, can describe both the spread of the heights above the mean surface and also the height variation along the surface!

The total area under the spectrum gives the variance (power), i.e.

$$\int_{-\infty}^{\infty} P(\vec{k}) d\vec{k} = \sigma^2$$

This result is a special case of a more general theorem (not shown) that the moments of a spectrum are related to the RMS averages for higher order surface derivatives.

The power spectrum for an anisotropic surface with a Gaussian correlation function is

$$P(k_1, k_2) = \frac{\sigma^2}{(2\pi)^2} \int_{-\infty}^{\infty} exp\left[ -(\frac{x^2}{\lambda_1^2} + \frac{y^2}{\lambda_2^2}) \right] \exp[i(k_1 x + k_2 y)]\, dxdy$$

$$= \frac{\sigma^2 \lambda_1 \lambda_2}{4\pi} \exp\,(-\frac{k_1^2 \lambda_1^2}{4}) \exp\,(-\frac{k_2^2 \lambda_2^2}{4})$$

Where $\lambda_1, \lambda_2$ are the correlation lengths in the x,y directions. $P(k_1, k_2)$ itself is a Gaussian, since the FT of a Gaussian is a Gaussian, with a standard deviation of $\sqrt{2}/\lambda_j$, where j=1,2 (or x,y).

The power spectrum for an anisotropic surface with an exponential correlation function is:

$$P(k_1, k_2) = \frac{\sigma^2}{(2\pi)^2} \int_{-\infty}^{\infty} exp\left[ -(\frac{|x|}{\lambda_1} + \frac{|y|}{\lambda_2}) \right] \exp[i(k_1 x + k_2 y)]\, dxdy$$

$$= \frac{\sigma^2}{\lambda_1 \lambda_2 \pi^2} \frac{1}{(\lambda_1^{-2} + k_1^2)} \frac{1}{(\lambda_2^{-2} + k_2^2)}$$

The last two terms in the last equation are “Lorentzian” functions. Each one is zero mean, and full width at half maximum is $2/\lambda_j$, j=1,2.

D. Higher order statistics

D.1. Two point height probability distribution

It is sometimes useful to examine $p_2(h,\vec{R}_1;h_0,\vec{R}_2)$, where $p_2(h,\vec{R}_1;h_0,\vec{R}_2)dhdh_0$ is the probability that the surface height is between h and h+dh at $\vec{R}_1$ and between $h_0 + dh_0$ at $\vec{R}_2$. If we consider an isotropic, stationary Gaussian surface (correlation dependent only on $R = |\vec{R}_2 - \vec{R}|$), we get the pdf in the form:

$$p_2(h,h_0,R) = \frac{1}{2\pi\sigma^2\sqrt{1-C^2(r)}}\exp\left\{-\left[\frac{h^2+h_0^2-2hh_0C(r)}{2\sigma^2\left[1-C^2(R)\right]}\right]\right\}$$

This has obvious limits:

$$p_2(h,h_0,R) \to p(h)p(h_0) \quad \text{as} \quad |R| \to \infty$$

$$p_2(h,h_0,R) \to p(h)\delta(h-h_0) \quad \text{as} \quad |R| \to 0$$

D.2. Higher order surface correlations/derivatives

There is a useful theorem for estimating RMS values of surface derivatives from the correlation function, namely

$$\left\langle \left( \frac{\partial^i h(\vec{r})}{\partial x^i} \right)^2 \right\rangle_S = \sigma^2 \left( \frac{\partial^{2i} C(\vec{r}_0 - \vec{r})}{\partial x^{2i}} \right) \Bigg|_{\vec{r}_0 = \vec{r}}$$

One can see that the existence of the derivative of the correlation function at R=0 is needed for the existence of the higher order derivatives of the surface heights.

