

**Problem 1: Wiper speed control**

Figure 1 shows an example of an optical system designed to detect the amount of water present on the windshield of a car to adjust the wiper speed. As shown in this figure, we can use the windshield as a waveguide to guide the light from a source located at one end (bottom of the windshield) to a detector located in the opposite end. The light suffers total-internal reflection (TIR) at the glass-air interface. However, when rain drops are present, some of the light will suffer frustrated TIR escaping outside the waveguide. Since we know the power of the light source, a given drop in power can be correlated to the amount of water present and used to adjust the wiper speed.

To quantify the feasibility of this design we start by computing the critical angles of the glass-air and glass-water interfaces,

$$\begin{aligned}\theta_{c_{g-a}} &= \arcsin\left(\frac{n_{air}}{n_{glass}}\right) = 41.8^\circ, \\ \theta_{c_{g-w}} &= \arcsin\left(\frac{n_{water}}{n_{glass}}\right) = 62.5^\circ.\end{aligned}$$

The incidence angle of a given ray propagating inside this waveguide is restricted to the following cases:

1. For  $\theta < \theta_{c_{g-a}}$ : The light will suffer frustrated TIR and escape out of the waveguide regardless of whether the interface is glass-air or glass-water.
2. For  $\theta_{c_{g-a}} < \theta < \theta_{c_{g-w}}$ : The light will suffer TIR at the glass-air interface and frustrated TIR at the glass-water interface.
3. For  $\theta_{c_{g-w}} < \theta$ : The light will suffer TIR at both interfaces.

**Problem 2: Telescope with infinite conjugates**

a) The geometry of this problem is shown in Figure 2. By inspection, we see that the first lens (L1) will focus the light at a distance of  $f_1$  at its back focal plane. This point now acts as a point source object for the second lens (L2) and since it is located at its front focal plane (provided that  $d = f_1 + f_2$ ), the light beam gets collimated and thus exits the optical system as a parallel ray bundle (plane wave).

b) To answer this question we use the matrix formulation. The output angle and height are related to the input angle and height according to,

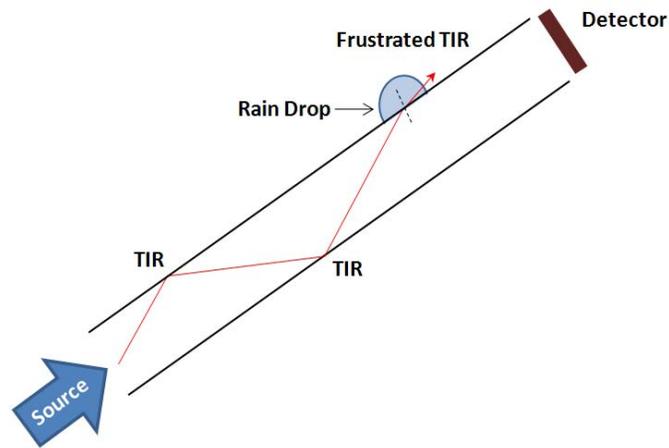


Figure 1: Wiper speed control problem.

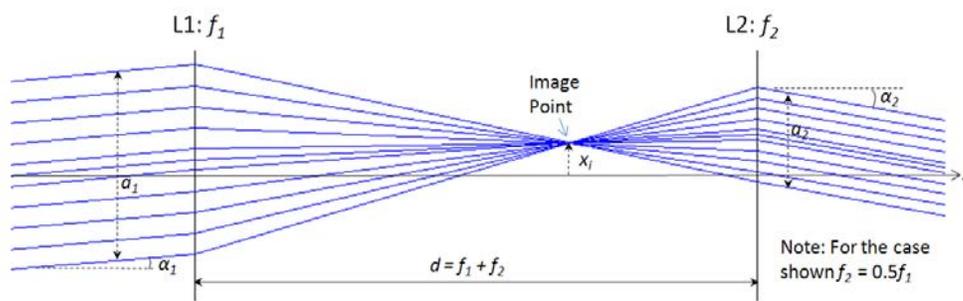


Figure 2: Telescope with infinite conjugates.

