MASSACHUSETTS INSTITUTE OF TECHNOLOGY

2.710 Optics **Spring '09**

Solutions to Problem Set #4

Due Wednesday, April 1, 2009

Problem 1: Knocking down one dimension: the Screen Hamiltonian

a) The main goal of this problem is to show how the 6×6 set of Hamiltonian equations can be simplified to a 4×4 set of ordinary differential equations known as *Screen Hamiltonian equations*. The Screen Hamiltonian equations describe the evolution of the intersection of the ray path with the screens, that are perpendicular to the optical axis, as z advances. The geometry of this problem is shown in Figure 1.

We begin by writing in an explicit form the set of Hamiltonian equations,

$$\begin{aligned} \frac{dq_x}{ds} &= \frac{\partial H}{\partial p_x}, & \frac{dp_x}{ds} &= -\frac{\partial H}{\partial q_x}, \\ \frac{dq_y}{ds} &= \frac{\partial H}{\partial p_y}, & \frac{dp_y}{ds} &= -\frac{\partial H}{\partial q_y}, \\ \frac{dq_z}{ds} &= \frac{\partial H}{\partial p_z}, & \frac{dp_z}{ds} &= -\frac{\partial H}{\partial q_z}, \end{aligned} \tag{1}$$

where the conserved Hamiltonian is,

$$H = n(\mathbf{q}) - \sqrt{p_x^2 + p_y^2 + p_z^2} = 0. \tag{2}$$

As shown in Figure 1, different points in the ray trajectory $(s_1, s_2, s_3, \cdots)$ have been projected to their corresponding axial coordinate $(z_1, z_2, z_3, \cdots)$ changing the parameterization of the ray from $[\mathbf{q}(s), \mathbf{p}(s)]$ to $[\mathbf{q}(z), \mathbf{p}(z)]$.We then apply the chain rule when taking the derivatives of the components of the position vector with respect to z and use the results of equations 1 and 2,

$$\begin{aligned} \frac{dq_x}{dz} &= \frac{dx}{dz} = \frac{dx}{ds}\frac{ds}{dz} = \left(\frac{\partial H}{\partial p_x}\right)\left(\frac{\partial p_z}{\partial H}\right) \\ &\Rightarrow \frac{dq_x}{dz} = \frac{p_x}{p_z}, \end{aligned} \tag{3}$$

$$\begin{aligned} \frac{dq_y}{dz} &= \frac{dy}{dz} = \frac{dy}{ds}\frac{ds}{dz} = \left(\frac{\partial H}{\partial p_y}\right)\left(\frac{\partial p_z}{\partial H}\right) \\ &\Rightarrow \frac{dq_y}{dz} = \frac{p_y}{p_z}, \end{aligned} \tag{4}$$

$$\frac{dq_z}{dz} = \frac{dz}{dz} = 1. \tag{5}$$

Similarly, we take the derivatives of the components of the momentum vector with respect to z,

$$\begin{aligned}\frac{dp_x}{dz} &= \frac{dp_x}{ds}\frac{ds}{dz} = \left(\frac{\partial H}{\partial q_x}\right)\left(\frac{\partial p_z}{\partial H}\right) \\ &\Rightarrow \frac{dp_x}{dz} = -\frac{|\mathbf{p}|}{p_z}\frac{\partial n}{\partial q_x},\end{aligned} \tag{6}$$

$$\begin{aligned}\frac{dp_y}{dz} &= \frac{dp_y}{ds}\frac{ds}{dz} = \left(\frac{\partial H}{\partial q_y}\right)\left(\frac{\partial p_z}{\partial H}\right) \\ &\Rightarrow \frac{dp_y}{dz} = -\frac{|\mathbf{p}|}{p_z}\frac{\partial n}{\partial q_y},\end{aligned} \tag{7}$$

$$\begin{aligned}\frac{dp_z}{dz} &= \frac{dp_z}{ds}\frac{ds}{dz} = \left(\frac{\partial H}{\partial q_z}\right)\left(\frac{\partial p_z}{\partial H}\right) \\ &\Rightarrow \frac{dp_z}{dz} = -\frac{|\mathbf{p}|}{p_z}\frac{\partial n}{\partial q_z}.\end{aligned} \tag{8}$$

b) From the Hamiltonian conservation principle of equation 2, we see that $|\mathbf{p}| = n(\mathbf{q})$, so equations 3 to 8 become,

$$\begin{array}{ll} \frac{dq_x}{dz} = \frac{p_x}{p_z}, & \frac{dp_x}{dz} = -\frac{n}{p_z}\frac{\partial n}{\partial q_x}, \\ \frac{dq_y}{dz} = \frac{p_y}{p_z}, & \frac{dp_y}{dz} = -\frac{n}{p_z}\frac{\partial n}{\partial q_y}, \\ & \frac{dp_z}{dz} = -\frac{n}{p_z}\frac{\partial n}{\partial q_z}. \end{array} \tag{9}$$

Solving for p_z from equation 2,

$$p_z = \sqrt{n(\mathbf{q})^2 - \left(p_x^2 + p_y^2\right)}. \tag{10}$$

c) We now set,

$$h(q_x, q_y, z; p_x, p_y) \equiv -p_z(q_x, q_y, z; p_x, p_y) = -\sqrt{n(\mathbf{q})^2 - \left(p_x^2 + p_y^2\right)}, \tag{11}$$

and use it to elliminate p_z, that appears in the equation set 9,

$$\begin{aligned}\frac{dq_x}{dz} &= \frac{p_x}{\sqrt{n(\mathbf{q})^2 - \left(p_x^2 + p_y^2\right)}} = \frac{\partial h}{\partial p_x}, \\ \frac{dq_y}{dz} &= \frac{p_y}{\sqrt{n(\mathbf{q})^2 - \left(p_x^2 + p_y^2\right)}} = \frac{\partial h}{\partial p_y}, \\ \frac{dp_x}{dz} &= -\frac{n}{\sqrt{n(\mathbf{q})^2 - \left(p_x^2 + p_y^2\right)}}\frac{\partial n}{\partial q_x} = -\frac{\partial h}{\partial q_x}, \\ \frac{dp_y}{dz} &= -\frac{n}{\sqrt{n(\mathbf{q})^2 - \left(p_x^2 + p_y^2\right)}}\frac{\partial n}{\partial q_y} = -\frac{\partial h}{\partial q_y}.\end{aligned} \tag{12}$$

