Lecture #15 Lecture Notes

The ocean water column is very much a 3-D spatial entity, and we need to represent that structure in an economical way to deal with it in calculations. We will discuss one way to do so, empirical orthogonal functions, in what follows.

A.1. EOF's

Let us assume a representation of the 3D soundspeed field (though it can actually be ANY ocean field) by

$$c(x,y,z) = \bar{c}(z) + \delta c(x,y,z)$$

As we have been doing with our other linear inverse problems, let's think of this in matrix form. In particular, let's consider a real world case where we have done six CTD profiles to get the ocean sourdspeed profile at six different (x,y) points. Let's also consider 4 vertical points of each profile as giving a fair sampling of the vertical structure. (We could always use more if the vertical profile has a lot of structure – we just pick four here as an example.) Then the six CTD's and four levels let us form a 4x6 matrix, as you can see in my beautiful handwriting below:

$$\text{ARRAY OF CTD's for } c(z) \Rightarrow \begin{pmatrix} c_1(z_1) & c_2(z_1) & \cdots & c_6(z_1) \\ c_1(z_2) & c_2(z_2) & & c_6(z_2) \\ c_1(z_3) & c_2(z_3) & & c_6(z_3) \\ c_1(z_4) & c_2(z_4) & \cdots & c_6(z_4) \end{pmatrix} = 4\times6 \text{ matrix}$$

Get the mean $\bar{c}(z)$ via $c(z_i) = \frac{1}{M}\sum_{j=1}^{M} c_j(z_i)$

Here, $\underbrace{i = 1 \text{ to } 4}_{N}, \underbrace{j = 1 \text{ to } 6}_{M}$

Now get a "delta c" variance matrix

$$\Delta c \equiv \begin{pmatrix} c_1(z_1)-\bar{c}(z_1) & c_2(z_1)-\bar{c}(z_1) & \cdots & c_6(z_1)-\bar{c}(z_1) \\ c_1(z_2)-\bar{c}(z_2) & c_2(z_2)-\bar{c}(z_2) & & c_6(z_2)-\bar{c}(z_2) \\ c_1(z_3-\bar{c}(z_3) & c_2(z_3)-\bar{c}(z_3) & & c_6(z_3)-\bar{c}(z_3) \\ c_1(z_4)-\bar{c}(z_4) & c_2(z_4)-\bar{c}(z_4 & & c_6(z_4)-\bar{c}(z_4) \end{pmatrix}$$

$= 4\times6$

Subtracting the mean soundspeed at each depth level (seen in equation above) from each measurement, we get the 4x6 soundspeed variance matrix seen above, i.e. the $\Delta c \equiv A$ matrix. This is OK, but we really want to work with square matrices. No problem – we just multiply the A matrix by its transpose, which is a 6x4 matrix, i.e.

$$A^T = \begin{pmatrix} \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \end{pmatrix} = 6\times4 \quad \Rightarrow AA^T = 4\times4 \text{ matrix}$$

"COVARIANCE MATRIX"

This is nice, because the dimension of the matrix is just the number of vertical levels we want to use, generally of order ~10. Often we have many CTD casts, or vertical profiles from towed CTD's like Scanfish, to deal with, so that M>>N in general. The big gain here is that

$$AA^T\Phi = \lambda\Phi$$

is a standard eigenvalue problem! The Φ_i (column vectors) belonging to each λ_i (the eigenvalues which represent the energy in each mode) are called the "Empirical Orthogonal Functions" and will be seen to be very useful. Let's discuss their properties for a second.

First, they are the modes of the covariance matrix – they express the variability of c(x,y,z) about the mean profile in a modal representation. Second, they are empirical because they are data based – no theory or model involved. And they are orthogonal because the relation $\Phi_i\Phi_j^T = \delta_{ij}$ holds. There are also two other properties of the EOF's that turn out to be very useful. First, they put the most energy possible into the lowest modes, i.e. they produce the "reddest" modal spectrum possible. This is useful because the ocean dynamical modes also have a red spectrum, so that the EOF's provide a good match to the ocean dynamics. And second, due to this red property, one can often truncate the modal sum with little or no harm to the results – a nice computational saving.