$$\begin{aligned}
\begin{bmatrix} \alpha_2 \\ x_2 \end{bmatrix} &= \begin{bmatrix} 1 & -\frac{1}{f_2} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ d & 1 \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{f_1} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ x_1 \end{bmatrix} \\
&= \begin{bmatrix} 1 - \frac{d}{f_2} & \frac{d}{f_1 f_2} - \frac{1}{f_1} - \frac{1}{f_2} \\ d & 1 - \frac{d}{f_1} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ x_1 \end{bmatrix}.
\end{aligned} \tag{1}$$

So the optical power of the composite optical system is,

$$P = \frac{d}{f_1 f_2} - \frac{1}{f_1} - \frac{1}{f_2} = -\frac{(f_1 + f_2) - d}{f_1 f_2}, \tag{2}$$

and the effective focal length is,

$$EFL = -\frac{1}{P} = \frac{f_1 f_2}{(f_1 + f_2) - d}. \tag{3}$$

Note that for the case of  $d = f_1 + f_2$ , equations 2 and 3 become:  $P = 0$ , and  $EFL = \infty$ . These type of optical systems are called *afocal*.

c) From equation 1, we find the equations for output angle,  $\alpha_2$ , and height,  $x_2$ ,

$$\begin{aligned}
\alpha_2 &= \left(1 - \frac{d}{f_2}\right) \alpha_1 + P x_1 \\
&= \left(1 - \frac{f_1 + f_2}{f_2}\right) \alpha_1 \\
&= -\alpha_1 \frac{f_1}{f_2},
\end{aligned} \tag{4}$$

$$\begin{aligned}
x_2 &= d \alpha_1 + \left(1 - \frac{d}{f_1}\right) x_1 \\
&= (f_1 + f_2) \alpha_1 - \frac{f_2}{f_1} x_1.
\end{aligned} \tag{5}$$

To find the beam width,  $a_2$ , we consider the case of normal incidence (i.e.  $\alpha_1 = 0$ ),

$$\begin{aligned}
a_2 &= 2x_2 = -2x_1 \frac{f_2}{f_1} \\
&= -a_1 \frac{f_2}{f_1}.
\end{aligned} \tag{6}$$

d) From Figure 2, we can see that a *real* image is formed of the off-axis object at infinity and is located at a distance  $f_1$  at the back focal plane of lens L1. The elevation of the image point respect to the optical axis is,

$$x_i = \alpha_1 f_1. \quad (7)$$

The second lens collimates this image point and the output angle is,

$$\alpha_2 = -\frac{x_i}{f_2} \quad (8)$$

The relations shown in equations 7 and 8 can also be used as a simple way to compute the angular and lateral magnifications. To compute the angular and lateral magnifications, we combine the previous relations and solve for the ratio between output and input angles,

$$\begin{aligned} M_A &= \frac{\alpha_2}{\alpha_1} = -\frac{f_1}{f_2}, \\ M_L &= \frac{1}{M_A} = -\frac{f_2}{f_1}, \end{aligned} \quad (9)$$

which are consistent with the results we found in part (c).

e) Figure 3 shows the new geometry of the system if we assume that L1 is negative ( $f_1 < 0$ ). We see that in order to have a collimated output beam, the separation between lenses is still  $d = f_1 + f_2$ ; however, to ensure a positive separation distance (i.e.  $d > 0$ ) we require that  $f_2 > |f_1|$ . The first lens now forms a virtual image at a distance  $f_1$  to the left of L1. The elevation of the virtual image respect to the optical axis again  $x_i = \alpha_1 f_1$ , but since  $f_1 < 0$  we see that now the image height is negative. For the second lens, L2, the light seems to be coming from a point source located a distance  $f_2$  at its front focal plane and thus it collimates the beam with the same output angle,  $\alpha_2$ , as that given by equation 4, but now  $\alpha_2 > 0$ . The size of the output beam,  $a_2$ , will also be the same as that given by equation 6 but since  $f_2 > |f_1|$ , we see that the beam gets expanded and not inverted. This type of telescope is known as *Galilean telescope*.