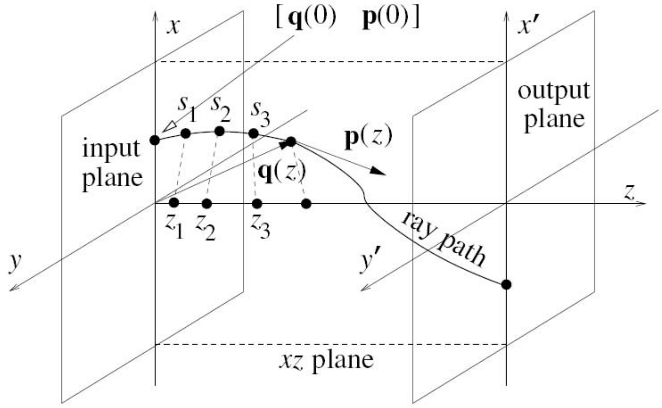

Figure 1: Screen Hamiltonian.

We also have the additional equation,

$$\frac{dp_z}{dz} = -\frac{n}{h}\frac{\partial n}{\partial z}. \tag{13}$$

d) The equation set 12, is a proper set of Hamiltonian equations with h as the *Screen Hamiltonian*; therefore, h is conserved in this 2D space. However, equation 13 shows that the Screen Hamiltonian is not conserved in general, unless $\partial n/\partial z = 0$, where the index is invariant along the optical axis. It is okay for the Screen Hamiltonian to not be conserved because the lateral momentum, (p_x, p_y), is not generally conserved; however, the 3D momentum, (p_x, p_y, p_z), must be conserved. This is also known as the *Phase Matching Condition*.

Problem 2: Rays in Harmonic Oscillation

a) In this problem we simplify the analysis by tracing rays in the xz plane as shown in Figure 1. The Screen Hamiltonian equations reduce to,

$$\begin{matrix} \frac{dq_x}{dz} = \frac{\partial h}{\partial p_x}, \\ \frac{dp_x}{dz} = -\frac{\partial h}{\partial q_x} \end{matrix} . \tag{14}$$

We are interested in considering the case of an optical element with an elliptical GRIN profile,

$$n(x) = \sqrt{n_o^2 - \kappa^2 q_x^2}, \tag{15}$$

and the Screen Hamiltonian becomes,

$$h = -\sqrt{n_o^2 - (\kappa^2 q_x^2 + p_x^2)}. \tag{16}$$

The harmonic solution of the Screen Hamiltonian solution is,

$$
\begin{aligned}
q_x(z) &= q_0 \cos\left(\frac{\kappa z}{h}\right) + \frac{p_0}{\kappa}\sin\left(\frac{\kappa z}{h}\right), \\
p_x(z) &= p_0 \cos\left(\frac{\kappa z}{h}\right) - \kappa q_0 \sin\left(\frac{\kappa z}{h}\right),
\end{aligned} \tag{17}
$$

where $q_0 = q_x(0)$ and $p_0 = p_x(0)$. To show that the Screen Hamiltonian is independent of the axial coordinate we need to take the derivative of h with respect to z,

$$
\begin{aligned}
\frac{\partial h}{\partial z} &= -\frac{1}{2}\frac{-2\kappa^2 q_x \frac{dq_x}{dz} - 2p_x \frac{dp_x}{dz}}{\sqrt{n_o^2 - (\kappa^2 q_x^2 + p_x^2)}} \\
&= \frac{1}{h}\left(\kappa^2 q_x \frac{p_x}{h} + p_x \frac{n}{h}\frac{\partial n}{\partial q_x}\right) \\
&= \frac{1}{h^2}\left(\kappa^2 q_x p_x - \kappa^2 q_x p_x\right) = 0.
\end{aligned} \tag{18}
$$

b) To verify that equation 17 is a solution of the Screen Hamiltonian equation we compute the derivative with respect to the axial coordinate,

$$
\begin{aligned}
\frac{dq_x}{dz} &= -q_0 \sin\left(\frac{\kappa z}{h}\right)\left(\frac{\kappa}{h} - \frac{\kappa z}{h^2}\frac{\partial h}{\partial z}\right) + \frac{p_0}{\kappa}\cos\left(\frac{\kappa z}{h}\right)\left(\frac{\kappa}{h} - \frac{\kappa z}{h^2}\frac{\partial h}{\partial z}\right) \\
&= -\frac{q_0 \kappa}{h}\sin\left(\frac{\kappa z}{h}\right) + \frac{p_0}{h}\cos\left(\frac{\kappa z}{h}\right),
\end{aligned} \tag{19}
$$

$$
\begin{aligned}
\frac{dp_x}{dz} &= -p_0 \sin\left(\frac{\kappa z}{h}\right)\left(\frac{\kappa}{h} - \frac{\kappa z}{h^2}\frac{\partial h}{\partial z}\right) - \kappa q_0 \cos\left(\frac{\kappa z}{h}\right)\left(\frac{\kappa}{h} - \frac{\kappa z}{h^2}\frac{\partial h}{\partial z}\right) \\
&= -\frac{p_0 \kappa}{h}\sin\left(\frac{\kappa z}{h}\right) - \frac{\kappa^2 q_0}{h}\cos\left(\frac{\kappa z}{h}\right),
\end{aligned} \tag{20}
$$

where we have used the fact that $\partial h/\partial z = 0$. We now compute the derivatives of the Screen Hamiltonian respect to the position and momentum components,

$$
\frac{\partial h}{\partial p_x} = \frac{p_x}{h} = \frac{p_0}{h}\cos\left(\frac{\kappa z}{h}\right) - \frac{\kappa q_0}{h}\sin\left(\frac{\kappa z}{h}\right), \tag{21}
$$

$$
\frac{\partial h}{\partial q_x} = \frac{\kappa^2 q_x}{h} = \frac{\kappa^2 q_0}{h}\cos\left(\frac{\kappa z}{h}\right) + \frac{\kappa p_0}{h}\sin\left(\frac{\kappa z}{h}\right). \tag{22}
$$