Let's look closer at the EOF modal representation, which is:

$$c(x,y,z) = \bar{c}(z) + \sum_{i=1}^{N} a_i(x,y)\Phi_i(z)$$

This expression is nice, but have we done anything better than write c(x,y,z) in a different way? Um,yeah. We had c(x,y,z) only at our measurement sites, and now with this representation, we can find c(x,y,z) *everywhere* in a way that is consistent with the ocean dynamics. To see this, let's first get the $a_i(x,y)$ at the measurement sites.

$$\Delta c(x,y,z) = \sum_{i=1}^{N} a_i(x,y)\Phi_i(z)$$

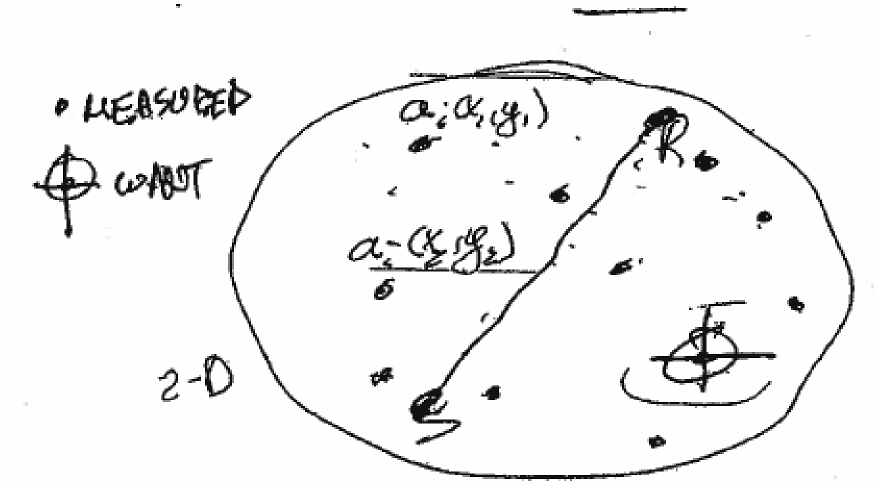
Following standard procedure, we use the orthogonality of the EOF's to project out the $\Phi_i(z)$ on the RHS, i.e. we multiply the equation above by $\Phi_j(z)$ and integrate over z. So we have

$$\int \Phi_j(z)\Delta c(x,y,z)dz = \sum_{i=1}^{N} a_i(x,y)\int \Phi_j(z)\Phi_i(z)dz$$

or

$$a_j(x,y) = \int \Delta c(x,y,z)\Phi_j(z)dz$$

We now have the EOF coefficients at the (x,y) coordinates of our measurements. This is what we need, as we can interpolate the $a_i(x,y)$ in the x-y plane on a mode by mode basis, as seen below!



In the little cartoon above, we want the 3D oceanography along the source to receiver acoustic line for some acoustics purpose or at the "want" spot. But we just have the $a_i(x,y)$ at the measurement points in the region. But, if we interpolate those coefficients in 2D (an easy enough exercise, as we will see) on a mode by mode basis, we can get the field anywhere we want, e.g. the "want" spot on the picture, or along the S/R track. And not only can we do this – this is perhaps the best, most consistent interpolation we can do, as we will see in the next section.

A.2. Gauss Markov interpolation - theory

We can interpolate the EOF coefficients in (x,y) via canned routine 2D interpolators to get a 3D ocean field, and there is no law to stop us. But there are better and worse ways to do the interpolation. In saying this, we're not talking about efficiency, but rather respecting the dynamic scales of the medium so that we don't interpolate entirely uncorrelated points! We will look here at the so-called "Gauss Markov" interpolator, which is both efficient, and also takes into account the scales of the ocean medium.

Assume that a measurement φ_r (e.g. the $a_i(x,y)$ that constitute our "measurement" at each r=(x,y) position) consists of a true value $\theta_r$plus some noise ϵ_r (which can be computed from the error/noise in our c(x,y,z) measurements). So

$$\varphi_r = \theta_r + \epsilon_r$$

If the error is spatially uncorrelated, and also not correlated to the true value, i.e.