### Problem 3: Telescope with finite conjugates

a) The geometry of this problem is shown in Figure 4. By inspection we see that since the off-axis point source is located in the front focal plane of the first lens, L1, it gets collimated. The second lens, L2, receives an off-axis plane wave that then focuses to form a real image on its back focal plane (i.e. a distance  $f_2$  to the right of L2).

b) To solve this part we use the matrix approach. The image height and angle are related to the object height and angle according to,

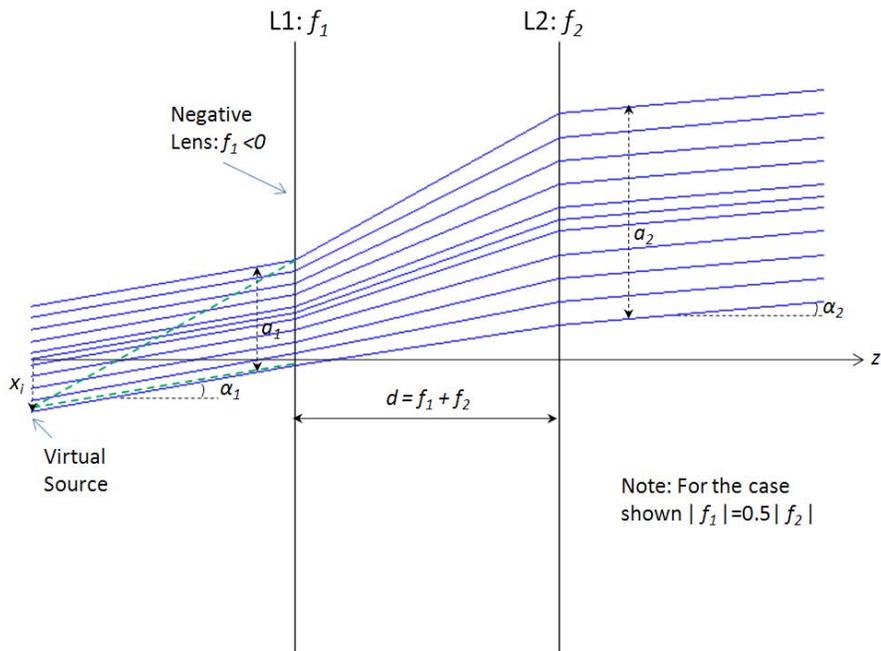


Figure 3: Galilean telescope.

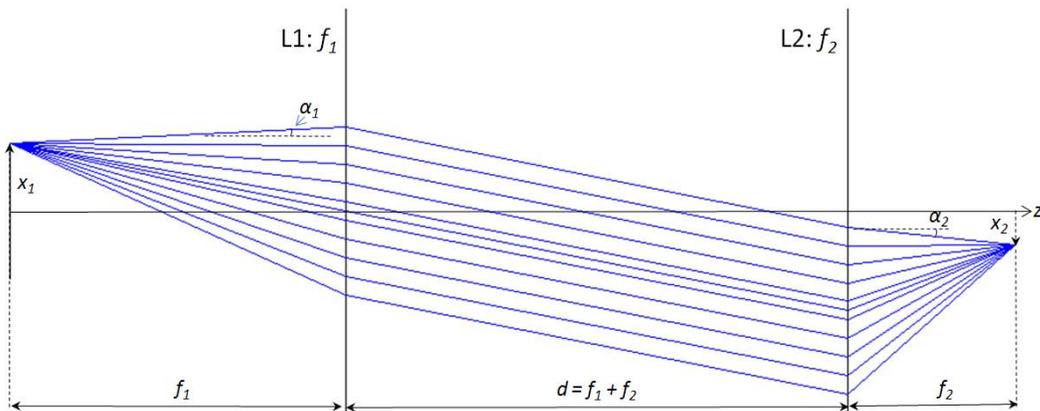


Figure 4: Telescope with finite conjugates.