If we compare equations 19 and 21, as well as 20 and 22, we see that the solution does satisfy the Screen Hamiltonian equations.

c) Figure 2 shows the ray position, $q_x(z)$, as a function of z for a collimated ray

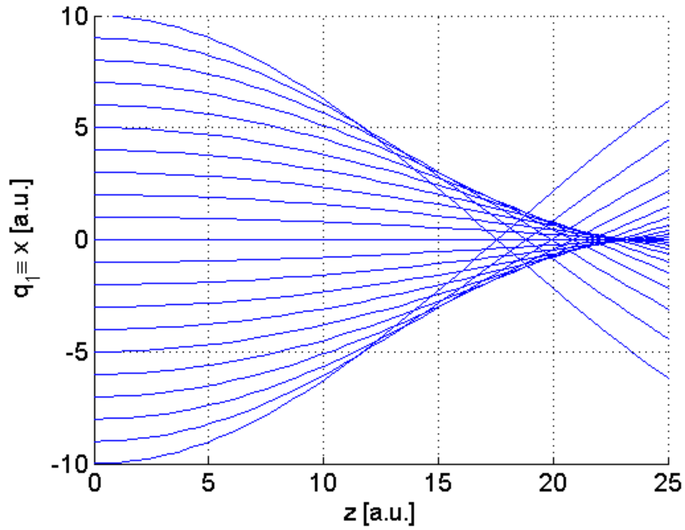

Figure 2: Elliptical GRIN lens.

bundle parallel to the optical axis z.

d) As you can see from Figure 2, the elliptical GRIN lens doesn't focus the incident parallel ray bundle satisfactory as it suffers a large degree of spherical aberration.

Problem 3: Mechanical Screen Hamiltonian

a) In this problem we consider a mechanical system whose Hamiltonian is given by,

$$H = \frac{p_x^2 + p_y^2 + p_z^2}{2m} + V(\mathbf{q}) = E, \tag{23}$$

where E is the energy of the system and is the conserved quantity. In the same way as in problem 1, we take the derivatives of the position and momentum vectors with respect to z,

$$\begin{aligned}
\frac{dq_x}{dz} &= \frac{dq_x}{dt}\frac{dt}{dz} = \left(\frac{\partial H}{\partial p_x}\right)\left(\frac{\partial p_z}{\partial H}\right) = \frac{p_x}{p_z}, \\
\frac{dq_y}{dz} &= \frac{dq_y}{dt}\frac{dt}{dz} = \left(\frac{\partial H}{\partial p_y}\right)\left(\frac{\partial p_z}{\partial H}\right) = \frac{p_y}{p_z}, \\
\frac{dp_x}{dz} &= \frac{dp_x}{dt}\frac{dt}{dz} = \left(-\frac{\partial H}{\partial q_x}\right)\left(\frac{\partial p_z}{\partial H}\right) = -\frac{m}{p_z}\frac{\partial V}{\partial q_z}, \\
\frac{dp_y}{dz} &= \frac{dp_y}{dt}\frac{dt}{dz} = \left(-\frac{\partial H}{\partial q_y}\right)\left(\frac{\partial p_z}{\partial H}\right) = -\frac{m}{p_z}\frac{\partial V}{\partial q_y}, \\
\frac{dp_z}{dz} &= \frac{dp_z}{dt}\frac{dt}{dz} = \left(-\frac{\partial H}{\partial q_z}\right)\left(\frac{\partial p_z}{\partial H}\right) = -\frac{m}{p_z}\frac{\partial V}{\partial q_z}.
\end{aligned} \tag{24}$$

We now solve for p_z from the conserved Hamiltonian,

$$p_z = \sqrt{2m\left(E - V(\mathbf{q})\right) - \left(p_x^2 + p_y^2\right)}, \tag{25}$$

and we define the Screen Hamiltonian of the mechanical system,

$$h(q_x, q_y, z; p_x, p_y) \equiv -\sqrt{2m\left(E - V(\mathbf{q})\right) - \left(p_x^2 + p_y^2\right)}. \tag{26}$$

The gradients of the Screen Hamiltonian are,

$$\begin{aligned}
\frac{\partial h}{\partial q_x} &= -\frac{-2m\frac{\partial V}{\partial q_x}}{2\sqrt{2m\left(E - V\right) - \left(p_x^2 + p_y^2\right)}} = \frac{m}{p_z}\frac{\partial V}{\partial q_x}, \\
\frac{\partial h}{\partial q_x} &= \frac{m}{p_z}\frac{\partial V}{\partial q_y}, \\
\frac{\partial h}{\partial p_x} &= \frac{p_x}{\sqrt{2m\left(E - V(\mathbf{q})\right) - \left(p_x^2 + p_y^2\right)}} = \frac{p_x}{p_z}, \\
\frac{\partial h}{\partial p_y} &= \frac{p_y}{p_z}.
\end{aligned} \tag{27}$$

b) The set of Screen Hamiltonian equations is,

$$\begin{matrix} \frac{dq_x}{dz} = \frac{\partial h}{\partial p_x} & \frac{dp_x}{dz} = -\frac{\partial h}{\partial q_x} \\ \frac{dq_y}{dz} = \frac{\partial h}{\partial p_y} & \frac{dp_y}{dz} = -\frac{\partial h}{\partial q_y} \end{matrix}, \tag{28}$$

and the equation of the evolution of the Screen Hamiltonian along the axial coordinate is,

$$\frac{dh}{dz} = -\frac{1}{p_z}\frac{\partial V}{\partial q_z}. \tag{29}$$

c) As in the optical case, the Screen Hamiltonian is only conserved if, $\partial V/\partial q_z = 0$. This makes physical sense as V doesn't impart momentum along z.

