

Problem 1: Spherical waves and energy conservation

In class we mentioned that the radiation from a point source can be characterized by a spherical wave in which the phase of the electric field is constant along a spherical surface (wavefront). As time evolves, the wavefronts expand outwards and thus the amplitude of the electric field at a given point changes. However, energy conservation requires the power to be constant at any radial distance from the point source after integrating over the spherical surface. The intensity of the field is defined as the power P over the area A (in Watts/m²),

$$I = \frac{P}{A},$$

where for a spherical surface $A = 4\pi r^2$. In addition, the intensity is proportional to the square of the electric field amplitude,

$$I = |E|^2.$$

Taking the square root on both sides of the previous equation, we arrive to the conclusion that the amplitude of the electric field for a spherical wave decays like $1/r$ and the intensity of the wave follows an “inverse-square law”.

Problem 2: Order of magnitude calculations

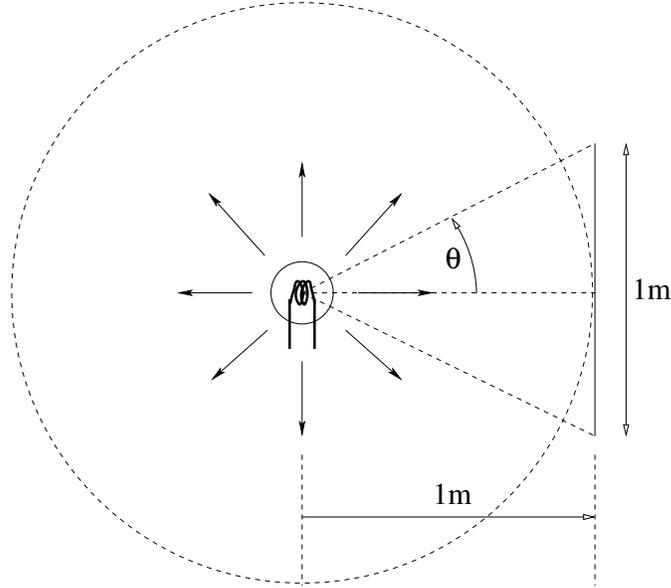
a) From the given website, we can see that the total solar irradiance falling outside the atmosphere of the earth is approximately 1370W/m². Consider a rectangular patch of 1m² surface area, shown in cross-section in the schematic below, illuminated by a light bulb placed 1m away from the patch's centre. From the geometry, we can see that the light bulb, approximated as a point radiator, subtends with respect to the patch a half-angle

$$\theta = \arctan \frac{0.5}{1} = 0.4636 \text{ rad.}$$

Let's assume that the light bulb is an isotropic radiator, and emits power P over the 4π solid angle. Then the power reaching the 1m² patch is $P(2\theta)^2/4\pi$. To match the irradiance received from the sun, we require

$$P \frac{(2\theta)^2}{4\pi} = 1370 \Rightarrow P \approx 2 \times 10^4 \text{ W} = 20 \text{ kW.}$$

A commercial light bulb has a power of 100W so it requires the power of approximately 200 light bulbs at a distance as close as 1m to match the solar irradiance.



b) The energy of a photon of wavelength λ is given by,

$$E(\lambda) = \frac{hc}{\lambda},$$

where h is Planck's constant (6.626×10^{-34} Js) and c is the speed of light in vacuum (3×10^8 m/s). We assume that the solar energy is uniformly distributed between the ultraviolet and infrared at $\lambda_1 = 300$ nm and $\lambda_2 = 1\mu$ m, respectively. Therefore, the solar energy at a given wavelength is $E_{s\lambda} = E_s/\Delta\lambda$, where $E_s = 1370$ W/m² as given in part (a) and $\Delta\lambda = \lambda_2 - \lambda_1$. The photon flux at a given wavelength is given by,

$$P_{flux\lambda} = \frac{E_{s\lambda}}{E(\lambda)}.$$

So, in order to find the total photon flux over the given spectral range we need to integrate the previous equation,