$$\begin{aligned}
\begin{bmatrix} \alpha_2 \\ x_2 \end{bmatrix} &= \begin{bmatrix} 1 & 0 \\ f_2 & 1 \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{f_2} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ d & 1 \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{f_1} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ f_1 & 1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ x_1 \end{bmatrix} \quad (10) \\
&= \begin{bmatrix} 1 & 0 \\ f_2 & 1 \end{bmatrix} \begin{bmatrix} 1 - \frac{d}{f_2} & \frac{d}{f_1 f_2} - \frac{1}{f_1} - \frac{1}{f_2} \\ d & 1 - \frac{d}{f_1} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ f_1 & 1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ x_1 \end{bmatrix} \\
&= \begin{bmatrix} f_1 P + \left(1 - \frac{d}{f_2}\right) & P \\ 0 & f_2 P + \left(1 - \frac{d}{f_1}\right) \end{bmatrix} \begin{bmatrix} \alpha_1 \\ x_1 \end{bmatrix},
\end{aligned}$$

where,

$$P = -\frac{(f_1 + f_2) - d}{f_1 f_2}, \quad (11)$$

and for the case of  $d = f_1 + f_2$ , the optical power  $P = 0$ , so equation 10 becomes,

$$\begin{bmatrix} \alpha_2 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 - \frac{d}{f_2} & 0 \\ 0 & 1 - \frac{d}{f_1} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ x_1 \end{bmatrix}. \quad (12)$$

We now compute the elevation  $x_2$  and the lateral magnification  $M_T$ ,

$$\begin{aligned}
x_2 &= \left(1 - \frac{d}{f_1}\right) x_1 = -x_1 \frac{f_2}{f_1}, \quad (13) \\
M_T &= \frac{x_2}{x_1} = -\frac{f_2}{f_1}.
\end{aligned}$$

c) Again from equation 12, the arrival angle,  $\alpha_2$ , and the angular magnification,  $M_A$ , are given by,

$$\begin{aligned}
\alpha_2 &= \left(1 - \frac{d}{f_2}\right) \alpha_1 = -\alpha_1 \frac{f_1}{f_2}, \quad (14) \\
M_A &= \frac{\alpha_2}{\alpha_1} = -\frac{f_1}{f_2} = \frac{1}{M_T}.
\end{aligned}$$

d) We now consider the case in which L2 is negative (i.e.  $f_2 < 0$ ). The geometry of this case is shown in Figure 5. Similar to the previous problem, in order to have a positive separation distance between both lenses ( $d > 0$ ), we require that  $f_1 > |f_2|$ . The same equations hold as for the case of a positive lens; however, we notice that the image formed by the second lens is now virtual and erect. The angular magnification is also the same as that of equation 14 except for the sign.

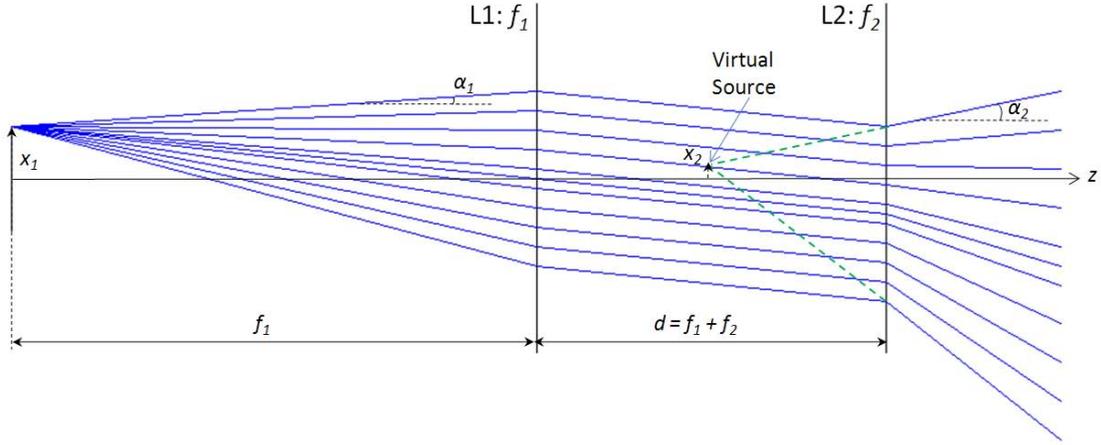


Figure 5: Galilean telescope.