Problem 4: Mechanical Harmonic Oscillator

a) In this problem, we deal with a special case of the mechanical Screen Hamiltonian of the previous problem where,

$$V(\mathbf{q}) = \frac{1}{2}kq_x^2. \tag{30}$$

As we can see from equation 30, the potential energy only depends on the x-component of the position vector, that is,

$$\frac{\partial V}{\partial q_z} = 0; \tag{31}$$

therefore, the Screen Hamiltonian is conserved. The Screen Hamiltonian equations are,

$$\begin{aligned} \frac{dq_x}{dz} &= \frac{p_x}{h}, \\ \frac{dp_x}{dz} &= -\frac{m}{h}\frac{\partial V}{\partial q_x} = -\frac{mkq_x}{h}, \end{aligned} \tag{32}$$

where,

$$h = \sqrt{2m\left(E - \frac{1}{2}kq_x^2 - \frac{1}{2m}p_x^2\right)} = \text{constant}. \tag{33}$$

The harmonic solution of the Screen Hamiltonian equations is,

$$\begin{aligned} q_x(z) &= q_0 \cos\left(\frac{\sqrt{mk}}{h}z\right) + \frac{p_0}{\sqrt{mk}} \sin\left(\frac{\sqrt{mk}}{h}z\right), \\ p_x(z) &= p_0 \cos\left(\frac{\sqrt{mk}}{h}z\right) - \sqrt{mk}q_0 \sin\left(\frac{\sqrt{mk}}{h}z\right), \end{aligned} \tag{34}$$

where $q_0 \equiv q_x(0)$ and $p_0 \equiv p_x(0)$ are the position and lateral momentum respectively, at $z = 0$.

b) To show that equation 34 is a solution of the Screen Hamiltonian equation 32, we compute the derivative of the position and lateral momentum with respect to the axial coordinate,

$$\begin{aligned} \frac{dq_x}{dz} &= -q_0 \sin\left(\frac{\sqrt{mk}}{h}z\right)\left(\frac{\sqrt{mk}}{h} - \frac{\sqrt{mk}z}{h^2}\frac{\partial h}{\partial z}\right) \\ &\quad + \frac{p_0}{\sqrt{mk}} \cos\left(\frac{\sqrt{mk}}{h}z\right)\left(\frac{\sqrt{mk}}{h} - \frac{\sqrt{mk}z}{h^2}\frac{\partial h}{\partial z}\right) \\ &= -\frac{q_0\sqrt{mk}}{h} \sin\left(\frac{\sqrt{mk}}{h}z\right) + \frac{p_0}{h} \cos\left(\frac{\sqrt{mk}}{h}z\right), \end{aligned} \tag{35}$$

$$\begin{aligned}\frac{dp_x}{dz} &= -p_0 \sin\left(\frac{\sqrt{mk}}{h}z\right)\left(\frac{\sqrt{mk}}{h} - \frac{\sqrt{mk}z}{h^2}\frac{\partial h}{\partial z}\right) \\ &\quad -\sqrt{mk}q_0 \cos\left(\frac{\sqrt{mk}}{h}z\right)\left(\frac{\sqrt{mk}}{h} - \frac{\sqrt{mk}z}{h^2}\frac{\partial h}{\partial z}\right) \\ &= -\frac{p_0\sqrt{mk}}{h}\sin\left(\frac{\sqrt{mk}}{h}z\right) - \frac{mkq_0}{h}\cos\left(\frac{\sqrt{mk}}{h}z\right),\end{aligned} \tag{36}$$

where again we have used the fact that the Screen Hamiltonian is conserved, that is $\partial h/\partial z = 0$. We now compute the derivatives of the Screen Hamiltonian respect to the position and momentum components,

$$\frac{\partial h}{\partial p_x} = \frac{p_x}{h} = \frac{p_0}{h}\cos\left(\frac{\sqrt{mk}}{h}z\right) - \frac{\sqrt{mk}q_0}{h}\sin\left(\frac{\sqrt{mk}}{h}z\right), \tag{37}$$

$$\frac{\partial h}{\partial q_x} = \frac{mkq_x}{h} = \frac{mkq_0}{h}\cos\left(\frac{\sqrt{mk}}{h}z\right) + \frac{p_0\sqrt{mk}}{h}\sin\left(\frac{\sqrt{mk}}{h}z\right). \tag{38}$$

If we compare equations 35 and 37, as well as equations 36 and 38, we see that equation 34 is a solution of the Screen Hamiltonian equations.

c) Note that the total energy $E > \frac{1}{2}kq_o^2 + \frac{1}{2m}p_o^2$ for the system to make sense (in math, we can see that if $h = 0$, the equations of motion blow up.) Physically, $E - \frac{1}{2}kq_o^2 - \frac{1}{2m}p_o^2 = \frac{p_z^2(0)}{2m}$ is the initial momentum setting the system in motion along the z axis (recall z is still a spatial coordinate!) If $p_z(0) = 0$, the system does not move along z, the initial momentum given to the system is conserved; in other words, the system moves with *constant velocity* along z, while it oscillates along x.

