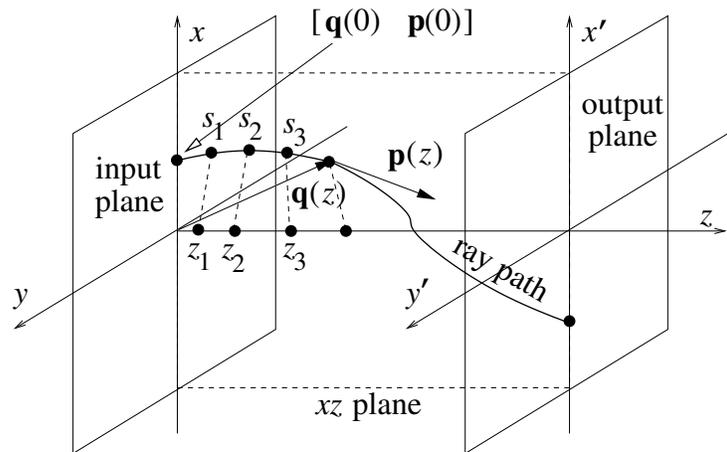


**1. Knocking down one dimension: the Screen Hamiltonian** In class, we derived the set of Hamiltonian equations in 3D space, where the position vector  $\mathbf{q} = (q_x, q_y, q_z)$  and momentum vector  $\mathbf{p} = (p_x, p_y, p_z)$  are parameterized by the variable  $s$ . The “time” variable  $s$  indexes location along the ray in the sense that as we advance  $s$  from 0 to  $s_1$  to  $s_2$  to  $s_3$  and so on, we progress along successive locations on the ray path (see the diagram below.) Using this parameterization, we arrived at the set of Hamiltonian equations

$$\frac{d\mathbf{q}}{ds} = \frac{\partial H}{\partial \mathbf{p}}, \quad \frac{d\mathbf{p}}{ds} = -\frac{\partial H}{\partial \mathbf{q}}, \quad \text{where} \quad H = |\mathbf{p}| - n(\mathbf{q})$$

is the Hamiltonian, and it is conserved because it physically expresses momentum conservation for the ray. In this problem, we will settle on the choice  $H = 0$  for the value of the (conserved) Hamiltonian.



The Hamiltonian equations are a  $6 \times 6$  set of coupled ordinary differential equations. In this problem we will show how this can be simplified to a  $4 \times 4$  set, essentially knocking down one spatial dimension. We begin by the observation that  $s$  seems to be somewhat superfluous: instead of it, we should be able to parameterize with respect to a spatial variable, say  $z$ . Graphically, this is also shown in the figure, where  $s_1, s_2, \dots$  have been projected to their corresponding  $z_1, z_2, \dots$  representing locations on the optical axis. Now the ray is parameterized as  $[\mathbf{q}(z), \mathbf{p}(z)]$  instead of  $[\mathbf{q}(s), \mathbf{p}(s)]$ .

Formally, we do this by eliminating  $s$  from the set of equations

$$x = q_x(s), \quad y = q_y(s), \quad z = q_z(s);$$

*i.e.*, solve the third equation for  $s(z)$  and substitute into the first two equations, which then become

$$x = q_x(z), \quad \text{and} \quad y = q_y(z).$$

We seek to derive a new set of ordinary differential equations to describe the trajectory  $[\mathbf{q}(z), \mathbf{p}(z)]$ .

a) Using the chain rule, show that the following equations are true,

$$\begin{aligned} \frac{dq_x}{dz} &= \frac{p_x}{p_z}, & \frac{dp_x}{dz} &= -\frac{|\mathbf{p}|}{p_z} \frac{\partial n}{\partial q_x}, \\ \frac{dq_y}{dz} &= \frac{p_y}{p_z}, & \frac{dp_y}{dz} &= -\frac{|\mathbf{p}|}{p_z} \frac{\partial n}{\partial q_y}, \\ & & \frac{dp_z}{dz} &= -\frac{|\mathbf{p}|}{p_z} \frac{\partial n}{\partial q_z}. \end{aligned}$$

b) Use the Hamiltonian conservation principle to rewrite these as

$$\begin{aligned} \frac{dq_x}{dz} &= \frac{p_x}{p_z}, & \frac{dp_x}{dz} &= -\frac{n}{p_z} \frac{\partial n}{\partial q_x}, \\ \frac{dq_y}{dz} &= \frac{p_y}{p_z}, & \frac{dp_y}{dz} &= -\frac{n}{p_z} \frac{\partial n}{\partial q_y}, \\ & & \frac{dp_z}{dz} &= -\frac{n}{p_z} \frac{\partial n}{\partial q_z}; \end{aligned}$$

and show that  $p_z$  itself, which so pervasively appears in all the denominators, is given by

$$p_z = \sqrt{n^2(\mathbf{q}) - (p_x^2 + p_y^2)}.$$

(Technically, both solutions  $p_z = \pm\sqrt{\cdot}$  are allowed, but we choose the “+” sign due to the convention that light rays should propagate toward the right hand side, *i.e.* towards positive  $z$ .)