#### Problem 4: Immersion lens

a) We begin by computing the composite matrix of the optical system,

$$\begin{aligned}
 \begin{bmatrix} n_2 \alpha_i \\ x_i \end{bmatrix} &= \begin{bmatrix} 1 & 0 \\ \frac{s_i}{n_2} & 1 \end{bmatrix} \begin{bmatrix} 1 & -\left(\frac{n_2 - n_0}{R_2}\right) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -\left(\frac{n_0 - n_1}{R_1}\right) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \frac{s_o}{n_1} & 1 \end{bmatrix} \begin{bmatrix} n_1 \alpha_o \\ x_o \end{bmatrix} \\
 &= \begin{bmatrix} 1 + \frac{s_o P}{n_1} & P \\ \frac{s_i}{n_2} + \frac{s_o}{n_1} \left(1 + \frac{s_i}{n_2} P\right) & 1 + \frac{s_i}{n_2} P \end{bmatrix} \begin{bmatrix} n_1 \alpha_o \\ x_o \end{bmatrix}, \quad (15)
 \end{aligned}$$

where,

$$P = -\frac{1}{f} = -\left[ n_0 \left( \frac{1}{R_1} - \frac{1}{R_2} \right) - \left( \frac{n_1}{R_1} - \frac{n_2}{R_2} \right) \right], \quad (16)$$

and we see that for the case of  $n_1 = n_2 = 1$ , equation 16 reduces to the "Lens Maker's" equation that we saw in class. Equation 16 is a more generalized version of this equation.

To find  $f_i$ , we consider a plane wave coming from the left such that  $\alpha_o = 0$ , and  $x_i = 0$  (i.e. for an object at infinity,  $s_o = \infty$ ),

$$\begin{aligned}
 0 &= \left( 1 + \frac{s_i}{n_2} P \right) x_o \\
 \Rightarrow f_i &= n_2 f. \quad (17)
 \end{aligned}$$

We now compare  $f_i$  with the focal length of the same lens immersed in air,  $f_a$ . Table 1 summarizes some of the possible conditions that the lens can have given the four variables,  $n_1, n_2, R_1$ , and  $R_2$ .

Table 1

$\mathbf{R_1 > 0}$ $\mathbf{R_2 < 0}$	$\mathbf{n_1 = 1}$	$\mathbf{n_2 = 1}$	$\mathbf{n_1 > 1}$	$\mathbf{n_2 > 1}$	$\mathbf{R_1 < 0}$ $\mathbf{R_2 > 0}$	$\mathbf{n_1 = 1}$	$\mathbf{n_2 = 1}$	$\mathbf{n_1 > 1}$	$\mathbf{n_2 > 1}$
$\mathbf{n_1 = 1}$		$B$		$C^{**}$	$\mathbf{n_1 = 1}$		$B$		$A^{**}$
$\mathbf{n_2 = 1}$	$B$		$C^*$		$\mathbf{n_2 = 1}$	$B$		$A^*$	
$\mathbf{n_1 > 1}$		$C^*$		$D$	$\mathbf{n_1 > 1}$		$A^*$		$D$
$\mathbf{n_2 > 1}$	$C^{**}$		$D$		$\mathbf{n_2 > 1}$	$A^{**}$		$D$	
$\mathbf{R_1 > 0}$ $\mathbf{R_2 > 0}$	$\mathbf{n_1 = 1}$	$\mathbf{n_2 = 1}$	$\mathbf{n_1 > 1}$	$\mathbf{n_2 > 1}$	$\mathbf{R_1 < 0}$ $\mathbf{R_2 < 0}$	$\mathbf{n_1 = 1}$	$\mathbf{n_2 = 1}$	$\mathbf{n_1 > 1}$	$\mathbf{n_2 > 1}$
$\mathbf{n_1 = 1}$		$B$		$A^{**}$	$\mathbf{n_1 = 1}$		$B$		$C^{**}$
$\mathbf{n_2 = 1}$	$B$		$C^*$		$\mathbf{n_2 = 1}$	$B$		$A^*$	
$\mathbf{n_1 > 1}$		$C^*$		$D$	$\mathbf{n_1 > 1}$		$A^*$		$D$
$\mathbf{n_2 > 1}$	$A^{**}$		$D$		$\mathbf{n_2 > 1}$	$C^{**}$		$D$	