d) To answer this part, we look at the original Hamiltonian,

$$\frac{\partial q_z}{\partial t} = \frac{\partial H}{\partial p_z} = \frac{p_z}{m} = \frac{p_z(0)}{m}, \tag{39}$$

is constant because $p_z(0)$ is the Screen Hamiltonian! Next, we integrate equation 39,

$$q_z(t) = \frac{p_z(0)}{m}t = \frac{h}{m}t, \tag{40}$$

where we assumed that $z = 0$ at $t = 0$. Finally, we express the harmonic solution for the lateral position as function of t,

$$\begin{aligned}q_x(t) &= q_0 \cos\left(\frac{\sqrt{mk}}{h}\frac{h}{m}t\right) + \frac{p_0}{\sqrt{mk}}\sin\left(\frac{\sqrt{mk}}{h}\frac{h}{m}t\right) \\ &= q_0\cos\left(\sqrt{\frac{k}{m}}t\right) + \frac{p_0}{\sqrt{mk}}\sin\left(\sqrt{\frac{k}{m}}t\right),\end{aligned} \tag{41}$$

which is consistent with the class notes.

Problem 5: Quadratic GRIN

a) The ray tracing plot for the elliptical GRIN is shown in Figure 2.

b) Now we solve the Hamiltonian equations for the case where the index is modulated quadratically,

$$n(x) = n_0^2 - \frac{\alpha}{2} q_x^2. \tag{42}$$

The Screen Hamiltonian becomes,

$$\begin{aligned} h &= -\sqrt{\left(n_0^2 - \frac{\alpha}{2} q_x^2\right)^2 - p_x} \\ &= -\frac{1}{2}\sqrt{4n_0^4 - 4\alpha n_0^2 q_x^2 + \alpha^2 q_x^4 - 4p_x^4}. \end{aligned} \tag{43}$$

Next, we take the derivative of the Screen Hamiltonian respect to the lateral component of the position vector,

$$\begin{aligned} \frac{\partial h}{\partial q_x} &= -\frac{1}{4} \frac{-8n_0^2 \alpha q_x + 4\alpha^2 q_x^3}{\sqrt{4n_0^4 - 4\alpha n_0^2 q_x^2 + \alpha^2 q_x^4 - 4p_x^4}} \\ &= \frac{2n_0^2 \alpha q_x - \alpha^2 q_x^3}{2h}. \end{aligned} \tag{44}$$

We use equation 44 to edit the `sgradh_quadratic_hw.m` function. Figure 3 shows a comparison of the ray tracing of the quadratic GRIN lens (blue-solid line) and the elliptical GRIN lens (red-dashed line). As shown in the figure, the quadratic GRIN lens also suffers from spherical aberrations that affect the focusing quality.

c) As indicated by equation 42, the refractive index only varies as function of x so that $\partial n / \partial z = 0$; therefore, the Screen Hamiltonian is conserved as we discussed in problem 1.

Problem 6: Plane waves and phasor representations

a) We begin by writing the general scalar form of a propagating plane wave in a phasor representation,

$$f(x, y, z, t) = Ae^{i\mathbf{k}\cdot\mathbf{r}} e^{-i\omega t}, \tag{45}$$

where $\mathbf{r}$ is the position vector, $\mathbf{k}$ is the wave vector with magnitude $|\mathbf{k}| = 2\pi/\lambda$, ω is the angular frequency and A is the amplitude of the wave. For a plane wave propagating at an angle of 30°
relative to the $\hat{\mathbf{z}}$ axis on the xz-plane, the wave vector becomes,

$$k = \frac{2\pi}{\lambda} \begin{bmatrix} \sin 30^\circ \\ 0 \\ \cos 30^\circ \end{bmatrix}. \tag{46}$$

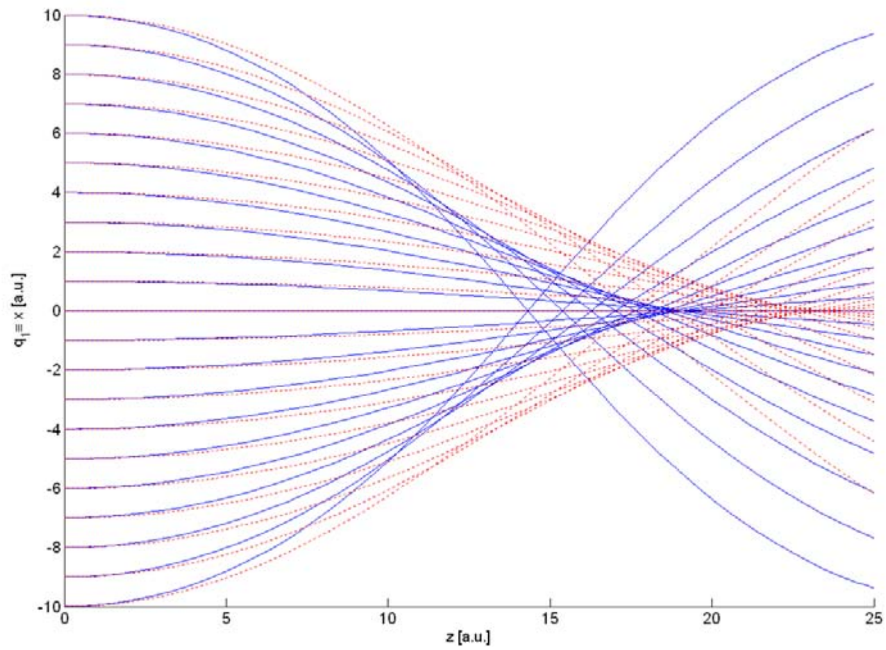

Figure 3: Comparison of quadratic and elliptical GRIN lenses.