c) Now set

$$h(q_x, q_y, z; p_x, p_y) \equiv -p_z(q_x, q_y, z; p_x, p_y) = -\sqrt{n^2(\mathbf{q}) - (p_x^2 + p_y^2)},$$

and show that the “lateral” dynamical variables  $[(q_x, q_y) (p_x, p_y)]$  obey the  $4 \times 4$  set of Hamiltonian equations (in terms of  $z$ )

$$\begin{aligned} \frac{dq_x}{dz} &= \frac{\partial h}{\partial p_x}, & \frac{dp_x}{dz} &= -\frac{\partial h}{\partial q_x}, \\ \frac{dq_y}{dz} &= \frac{\partial h}{\partial p_y}, & \frac{dp_y}{dz} &= -\frac{\partial h}{\partial q_y}. \end{aligned}$$

The quantity  $h$  is referred to as “Screen Hamiltonian,” in the following sense: if we slice the ray trajectories with successive planes (“screens”) oriented

perpendicular to the optical axis  $z$ , then the set of Screen Hamiltonian equations describes the evolution of the *intersection* of the ray path with the screens as  $z$  advances. That is, we can solve the Screen Hamiltonian equations starting with the initial ray location  $(q_x, q_y)$  and momentum (angle of propagation)  $(q_x, q_y)$  at an “input plane”  $z = 0$  and solve for the location and angle at any “output plane” located at  $z > 0$  (see again the figure on the first page.) Therefore, the Screen Hamiltonian equations do the same thing as the ray matrices that we learnt how to use earlier in Geometrical Optics; in fact, it can be shown that if the successive surfaces that we used then approach to within infinitesimally small spacing, then the same Screen Hamiltonian equations can be derived using the matrix approach (but we won’t do that here.)

- d) Show that the Screen Hamiltonian is not conserved in general, but there is a special case when it is conserved. Argue that both cases are physically correct, *i.e.* the non-conservation in the general case does not violate any physical principles; and that the conservation in the special case reverts back to a physical interpretation of momentum conservation in a 2D space.

**2. Rays in Harmonic Oscillation** As application of the Screen Hamiltonian, we will ray trace a case where the index of refraction is independent of  $z$ . We further simplify by assuming that it is independent of  $y$  as well, so it is sufficient to ray trace on the  $xz$  plane (see the figure in Problem 1.) The Screen Hamiltonian equations are then reduced to

$$\frac{dq_x}{dz} = \frac{\partial h}{\partial p_x}, \quad (1)$$

$$\frac{dp_x}{dz} = -\frac{\partial h}{\partial q_x}. \quad (2)$$

The ray trajectory is the solution  $\{q_x(z), p_x(z)\}$  of these coupled differential equations.

The Screen Hamiltonian equations can rarely be solved analytically. One case that indeed admits an analytical solution is the “elliptical GRIN” profile

$$n(x) = \sqrt{n_0^2 - \kappa^2 q_x^2}. \quad (3)$$

The solution is

$$q_x(z) = q_0 \cos\left(\frac{\kappa z}{h}\right) + \frac{p_0}{\kappa} \sin\left(\frac{\kappa z}{h}\right), \quad (4)$$

$$p_x(z) = p_0 \cos\left(\frac{\kappa z}{h}\right) - \kappa q_0 \sin\left(\frac{\kappa z}{h}\right), \quad (5)$$

where  $q_0 \equiv q_x(0)$  and  $p_0 \equiv p_x(0)$  are the initial ray position and momentum, respectively, and

$$h = -\sqrt{n_0^2 - (\kappa^2 q_x^2 + p_x^2)} \quad (6)$$

is the Screen Hamiltonian, as defined in Problem 1.

- a) Show that  $h$  is independent of  $z$ .
- b) Using the result of part (a) to simplify your algebra, show that indeed (4) and (5) are the solution to the Hamiltonian equations (2).
- c) Plot the ray position  $q_x(z)$  as function of  $z$  for a collimated ray bundle parallel to the optical axis  $z$  and an elliptical GRIN with  $n_0 = 1.5$ ,  $\kappa = 0.1$ . Plot several traces for values of  $q_0$  ranging from  $-10 : 10$  and for  $0 \leq z \leq 25$ .
- d) In your opinion, does the elliptical GRIN focus the incident parallel ray bundle satisfactorily? What is the dominant aberration that you observe?