\* Note: This hold for  $R_1 > R_2$ ; for  $R_1 < R_2$ ,  $f_i = f(n_1, n_2, R_1, R_2)$

\*\* Note: This hold for  $R_1 < R_2$ ; for  $R_1 > R_2$ ,  $f_i = f(n_1, n_2, R_1, R_2)$

Where,

$$\begin{aligned}
 A & : f_i < f_a \\
 B & : f_i = f_a \\
 C & : f_i > f_a \\
 D & : f_i = f(n_1, n_2, R_1, R_2).
 \end{aligned}$$

b) To derive the imaging condition, we note that all the rays, regardless of their originating angle  $\alpha_o$ , arrive at the image point  $x_i$ . From equation 15, we want to find the condition when  $\partial x_i / \partial \alpha_o = 0$ ,

$$\begin{aligned}
 \left( \frac{s_i}{n_2} + \frac{s_o}{n_1} \left( 1 + \frac{s_i}{n_2} P \right) \right) n_1 & = 0 \Rightarrow \\
 \frac{n_1}{s_o} + \frac{n_2}{s_i} & = \frac{1}{f}.
 \end{aligned} \tag{18}$$

c) The angular magnification is given by,

$$\begin{aligned}
M_A &= \frac{\partial \alpha_i}{\partial \alpha_o} \\
&= \frac{n_1}{n_2} \left( 1 - \frac{s_o}{n_1 f} \right) \\
&= \frac{n_1}{n_2} \left( \frac{n_1 f - s_o}{n_1 f} \right).
\end{aligned} \tag{19}$$

From the imaging condition of equation 18, we solve for  $s_i$ ,

$$\frac{1}{s_i} = \frac{s_o - n_1 f}{n_2 f s_o}. \tag{20}$$

Using equation 20 in equation 19 we get,

$$M_A = -\frac{s_o}{s_i}. \tag{21}$$

Similarly, we derive the lateral magnification using the equation for  $x_i$  for an on-axis ray (i.e.  $\alpha_o = 0$ ),

$$x_i = \left( 1 - \frac{s_i}{n_2 f} \right) x_o \tag{22}$$

$$M_L = \frac{x_i}{x_o} = \left( 1 - \frac{s_i}{n_2 f} \right) = -\frac{n_1 s_i}{n_2 s_o}. \tag{23}$$

d) We consider the triangle formed by a ray that originates at the object's tip and aims at the optical center of the lens. The angle of the ray respect to the optical axis is,

$$\tan \alpha_o \approx \alpha_o = -\frac{x_o}{s_o}. \tag{24}$$

From equation 15 we see that the object and image angles are related according to,

$$\begin{aligned}
n_2 \alpha_i &= \left( 1 - \frac{s_o}{n_1 f} \right) n_1 \alpha_o - \frac{x_o}{f} \\
&= -\left( 1 - \frac{s_o}{n_1 f} \right) \frac{n_1 x_o}{s_o} - \frac{x_o}{f} \\
&= -\frac{x_o}{s_o} n_1 = \alpha_o n_1,
\end{aligned} \tag{25}$$

which is Snell's law!

**Problem 5: Hyperboloidal refractor**

a) The geometry for this problem is shown in Figure 6. Fermat's principle requires that the propagation length of ray 1,  $l'$ , is equal to that of ray 2,  $l$ , where,

$$\begin{aligned} l' &= f + ns, \text{ and} \\ l &= c + d. \end{aligned} \tag{26}$$

By similar triangles we see that,

$$\begin{aligned} \frac{f}{c} &= \frac{s}{d} \Rightarrow d = \frac{sc}{f}, \\ \frac{a}{f} &= \frac{b}{s} \Rightarrow b = \frac{as}{f}. \end{aligned} \tag{27}$$