$$\begin{aligned} P_{flux} &= \int_{\lambda_1}^{\lambda_2} \frac{E_{s\lambda}}{hc} \lambda d\lambda = \frac{E_{s\lambda}}{hc} \left(\frac{\lambda_2^2}{2} - \frac{\lambda_1^2}{2} \right) \\ &= \frac{E_s}{2hc} \left(\frac{\lambda_2^2 - \lambda_1^2}{\lambda_2 - \lambda_1} \right) = \frac{E_s}{2hc} \left[\frac{(\lambda_2 - \lambda_1)(\lambda_2 + \lambda_1)}{(\lambda_2 - \lambda_1)} \right] \\ &= \frac{E_s}{2hc} (\lambda_2 + \lambda_1) = 4.4798 \times 10^{21} \text{Photons/m}^2\text{s}. \end{aligned}$$

Problem 3: The lifeguard problem

This problem essentially asks you to derive Snell's law using the analogy of the lifeguard. The lifeguard must get to the swimmer as quickly as possible in order to

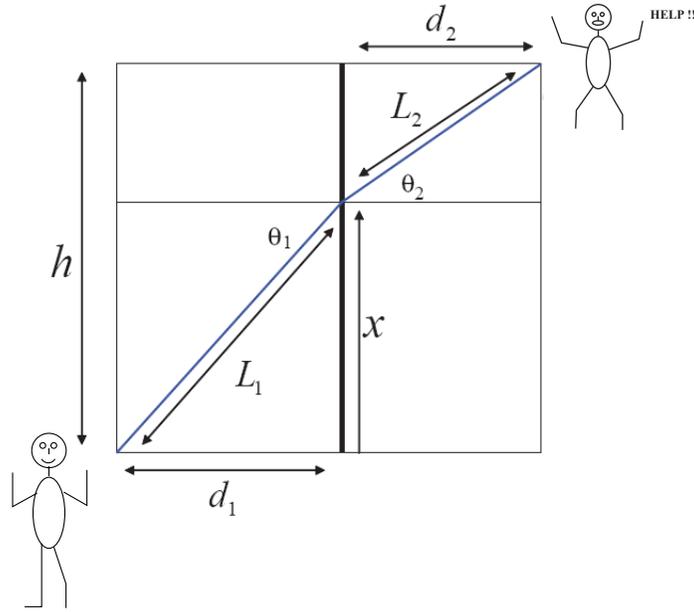


Figure 1: The lifeguard problem.

prevent him/her from drowning. The lifeguard can achieve different speeds running on the sand and swimming in the water. We seek to find the optimal path (i.e. the direction and distance that the lifeguard must travel on the sand and water) that minimizes the time required to reach the swimmer. Similarly, Fermat's principle tells us that a photon seeks to minimize the time of travel between two points. For the simple case of light travelling from one medium to another, the problem is exactly the same as the lifeguard problem. The geometry for this problem is shown in Figure 1. From this figure we see that

$$L_1 = \sqrt{x^2 + d_1^2}, \text{ and}$$

$$L_2 = \sqrt{(h - x)^2 + d_2^2}.$$

Let the running speed on sand be $v_{sand} = c$, and the swimming speed $v_{swim} = c/n$. Since velocity is distance over time, the time required to travel a distance x at a velocity v is $t = x/v$. So the total time is

$$t = \frac{1}{c} (L_1 + nL_2).$$

To minimize the total time, we differentiate the above equation with respect to x and

set the derivative to zero:

$$\begin{aligned} \frac{\partial t}{\partial x} &= \frac{1}{c} \left(\frac{\partial L_1}{\partial x} + n \frac{\partial L_2}{\partial x} \right) \\ &= \frac{1}{c} \left(\frac{x}{\sqrt{x^2 + d_1^2}} - \frac{n(h-x)}{\sqrt{(h-x)^2 + d_2^2}} \right) = 0 \\ &\Rightarrow \sin \theta_1 = n \sin \theta_2, \end{aligned}$$

which is Snell's law!