The solution (4–5) is analogous to that of the mechanical harmonic oscillator that we saw in the **Lecture 9, p. 22** with natural frequency  $\kappa/h$ .

We will pursue the analogy further in the next two problems.

- 3. Mechanical Screen Hamiltonian** Consider a mechanical system whose Hamiltonian is given as the sum of kinetic and potential energies

$$H = \frac{|\mathbf{p}|^2}{2m} + V(\mathbf{q}) = \frac{p_x^2 + p_y^2 + p_z^2}{2m} + V(\mathbf{q}) \equiv E,$$

where  $E$  is the energy, and it is conserved. Follow the same procedure as we did in Problem 1 in the optical case to

- a) show that in the mechanical case we may also parameterize the trajectory in terms of  $z$  (instead of time  $t$ ) and the Screen Hamiltonian of the mechanical system is

$$h = -\sqrt{2m \left( E - V(\mathbf{q}) \right) - (p_x^2 + p_y^2)};$$

- b) write down the Screen Hamiltonian equations for the mechanical system;
- c) argue the physical meaning of the mechanical Screen Hamiltonian, and a physical justification for the general case when it is not conserved, as well as for the special case when it is conserved.

- 4. Mechanical Harmonic Oscillator** This is a special case of the mechanical Screen Hamiltonian of the previous problem, where the potential is

$$V(\mathbf{q}) = \frac{1}{2}kq_x^2.$$

As in Problem 2, for simplicity we have again restricted motion to the  $xz$  plane (*i.e.*, we ignore all  $y$  derivatives.)

- a) Show that the mechanical Screen Hamiltonian  $h$  is conserved in this case.
- b) Show that the solution of the dynamical equations is

$$q_x(z) = q_0 \cos\left(\frac{\sqrt{mk}}{h}z\right) + \frac{p_0}{\sqrt{mk}} \sin\left(\frac{\sqrt{mk}}{h}z\right),$$

$$p_x(z) = p_0 \cos\left(\frac{\sqrt{mk}}{h}z\right) - \sqrt{mk} q_0 \sin\left(\frac{\sqrt{mk}}{h}z\right),$$

where  $q_0 \equiv q_x(0)$  and  $p_0 \equiv p_x(0)$  are the position and lateral momentum, respectively, at  $z = 0$ . (“Lateral” means momentum in the  $x$  direction.)

- c) Discuss why we are safe from the danger of  $h = 0$  blowing up our solution, and explain physically any assumptions that are required for this argument.
- d) Given that indeed  $h \neq 0$ , how is the Screen Hamiltonian solution consistent with the harmonic oscillator solution from the Notes (Lecture 9, p. 22) that had natural frequency  $\sqrt{k/m}$ ?

**5. Quadratic GRIN** Finally, we turn back to Optics and the case of the GRIN profile that is most often implemented in practice,

$$n(x) = n_0^2 - \frac{\alpha}{2}x^2. \quad (7)$$

This does not yield an analytical solution to the Hamiltonian equations, so we will integrate it numerically. Download from the class website the Matlab functions `bundle_raytrace.m`, `screen_raytrace.m`, `sgradh_elliptical.m`, and `sgradh_quadratic_hw.m`. The main function `screen_raytrace` is the solver that computes ray trajectories numerically from the set of Screen Hamiltonian equations, while `bundle_raytrace` calls the solver repeatedly for a collimated ray bundle parallel to the optical axis  $z$  and several values of  $q_0$ .

Note that the Matlab functions are for two-dimensional screens  $\mathbf{q} = (q_x, q_y)$ ,  $\mathbf{p} = (p_x, p_y)$  but you can ignore the 2<sup>nd</sup> ( $y$ ) dimension by setting all arguments equal to zero.

- a) Call `bundle_raytrace` with `sgradh_elliptical` as argument to numerically ray-trace the elliptical GRIN (3) that we solved analytically earlier. Verify that the numerical traces agree with your plots from part (d). Make sure that you study and understand the structure of `sgradh_elliptical.m`.
- b) Now consider `sgradh_quadratic_hw.m` for solving the quadratic GRIN (7). Edit the function to enter the proper value of the derivative  $dp_x/dz$ . Ray trace again by calling `bundle_raytrace` with `sgradh_quadratic` as argument. How does the performance of the quadratic GRIN compare with the elliptical GRIN in terms of focusing quality?