$E[\epsilon_r \epsilon_s] = \epsilon^2 \delta_{rs}$ where ϵ^2 is the error variance

$E[\epsilon_r \theta_r] = 0$

Then (without showing the proof) the least squares optimal estimator of θ at any point t is given by

$$\widehat{\theta_t} = \sum_{r=1}^{n} B_{tr} \left[\sum_{s=1}^{n} A_{rs}^{-1}\, \varphi_s \right]$$

In the above, n is the number of measurements (data points) one uses in framing the estimate and the A,B matrices are given by:

$$A_{rs} = E[\varphi_r \varphi_s] = C(\overrightarrow{x_t} - \vec{x}_s) + \epsilon^2 \delta_{rs}$$

$$B_{tr} = X(\overrightarrow{x_t} - \vec{x}_r)$$

where C is the spatial correlation function. We see that this formalism explicitly injects the spatial scale(s) of the ocean medium into the problem, so that the estimate for the soundspeed (or temperature or whatever) field has both reasonable modal structure (the EOF modes and ocean modes are similar) and also the correct correlation scales for the ocean modes. Moreover, in doing the sums above, one only has to consider the n data points that are within a correlation length of the point we are trying to interpolate to. This makes the scheme more numerically efficient as well.

Often, for ocean work, we can use a simple radial form for the correlation function, e.g. a Gaussian, where the width of the Gaussian is the radial spatial correlation length. In considering ocean eddies (as an example), we can take this correlation length to be the "Rossby radius" of deformation, which describes how the Coriolis effect moves the water in a circular path given the latitude and speed of the feature. Specifically,

$$R_{Rossby} = c/|f|$$

$$f = 2\Omega \sin(latitude)$$

c =phase velocity of the feature of interest

As an example, we can use R_{Rossby} ~ 50 km at 45 degrees for deep ocean eddies, and set the correlation length to this number.

Another way to get the correlation function, if one has lots of data, is the data-data correlation function, which we define via

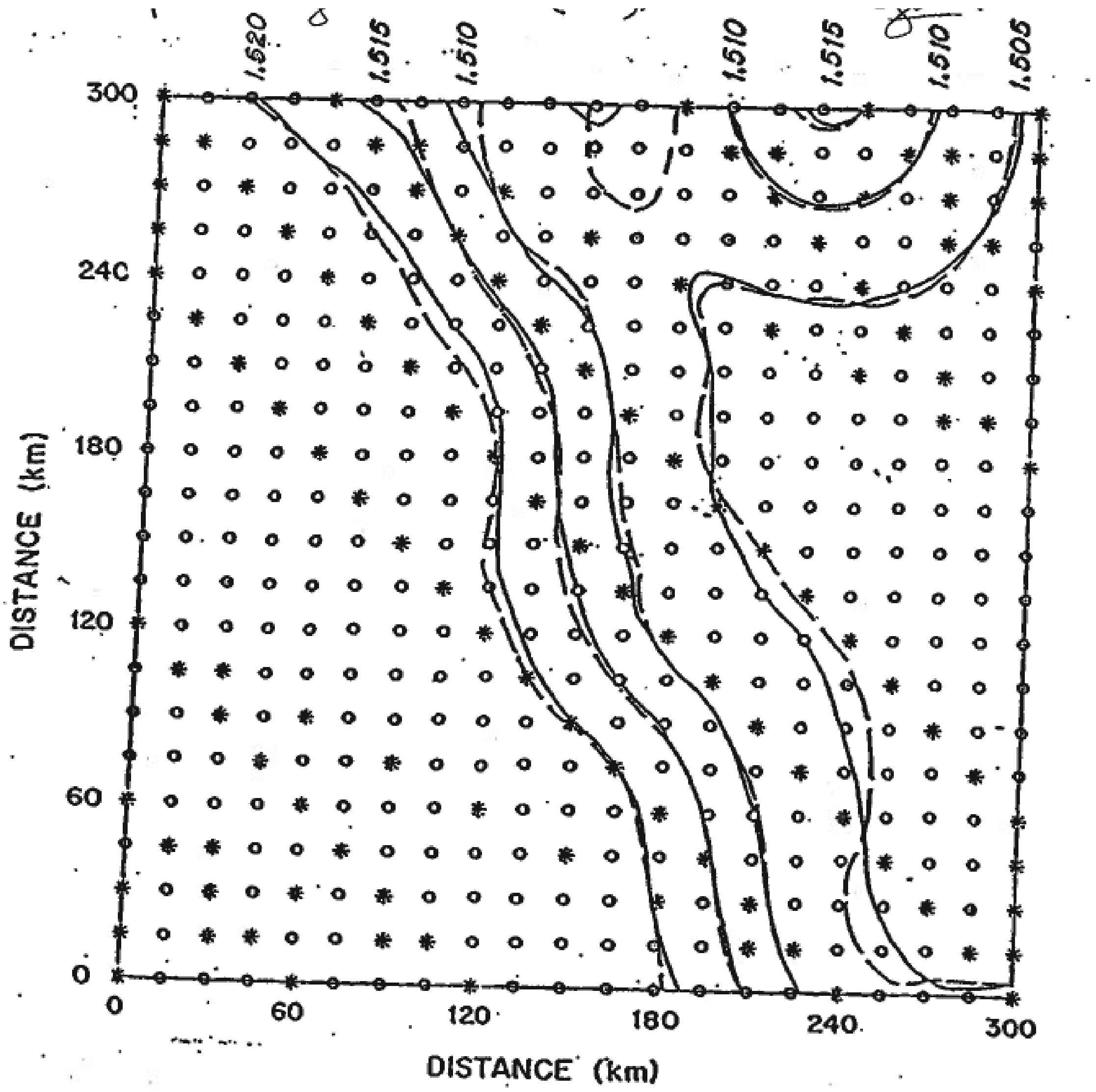
$$C(r) = C(|\vec{r}_{\bar{\bar{n}}} - \overrightarrow{r_n}|)$$