Problem 4: Refraction from a dielectric stack

In general, given the angle entering a layered media to compute the exit angle we only need to account for the refractive indices of the first, n_0 , and last, n_7 layers. This is because the quantity $n \sin \theta$ is conserved at each refraction stage, according to Snell's law, *i.e.*

$$n_1 \sin \theta_1 = n_2 \sin \theta_2 = \dots = n_7 \sin \theta_7 \dots$$

However, notice that $n_0 > n_1 > n_2 > \dots > n_7$. This suggests that a given ray gradually transitions from a high-index to a low-index medium. In class we learned that in each case, total internal reflection (TIR) may occur if the ray attempts to enter the lower index medium at angle larger than the critical angle. The critical angle is determined by the refractive indices of the incident, n_{inc} , and transmitted, n_{trans} , media

$$\theta_c = \arcsin \left(\frac{n_{trans}}{n_{inc}} \right).$$

From Snell's law, we see TIR occurs at layer 7, since

$$\frac{n_0}{n_7} \sin \theta_0 > 1.$$

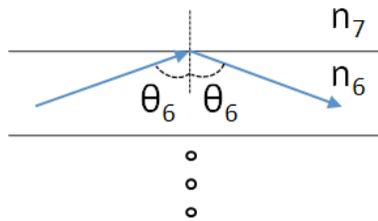


Figure 2: TIR at layer 6.

As shown in Figure 2, the ray arrives at layer 6 at an angle $\theta_6 = \arcsin(n_0/n_6 \cdot \sin \theta_0) = 83.39^\circ$. The critical angle in transitioning between layers 6 and 7 is $\theta_c = 82.48^\circ$, so we see that $\theta_6 > \theta_c$. After suffering TIR at this interface, the ray propagates back along the layered media exiting at the same angle as the entrance angle, $\theta_{exit} = \theta_0 = 76^\circ$.

Problem 5: Trapezoidal prism

The geometry for this problem is shown in Figure 3. Snell's law at the entrance interface yields

$$\sin \theta' = \left(\frac{1}{n} \sin \theta \right), \quad (1)$$

while the angles inside the triangle are related as

$$\psi + \alpha + \left(\frac{\pi}{2} + \theta' \right) = \pi. \quad (2)$$

The TIR condition at the next interface is

$$n \sin \alpha' > 1 \Leftrightarrow n \sin \left(\frac{\pi}{2} - \alpha \right) > 1. \quad (3)$$

Combining (2) and (3) we obtain

$$\begin{aligned} n \sin (\theta' + \psi) &> 1 \Rightarrow \\ n [\sin \theta' \cos \psi + \cos \theta' \sin \psi] &> 1 \end{aligned} \quad (4)$$

Combining (1) into (4) we obtain

$$\begin{aligned} n \left[\frac{\sin \theta}{n} \cos \psi + \sqrt{1 - \frac{\sin^2 \theta}{n^2}} \sin \psi \right] &> 1 \Rightarrow \\ \sqrt{n^2 - \sin^2 \theta} \sin \psi &> 1 - \sin \theta \cos \psi \Rightarrow \\ (n^2 - \sin^2 \theta) \sin^2 \psi &> 1 + \sin^2 \theta \cos^2 \psi - 2 \sin \theta \cos \psi \Rightarrow \\ n^2 \sin^2 \psi &> \sin^2 \psi + (\cos \psi - \sin \theta)^2 \Rightarrow \\ n^2 &> 1 + \left(\frac{\cos \psi - \sin \theta}{\sin \psi} \right)^2. \end{aligned} \quad (5)$$

As a sanity check on (5) we try the case of normal incidence $\theta = 0$, which yields

$$n^2 > 1 + \left(\frac{\cos \psi}{\sin \psi} \right)^2 = \frac{1}{\sin^2 \psi},$$

i.e. the usual TIR condition for incidence at angle $\alpha' = \frac{\pi}{2} - \alpha = \psi$ since $\theta = 0$.