- c) Is the Screen Hamiltonian conserved in your numerical results? Is that consistent with your conclusion about the conservation (or not) of the Screen Hamiltonian from Problem 1?

**6. Plane waves and phasor representations** Throughout this problem, by “real expression” of a wave we mean the space-time representation, e.g.  $f(x, y, z, t) = A \cos(kz - \omega t)$  is a plane wave of wave-vector magnitude  $k$  and frequency  $\omega$  propagating in the direction of the  $\hat{\mathbf{z}}$  coordinate axis. By “phasor expression” we mean the complex representation of the wave, e.g.  $Ae^{ikz}$  for the same wave.

- 6.a)** Write the real and phasor expressions for a plane wave  $f_1(x, y, z, t)$  propagating at an angle  $30^\circ$  relative to the  $\hat{\mathbf{z}}$  axis on the  $xz$ -plane (*i.e.*, the plane  $y = 0$ ). The wavelength is  $\lambda = 1\mu\text{m}$ , and the wave speed is  $c = 3 \times 10^8 \text{m} \cdot \text{sec}^{-1}$ .
- 6.b)** Write the real and phasor expressions for a plane wave  $f_2(x, y, z, t)$  of the same wavelength and wave speed as  $f_1$  but propagating at angle  $60^\circ$  relative to the  $\hat{\mathbf{z}}$  axis on the  $yz$ -plane.
- 6.c)** Use the real expression to plot  $f_1(x, y, z = 0, t = 0)$  and  $f_2(x, y, z = 0, t = 0)$  using MATLAB. (*Note:* you will probably need to use `surf` or an equivalent command.)
- 6.d)** The plane  $z = 0$  is illuminated by the superposition of the two waves  $f_1$  and  $f_2$ . Plot the waveform received at points A, B, C, D, E with Cartesian coordinates, respectively,

$$(0, 0, 0), \left(\frac{1}{4}, -\frac{1}{4\sqrt{3}}, 0\right), \left(\frac{1}{2}, -\frac{1}{2\sqrt{3}}, 0\right), \left(\frac{3}{4}, -\frac{3}{4\sqrt{3}}, 0\right), \left(1, -\frac{1}{2\sqrt{3}}, 0\right)$$

(all units in microns) as function of time. What do you observe?

**7. Wave superposition** Consider the following two waves,

$$f_1(x, z, t) = 5 \cos\left(\frac{2\pi}{17} \left[z + \frac{x^2}{2z}\right] - 2\pi 10t\right)$$

$$f_2(x, z, t) = 5 \cos\left(\frac{2\pi}{17} \left[z + \frac{(x-5)^2}{2z}\right] - 2\pi 10t + \frac{\pi}{3}\right)$$

- a) What is the physical interpretation of these waves? Be as detailed as possible.
- b) If these waves approximately satisfy the (Helmholtz) Wave Equation, what is the phase velocity?
- c) Using trigonometric identities, calculate the wave that results from the “coherent superposition” of the two waves, *i.e.*  $f(x, z, t) \equiv f_1(x, z, t) + f_2(x, z, t)$ .

- d) Now express each wave as phasor, add the two phasors and compare the result to the phasor of  $f(x, z, t)$  from part (b).

**8. Dispersive waves** Consider the metallic waveguide from Lecture 13, p. 7

- a) An electromagnetic wave of frequency  $\omega = 1.5 \times 10^{15}$  rad/sec is launched into the waveguide such that at  $t = 0$  the amplitude is maximum at a distance  $0.4 \mu\text{m}$  inside the guide. Plot the wave 1.05 fsec, 2.1 fsec, and 3.15 fsec later.
- b) Compare the distance traveled by a point of constant phase on the wavefront in 4.2 fsec to the distance traveled in the same time by a point of constant phase on the wavefront of a wave of the same frequency but propagating in free space.
- c) Show that the group velocity is

$$v_g = c \sqrt{1 - \left(\frac{m\pi c}{a\omega}\right)^2}.$$

**9. Schroedinger's Equation** The equation describing the wavepacket associated with a particle in Quantum Mechanics is

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} = -i \frac{2m}{\hbar} \frac{\partial \psi}{\partial t},$$

where  $m$  is the particle mass,  $\hbar = h/2\pi$  and  $h$  is Planck's constant.

- a) Interpret the  $i$  appearing in this equation.
- b) Use a procedure similar to Lecture 13, p. 4 to derive the dispersion relation for Schroedinger waves and plot it as a dispersion diagram.

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