The above theorem can quickly provide a useful parameter of the rough surface, the RMS gradient. Assuming an isotropic surface with a Gaussian correlation function, we get

$$\sigma_S = \sqrt{\left\langle \left( \frac{\partial h}{\partial x} \right)^2 \right\rangle} = \sqrt{\left\langle \left( \frac{\partial h}{\partial y} \right)^2 \right\rangle} = \frac{\sigma\sqrt{2}}{\lambda_0}$$

We see that an increase in the RMS roughness or a decrease in the correlation length gives a larger RMS gradient, as expected.

.D.3. Radius of curvature and other considerations

The radius of curvature $r_c$ at a point on a rough surface is given by

$$\frac{d\hat{t}}{ds} = \frac{\hat{n}}{r_c}$$

where $\hat{n}$ is the unit surface normal, $\hat{t}$ is the unit surface tangent, and s is the arc length along the surface. For a one-dimensional surface, this can be shown to be:

$$r_c = -\left[1 + \left(\frac{dh}{dx}\right)^2\right]^{3/2} \left(d^2h/dx^2\right)^{-1}$$

The average of the modulus of the radius of curvature will be infinite, due to points where $\left(\frac{dh}{dx}\right)^2$ goes to zero. However, if we replace the derivatives by their averages, we get a finite result

$$r_c^{der-avg} = \frac{\lambda_0^2}{2\sigma\sqrt{3}}\left(1 + \frac{2\sigma^2}{\lambda_0^2}\right)^{3/2}$$

E. Isotropy, Stationarity, and Ergodicity

ISOTROPY – Rotational invariance, i.e. the statistics of the surface are independent of the direction along the surface. This simplifies the mathematics considerably, but is not always correct. If the processes forming the surface are directionally dependent, then the surfaces formed will also be. In that case, the correlation function will display such dependence, e.g. for an exponential

$$C(x,y) = \exp\left[-\left(\frac{|x|}{\lambda_x} + \frac{|y|}{\lambda_y}\right)\right]$$

and

$$C(x,y) = \exp\left[-\left(\frac{x^2}{\lambda_x^2} + \frac{y^2}{\lambda_y^2}\right)\right]$$

A good example (especially for this course!) of a directional spectrum is the Pierson (also called Neumann-Pierson) power spectrum for ocean surface waves. The winds force a very directional spectrum, given by (for fully developed seas far from a shoreline, i.e. "infinite fetch")

$$P(k_1,k_2) \propto \frac{1}{k^{9/2}} \exp\left[\frac{g}{v^2 k}\right] \cos^2\left[\tan^{-1}\left(\frac{k_2}{k_1}\right)\right]$$

where g is the accelaeration of gravity, v is the windspeed, $k = \left(k_1^2 + k_2^2\right)^{1/2}$, and k1 and k2 are mutually perpendicular wavenumber directions, with k1 being in the wind direction.

STATIONARITY – Very simply, the translational invariance of a statistic or property of the surface. (Temporal stationarity is the invariance wrt a time displacement, but we will not pursue temporal issues here.) For two point statistics (e.g. height correlations), the property depends only on the difference of the two points in range (separation), and not on their absolute positions.

ERGODICITY- "Any statistical average taken over many different parts of one surface realization (spatial averaging) is the same as the average over many different realizations (ensemble averaging)." For time varying surfaces, the concept is extended to averaging over the same part of the surface at many different times (temporal averaging), which will produce the same statistical description as spatial or ensemble averaging. In much of the work we do, we assume ergodicity. Also, ergodicity implies stationarity.

E. Fractal Surfaces

We have become increasingly familiar with fractal surfaces over the last quarter century or so. Surfaces whose structure looks similar at any "level of magnification" or scales (so-called "self similar surfaces") are fractals. Rough surfaces like coastlines, the ocean bottom, cliffs and mountains, etc. often show a degree of self-similar behavior. It is thus obvious that fractal mathematics needs to be incorporated into rough surface scattering theory. [A classic paper on how fractals were introduced into the description of ocean bottom roughness is: J.A. Goff and T.H. Jordan, "Stochastic modeling of seafloor morphology:inversion of Sea Beam data for second order statistics", J. Geophysical Research 93, B11, pp 13589-13608, Nov 10, 1988. It is part of the background reading for this course.]