Problem 6: Ellipsoidal refractor

a) In this problem we are told that the ellipsoidal refractor is designed to focus perfectly a plane wave at a nominal wavelength of $\lambda_1 = 1.5 \mu\text{m}$. At this wavelength, the refractive index of SF6 glass is $n_1 = 1.7649$. We find this either by interpolating the numerical data given in the website, or applying the formula given in the same website. Here, we have followed the latter approach. The MATLAB script that generates the index of refraction for SF6 as function of wavelength is `n_sf6.m` and has been posted in the class Stellar website (under `Materials`→`MATLAB CODE`.)

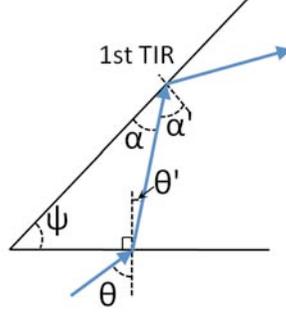


Figure 3: Trapezoidal prism problem.

We revisit the ellipsoidal refractor equation presented in class,

$$\left(s - \frac{n}{n+1}f\right)^2 + \left(\frac{n^2}{n^2-1}\right)x^2 = \left(\frac{n}{n+1}f\right)^2.$$

We want to rewrite the previous equation into the same form of the equation of a shifted ellipse,

$$\frac{(s-h)^2}{a^2} + \frac{x^2}{b^2} = 1$$

where a and b are the ellipse major and minor semi-axes, respectively, and h is the displacement of the ellipse centre along the horizontal axis (optical axis) with respect to the coordinate origin. For the ellipsoidal refractor,

$$a = h = \frac{n}{n+1}f, \text{ and}$$

$$b = f\sqrt{\frac{n-1}{n+1}}.$$

Since the focal length at the nominal wavelength is $f_1 = 15\text{mm}$, the major axis of the ellipsoidal refractor is

$$a_1 = \frac{n_1}{n_1+1}f_1,$$

and that is *fixed*, since it represents the shape of the optical element.

Suppose that we illuminate the previous refractor with a plane wave at a wavelength $\lambda_2 \neq \lambda_1$. The ellipsoidal refractor now has a different refractive index n_2 , which can be computed from the SF6 data at the new wavelength λ_2 . Since the shape of the lens remains the same,

$$a_1 = a_2$$

$$\frac{n_1}{n_1+1}f_1 = \frac{n_2}{n_2+1}f_2$$

$$\Rightarrow f_1 = f_2 \left(\frac{n_2}{n_1}\right) \left(\frac{n_1+1}{n_2+1}\right).$$

For perfect focusing at this new wavelength, the second parameter of the ellipse (i.e. b) needs to also remain constant. In other words, we want to verify if,

$$\begin{aligned}
 & b_1 \stackrel{=?}{=} b_2 \\
 & f_1 \sqrt{\frac{n_1 - 1}{n_1 + 1}} \stackrel{=?}{=} f_2 \sqrt{\frac{n_2 - 1}{n_2 + 1}} \\
 & f_2 \left(\frac{n_2}{n_1} \right) \left(\frac{n_1 + 1}{n_2 + 1} \right) \sqrt{\frac{n_1 - 1}{n_1 + 1}} \stackrel{=?}{=} f_2 \sqrt{\frac{n_2 - 1}{n_2 + 1}} \\
 & \Rightarrow \frac{(n_1 + 1)}{n_1} \sqrt{\frac{n_1 - 1}{n_1 + 1}} \neq \frac{(n_2 + 1)}{n_2} \sqrt{\frac{n_2 - 1}{n_2 + 1}}.
 \end{aligned}$$

So we conclude that the answer is *no*, if the operating wavelength changes but the refractor shape does not change, then perfect focusing does no more occur. In order to have perfect focusing with the new wavelength, the shape of the refractor has to change. Figure 4 shows the case when the ellipsoidal refractor is illuminated with the nominal wavelength. Figure 5 shows the case of the same refractor but illuminated with a wavelength different than the nominal. These figures were simulated using the ray tracing program called Zemax.