For a wavelength $\lambda = 1\mu$m, the wave number is, $k = |\mathbf{k}| = 6.28 \times 10^6 \text{m}^{-1}$. The angular frequency is related to the wavelength by means of the dispersion relation,

$$\begin{aligned} c &= \lambda\nu = \lambda 2\pi\omega \\ \Rightarrow \quad & \omega = \frac{c}{\lambda 2\pi} = 4.77 \times 10^{13} \text{rad} \cdot \text{sec}^{-1} \,. \end{aligned} \tag{47}$$

The phasor representation of the wave is,

$$f_1(x, y, z, t) = A \exp\left[ik\left(\sin 30^\circ x + \cos 30^\circ z\right)\right] \exp(-i\omega t), \tag{48}$$

and the space-time representation is,

$$f_1(x, y, z, t) = A \cos[k\left(\sin 30^\circ x + \cos 30^\circ z\right) - \omega t]. \tag{49}$$

b) Similar to part (a), the phasor representation of a plane wave propagating at an angle of 60° relative to the optical axis on the yz-plane is,

$$f_2(x, y, z, t) = A \exp\left[ik\left(\sin 60^\circ y + \cos 60^\circ z\right)\right] \exp(-i\omega t), \tag{50}$$

and the space-time representation is,

$$f_2(x, y, z, t) = A \cos[k\left(\sin 60^\circ y + \cos 60^\circ z\right) - \omega t]. \tag{51}$$

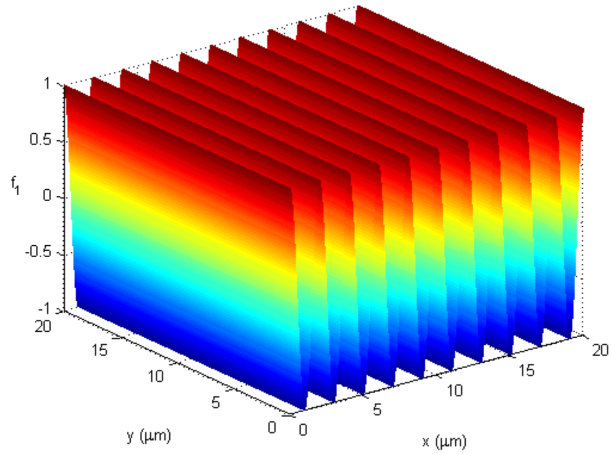

Figure 4: Plane wave propagating at 30° in the xz-plane.

c) Figures 4 and 5 show the waves for $f_1(x, y, z = 0, t = 0)$ and $f_2(x, y, z = 0, t = 0)$.

d) The plane $z = 0$ is illuminated by the superposition of the two waves, f_1 and f_2, and we are interested in plotting the evolution of the resulting wave received at points A, B, C, D, E,

$$(0, 0, 0), \left(\frac{1}{4}, -\frac{1}{4\sqrt{3}}, 0\right), \left(\frac{1}{2}, -\frac{1}{2\sqrt{3}}, 0\right), \left(\frac{3}{4}, -\frac{3}{4\sqrt{3}}, 0\right), \left(1, -\frac{1}{2\sqrt{3}}, 0\right).$$

The evolution of the resulting wave is shown in Figure 6. As shown in this figure, at point A the waves f_1 and f_2 are *in phase* so they interfere constructively. In contrast, at point B, the waves are *out-of-phase* and they interfere destructively. An interference pattern is produced at the plane $z = 0$ as a result of the superposition of both waves.

Problem 7: Wave superposition

a) Consider the following two waves,

$$\begin{aligned} f_1(x, z, t) &= 5\cos\left(\frac{2\pi}{17}\left[z + \frac{x^2}{2z}\right] - 2\pi 10t\right), \\ f_2(x, z, t) &= 5\cos\left(\frac{2\pi}{17}\left[z + \frac{(x-5)^2}{2z}\right] - 2\pi 10t + \frac{\pi}{3}\right). \end{aligned} \tag{52}$$

As described in the class notes (Lecture 13, p. 6), the waves of equation 52 are paraxial approximations of spherical waves. For the case of f_1, the originating point source is centered at $(0, 0)$, and the additional parameters are: $A = 5$, $\lambda = 17$, $\nu = 10$. The

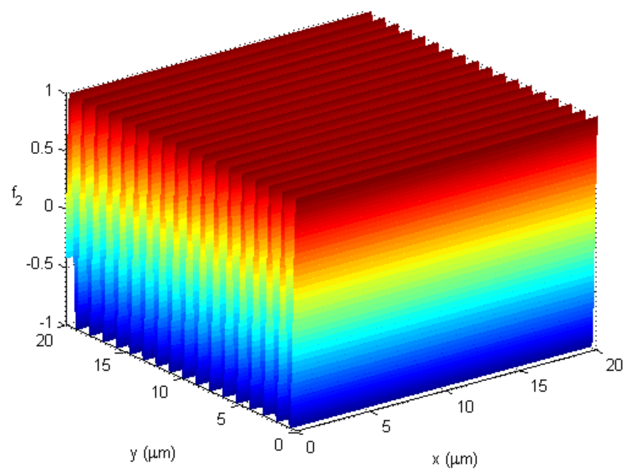

Figure 5: Plane wave propagating at 60° in the xz-plane.