$$= \frac{a_n^m(\vec{r}_n)a_{\bar{n}}^m(\overrightarrow{r_{\bar{n}})}}{|a_n^m(\vec{r}_n)|^2}$$

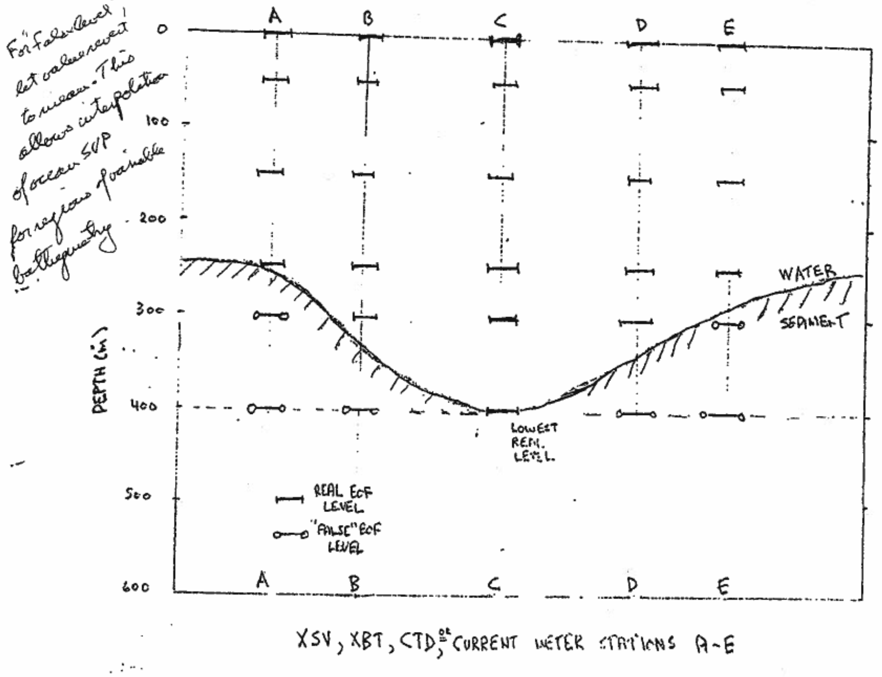
We will generally get a more irregularly shaped correlation function from the real data, but it will be a correct one. We also not in passing that for the EOF's (and any other such modal representation), each mode has its own correlation length, i.e. R_{corr}^m.