(b) To obtain an *estimate* of the focal length at wavelengths other than the nominal, we may still use the approximate expression

$$f(\lambda) \approx \left(1 + \frac{1}{n(\lambda)} \right) a_1,$$

where a_1 is the ellipse's major axis at the nominal wavelength $\lambda_1 = 1.5\mu\text{m}$. This equation is valid in the paraxial approximation, as can be seen in Figure 5. The result is shown in Figure 6.

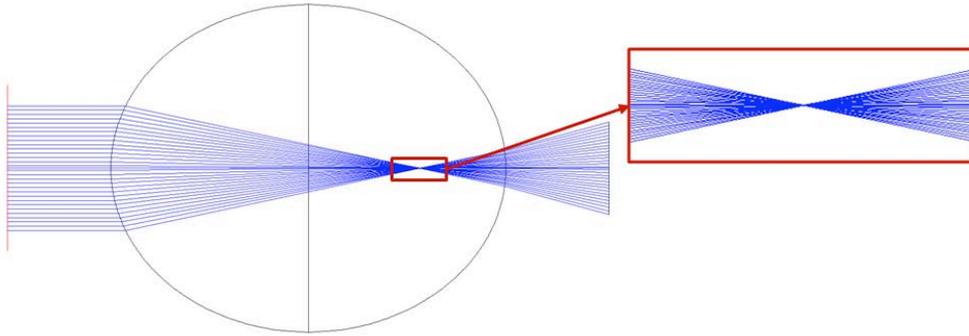


Figure 4: Ellipsoidal refractor illuminated at the nominal wavelength.

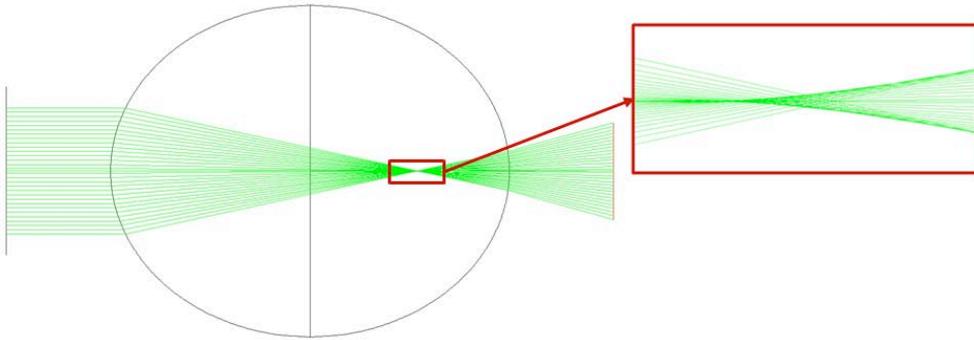


Figure 5: Ellipsoidal refractor illuminated at $\lambda = 700\text{nm}$.

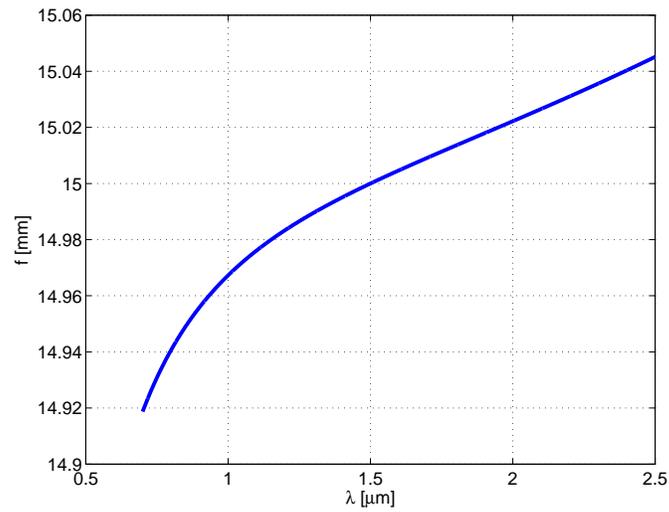


Figure 6: Approximate focal length of an ellipsoidal refractor.

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