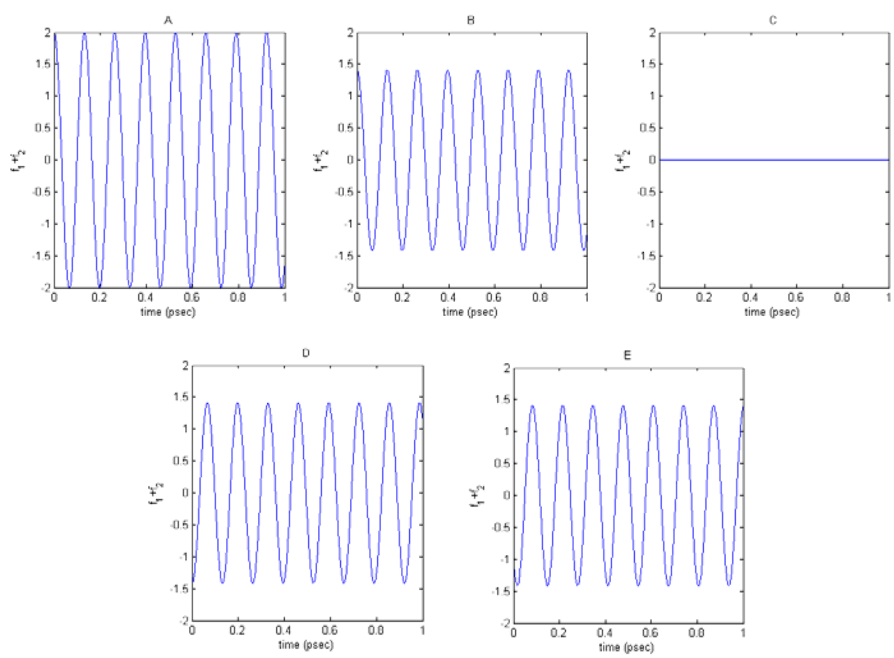

Figure 6: Wave superposition.

second wave, f_2, shares the same parameters as f_1; however, the originating point souce is shifted at $x_s = 5$ and the wave is phase shifted by $\phi = \pi/3$.

b) The phase velocity is given by $v_p = \omega/k$. For the two waves of equation 52 their corresponding phase velocities are,

$$v_{p1} = v_{p2} = \lambda\nu = 170. \tag{53}$$

c) Now we are interested in computing the coherent superposition of the two waves,

$$\begin{aligned}
f(x,z,t) &= f_1(x,z,t) + f_2(x,z,t) \\
&= 5[\cos\left(\frac{2\pi}{17}\left[z + \frac{x^2}{2z}\right] - 2\pi 10t\right) \\
&\quad + \cos\left(\frac{2\pi}{17}\left[z + \frac{(x-5)^2}{2z}\right] - 2\pi 10t + \frac{\pi}{3}\right)] \\
&= 5\left[\cos\left(\phi_1\right) + \cos\left(\phi_2\right)\right] \\
&= 10\left[\cos\left(\frac{\phi_1+\phi_2}{2}\right)\cos\left(\frac{\phi_1-\phi_2}{2}\right)\right] \\
&= 10[\cos\left(\frac{12\pi z^2 + 6\pi x^2 - 2040\pi tz - 30\pi x + 75\pi + 17\pi z}{102z}\right) \\
&\quad \cdot\cos\left(\frac{30\pi x - 75\pi - 17\pi z}{102z}\right)].
\end{aligned} \tag{54}$$

d) The two waves in phasor notation are,

$$\begin{aligned}
f_{p1}(x,z,t) &= 5\exp\left(i\frac{2\pi}{17}\left[z + \frac{x^2}{2z}\right] - i2\pi 10t\right), \\
f_{p2}(x,z,t) &= 5\exp\left(i\frac{2\pi}{17}\left[z + \frac{(x-5)^2}{2z}\right] - i2\pi 10t + i\frac{\pi}{3}\right).
\end{aligned} \tag{55}$$

The coherent superposition of the two waves is,

$$\begin{aligned}
f_p(x,z,t) &= f_{p1}(x,z,t) + f_{p2}(x,z,t) \qquad (56)\\
&= 5[\exp\left(i\frac{2\pi}{17}\left[z+\frac{x^2}{2z}\right] - i2\pi 10t\right)\\
&\quad + \exp\left(i\frac{2\pi}{17}\left[z+\frac{(x-5)^2}{2z}\right] - i2\pi 10t + i\frac{\pi}{3}\right)]\\
&= 5[\exp\left(i\frac{2\pi}{17}\left[z+\frac{x^2}{2z}\right] - i2\pi 10t\right)\\
&\quad + \exp\left(i\frac{2\pi}{17}\left[z+\frac{x^2}{2z}\right] - i2\pi 10t\right)\exp\left(i\frac{2\pi}{17}\left[-\frac{5x}{2z}+\frac{25}{2z}\right] + i\frac{\pi}{3}\right)]\\
&= 5\exp(i\phi_1)[1+\cos\left(\frac{2\pi}{17}\left[\frac{25-5x}{2z}\right]+\frac{\pi}{3}\right)\\
&\quad + i\sin\left(\frac{2\pi}{17}\left[\frac{25-5x}{2z}\right]+\frac{\pi}{3}\right)]\\
&= 5\left[\cos(\phi_1) + i\sin(\phi_1)\right]\left[1+\cos(\phi_3)+i\sin(\phi_3)\right].
\end{aligned}$$

If we take the real part of equation 56,

$$\begin{aligned}
f(x,z,t) &= \mathrm{Re}\{f_p(x,z,t)\} \qquad (57)\\
&= 5\left[\cos(\phi_1) + \cos(\phi_1)\cos(\phi_3) - \sin(\phi_1)\sin(\phi_3)\right]\\
&= 5\left[\cos(\phi_1) + \cos(\phi_1+\phi_3)\right]\\
&= 5\left[\cos(\phi_1) + \cos(\phi_2)\right],
\end{aligned}$$

which is the same as in equation 54.