A.3. Gauss Markov interpolation – some examples

Let me post some old calculated examples of using the Gauss Markov interpolator in various circumstances. The first example is a Harvard Open Ocean model section of the Gulf Stream. We look at the soundspeed contours at some depth (which doesn't so much matter for now), and get the solid line result from the model. Now, if we remove all the grid points of the model except for the ones with asterisks (a drastic decimation of the data), we want to retrieve an "interpolated" version of the Gulf Stream SSP using a Gauss Markov interpolation. Using the data-data correlation function, we were able to get the dashed line result – a very good match to the original 3D field!



Another useful example to show is how one can “interpolate through bathymetric change” using the Gauss Markov interpolator. By looking at the average soundspeed at each depth level, which can have a different number of points, and assigning the water soundpeed in the sediment to be the average of the water column points at that level, we get reasonable answers, e.g.



For "false level", let values revert to mean. This allows interpolation of ocean SVP for regions of variable bathymetry.

FIG 1. Variable bathymetry region EOF interpolation for water column soundspeed (or current) profiles. Note positioning of false and real levels.

An example of this is interpolating in the vicinity of a seamount. This is a crude old MATLAB result, but shows quite well that the interpolation with and without bathymetry is consistent. The only difference in the results is inside the seamount, where the result is meaningless anyway!

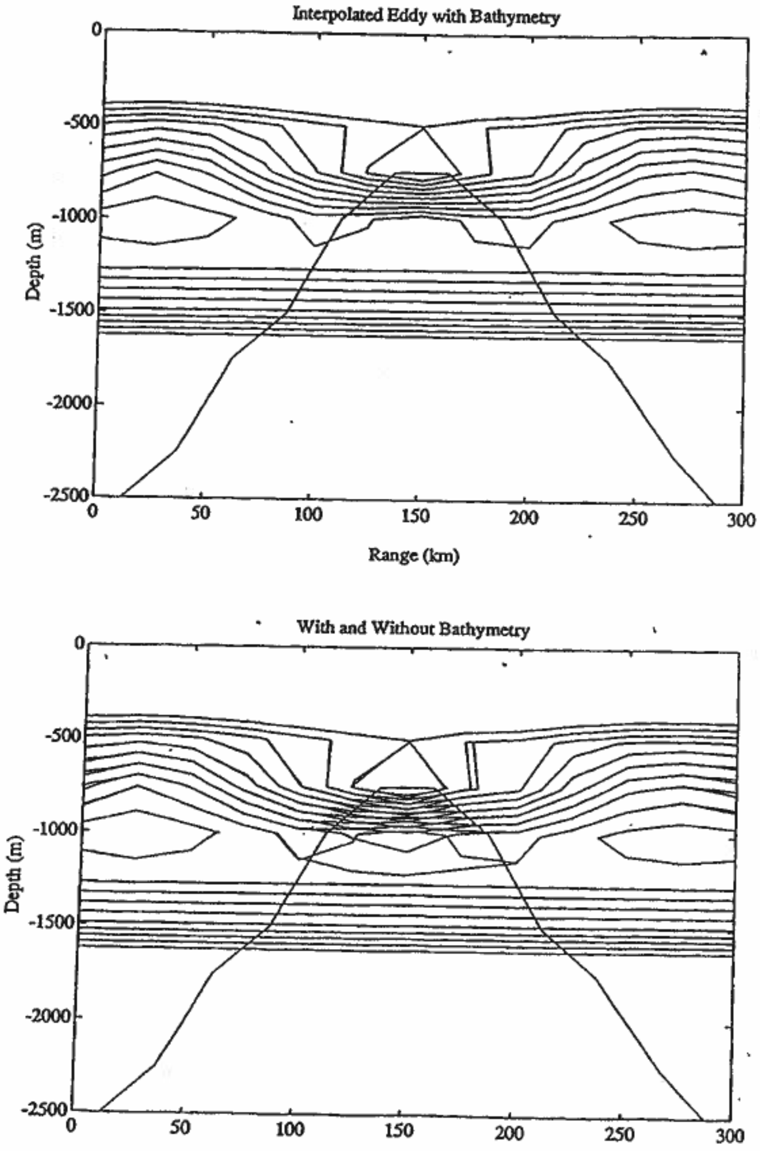


FIG 6. Interpolated profiles with and without "seamount" bathymetric feature. Interpolation is largely unaffected by presence of bathymetry, except inside the seamount where the result is meaningless anyway.

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