A fractal structure may enclose a finite volume, but have infinite area. This property is described by the "Haussdorf dimension" D+1, where D is the dimension of the surface when cut by a plane. Non fractal surfaces have D=1, but fractals have dimensionality 1<D<2 (a non-integer dimension.)

For rough surface scattering, we can relate D to the surface height function. Let's assume that a one-dimensional rough surface has the property:

$$|h(x+\Delta) - h(x)| \sim \Delta^{\alpha}$$

for small delta. Alpha is called the Lipschitz exponent. For the range $0 < \alpha < 1$, the surface is continuous but not differentiable, and therefore is fractal. The exponent alpha is related to D through $\alpha = 2 - D$.

Fractal surfaces cannot be described using many of the standard statistical metrics. For instance, $\langle h^2 \rangle$ is infinite because of the existence of structure down to infinitely small scales. The existence of infinitely long order also leads to infinite correlation lengths.

However, all is not lost – the power spectrum is still useful for fractals! Let's look at forms:

$$P(k) = 1/k^{\nu}$$

where nu is related to the dimensionality of the fractal by

$$D = \frac{5-\nu}{2}$$

Thus $1 < \nu < 3$. For this range of nu, the Fourier transform of P(k) is infinite, implying an infinite RMS surface height and infinite correlation function.

Finally, we note that the fractal nature of a surface usually holds only over some restricted range of k: $k_1 < k < k_2$. The limits of k are the resolution of the sampling (at the smallest) and the sample size (at the largest.)

F. Some probability integrals

Since Gaussian statistics are so commonly invoked (whether correctly or not!), we often need to evaluate Gaussian moment integrals. There are 1-2 simple tricks to doing that that should be in ones tool kit, which we will describe here for completeness. To begin with, consider:

$$I_n = \int_0^{\infty} x^n \exp(-\lambda x^2) dx$$

where n is an integer. The odd moment integrals vanish when we take the limits from minus infinity to infinity (which is what we want), so we just look at the even moments. Let's consider $I_n$ to be a function of $\lambda$, and take the derivative wrt $\lambda$. We obtain:

$$\frac{dI_n}{d\lambda} = \int_0^{\infty} -x^2 x^n \exp(-\lambda x^2) dx$$

If $I_0$ is known, then all the $I_n$ for even n can be easily obtained. To get $I_0$, we use a simple polar form:

$$I_0^2 = \int_0^{\infty} \exp(-\lambda x^2) dx \int_0^{\infty} \exp(-\lambda y^2) dy = \iint \exp(-\lambda(x^2 + y^2)) dxdy$$

We use the polar forms $r^2 = x^2 + y^2$ and $\tan\theta = y/x$and let the element $dxdy = rdrd\theta$. The integration limits become $\theta = 0 \rightarrow \pi/2$ and r=0 to infinity. Thus we get

$$I_0^2 = \iint \exp(-\lambda r^2) rdrd\theta = \frac{\pi}{2} I_1$$

$I_1$ is easily evaluated using the substitution $u = \lambda x^2$ and $du = 2\lambda x dx$ to give $I_1 = \dfrac{1}{2\lambda}$.

Thus we get that $I_0 = \dfrac{1}{2}\sqrt{\dfrac{\pi}{\lambda}}$ .Differentiating this , we get

$I_2 = \dfrac{\sqrt{\pi}}{4}\lambda^{-3/2}$ and $I_4 = \dfrac{3\sqrt{\pi}}{8}\lambda^{-5/2}$ etc.

The extension from minus infinity to infinity just involves multiplying the above by a factor of two (by symmetry about the origin). Also, we must remember that $\lambda = 1/2\sigma^2$ for the standard Gaussian form.