Problem 8: Dispersive waves

a) Recall frrom the class notes (Lecture 13, p. 7)that the dispersion relation for a metallic waveguide is,

$$\left(\frac{m\pi}{a}\right)^2 + k^2 = \left(\frac{\omega}{c}\right)^2, \qquad (58)$$

where for this problem $\omega = 1.5\times 10^{15}$rad/sec, $a = 1\mu$m and $m = 1$ since only one mode is allowed. Solving for k,

$$\begin{aligned}
k &= \sqrt{\left(\frac{\omega}{c}\right)^2 - \left(\frac{m\pi}{a}\right)^2} \qquad (59)\\
&= 3.8898\times 10^6\mathrm{m}^{-1}\\
&\Rightarrow \lambda_{wg} = \frac{2\pi}{k} = 1.6153\times 10^{-6}\mathrm{m}.
\end{aligned}$$

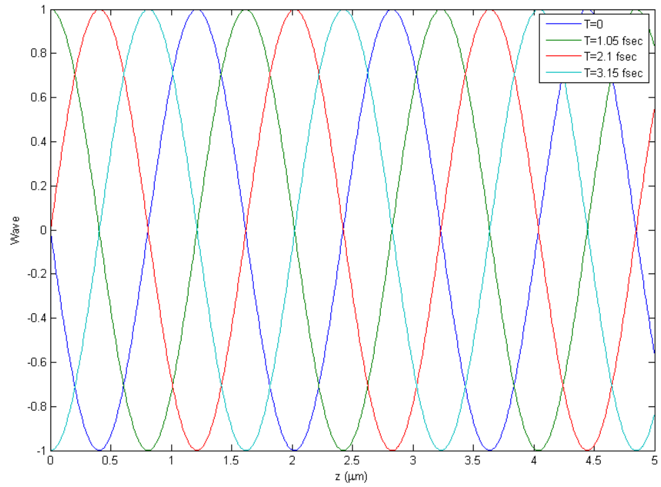

Figure 7: Single mode inside a metallic waveguide.

Comparing equation 59 with the free space wavelength,

$$\lambda_{fs} = \frac{2\pi c}{\omega} = 1.2566 \times 10^{-6}\text{m}. \tag{60}$$

The temporal period is,

$$T = \frac{2\pi}{\omega} = 4.188 \text{ f sec}\,. \tag{61}$$

Since in the problem statement we are told that the amplitude of the wave is maximum at a distance 0.4μm inside the waveguide at $t = 0$, that is $0.4\mu\text{m} \approx \lambda_{wg}/4$, the wave is initially phase advanced by $\pi/2$. Figure 7 shows the evolution of the wave at times 1.05fsec, 2.1fsec and 3.15fsec after the wave is launched.

b) As discussed in part (a), the distance traveled by a point of constant phase on the wavefront after 4.2 fsec (temporal period) equals λ_{wg}.The distance traveled by the same point for a wave propagating in free space equals λ_{fs}.

c) The group velocity is given by,

$$
\begin{aligned}
v_g &= \frac{\partial \omega}{\partial k} \\
&= \frac{\partial \left(c\sqrt{\left(\frac{m\pi}{a}\right)^2 + k^2} \right)}{\partial k} \\
&= \frac{c^2 k}{c\sqrt{\left(\frac{m\pi}{a}\right)^2 + k^2}} = \frac{c^2 k}{\omega},
\end{aligned} \tag{62}
$$

since k is given by equation 59,

$$
\begin{aligned}
v_g &= \frac{c^2\sqrt{\left(\frac{\omega}{c}\right)^2 - \left(\frac{m\pi}{a}\right)^2}}{\omega} \\
&= c\sqrt{1 - \left(\frac{m\pi c}{a\omega}\right)^2}.
\end{aligned} \tag{63}
$$

Problem 9: Schroedinger's Equation

a) The equation describing the wavepacket associated with a particle in Quantum Mechanics is,

$$
\frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2} + \frac{\partial^2 \Psi}{\partial z^2} = -i\frac{2m}{\hbar}\frac{\partial \Psi}{\partial t}, \tag{64}
$$

where, m is the particle mass, $\hbar = h/2\pi$ and h is Planck's constant. Consider a trial solution of the form,

$$
\begin{aligned}
\Psi(x, y, z, t) &= e^{i\mathbf{k}\cdot\mathbf{r}} e^{-i\omega t}, \\
&\Rightarrow \frac{\partial \Psi}{\partial t} = -i\omega\Psi.
\end{aligned} \tag{65}
$$

Since $i = \exp(i\pi/2)$, the term $\partial\Psi/\partial t$ should be $\pi/2$ phase shifted with respect to the Laplacian, $\nabla^2\Psi$.

b) We compute the dispersion relation using the plane wave solution of equation 65,

$$
\begin{aligned}
-(k_x^2 + k_y^2 + k_z^2) &= -\frac{2m\omega}{\hbar} \\
|\mathbf{k}|^2 &= \frac{2m\omega}{\hbar}.
\end{aligned} \tag{66}
$$

An example of a dispersion diagram is shown in Figure 8.

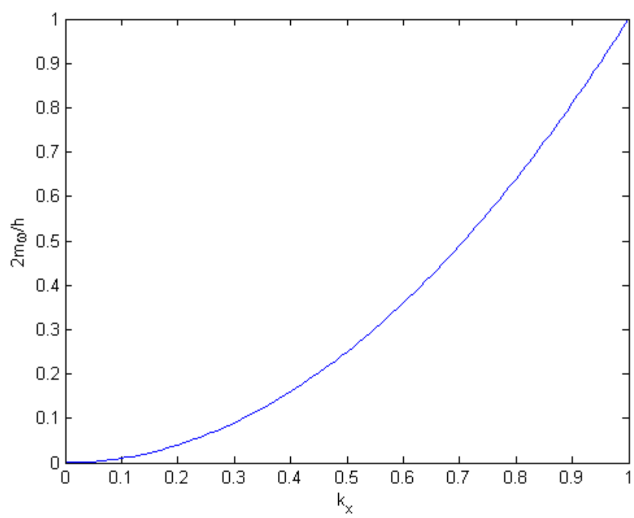

Figure 8: Dispersion diagram.