Since  $x = a + b$ , and  $c^2 = f^2 + a^2$ ,

$$\begin{aligned} x &= a \left(1 + \frac{s}{f}\right) \Rightarrow a = \frac{x}{\left(1 + \frac{s}{f}\right)}, \\ c^2 &= f^2 + \frac{x^2}{\left(1 + \frac{s}{f}\right)^2} \Rightarrow c = \left[ f^2 + \frac{x^2}{\left(1 + \frac{s}{f}\right)^2} \right]^{1/2}. \end{aligned} \tag{28}$$

Combining equation 28 with equation 26 we get,

$$\begin{aligned} l' &= l \\ f + ns &= \left[ f^2 \left(1 + \frac{s}{f}\right)^2 + x^2 \right]^{1/2} \Rightarrow \\ (f + ns)^2 &= f^2 \left(1 + \frac{s}{f}\right)^2 + x^2 \Rightarrow \\ (f + ns)^2 &= (f + s)^2 + x^2 \Rightarrow \\ f^2 + (ns)^2 + 2fns &= f^2 + s^2 + 2fs + x^2 \Rightarrow \\ s^2 + 2fs \left(\frac{n-1}{n^2-1}\right) &= \frac{x^2}{n^2-1} \Rightarrow \\ s^2 + 2s \left(\frac{f}{n+1}\right) + \left(\frac{f}{n+1}\right)^2 &= \frac{x^2}{n^2-1} + \left(\frac{f}{n+1}\right)^2 \Rightarrow \\ \left(s + \frac{f}{n+1}\right)^2 - \frac{x^2}{n^2-1} &= \left(\frac{f}{n+1}\right)^2. \end{aligned} \tag{29}$$

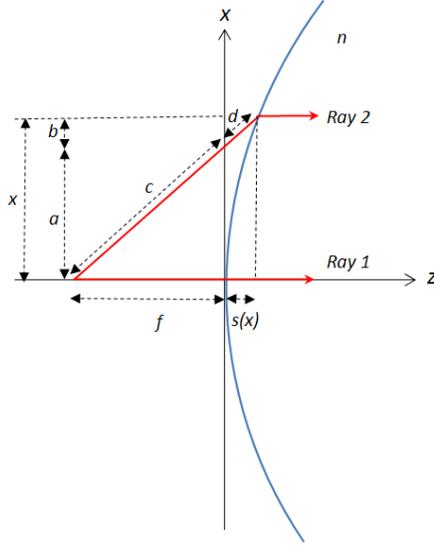


Figure 6: Hyperboloidal refractor problem.

b) We begin by rewriting equation 29 to match the form of a shifted hyperbola,

$$\frac{(s - h)^2}{a^2} - \frac{x^2}{b^2} = 1, \quad (30)$$

where,

$$\begin{aligned} a &= -h = \frac{f}{n+1}, \text{ and} \\ b &= f\sqrt{\frac{n-1}{n+1}}. \end{aligned} \quad (31)$$

From equation 30 we solve for  $s$ ,

$$s(x) = a\sqrt{1 + \left(\frac{x}{b}\right)^2} + h. \quad (32)$$

We expand equation 32 into a Taylor series evaluated at  $x = 0$ ,

$$s(x) \approx s(0) + s'(0)x + s''(0)\frac{x^2}{2} + \text{high order terms}, \quad (33)$$

where,

$$\begin{aligned}
s'(0) &= \frac{ax}{\left[1 + \left(\frac{x}{b}\right)^2\right]^{1/2} b^2} = 0, \\
s''(0) &= \frac{a}{\left[1 + \left(\frac{x}{b}\right)^2\right]^{1/2} b^2} - \frac{ax^2}{\left[1 + \left(\frac{x}{b}\right)^2\right]^{3/2} b^4} = \frac{a}{b^2}.
\end{aligned} \tag{34}$$

So equation 33 becomes,

$$s(x) \simeq (a + h) + \left(\frac{a}{2b^2}\right) x^2 + \text{high order terms},$$

and we see that for small  $|x|$  compared to the radii of curvature, (i.e. in the paraxial region), the hyperboloidal surface is approximated as a paraboloidal surface,  $s = cx^2$  (the constant term cancels out as  $a = -h$  for our case).

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Spring 2009

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