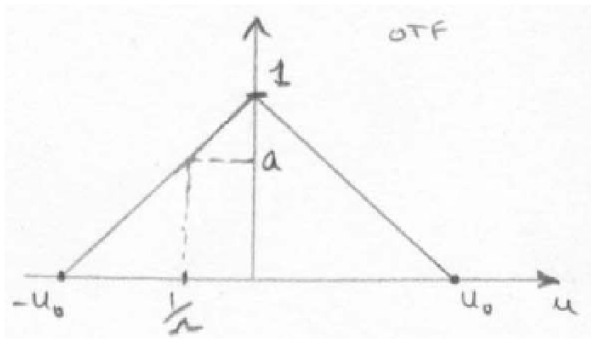
1. (a) For a diffraction limited system the slopes of the OTF are constant.

$$m\Big|_{u=25\text{mm}^{-1}} = 68.75\% = 0.6875$$



$$I_{\text{in}} = \frac{1}{2}\left[1+\cos\left(2\pi\frac{x}{\Lambda}\right)\right]$$
$$\Rightarrow \hat{I}_{\text{in}} = \frac{1}{2}\delta(u)+\frac{1}{4}\delta\left(u-\frac{1}{\Lambda}\right)+\frac{1}{4}\delta\left(u+\frac{1}{\Lambda}\right)$$
$$\hat{I}_{\text{out}} = \hat{I}_{\text{in}}\cdot\text{OTF} = \frac{1}{2}\delta(u)+\frac{a}{4}\delta\left(u-\frac{1}{\Lambda}\right)+\frac{a}{4}\delta\left(u+\frac{1}{\Lambda}\right)$$
$$I_{\text{out}}(x') = \frac{1}{2}\left[1+a\cos\left(2\pi\frac{x'}{\Lambda}\right)\right]$$
$$m\Big|_{u=\frac{1}{\Lambda}} = \frac{(\frac{1}{2}+\frac{a}{2})-(\frac{1}{2}-\frac{a}{2})}{(\frac{1}{2}+\frac{a}{2})+(\frac{1}{2}-\frac{a}{2})} = a$$

$\therefore$ the contrast is the normalized value of the OTF at that frequency. Using similar triangles, if $m\Big|_{u=25\text{mm}^{-1}} = 0.6875 = (1-0.3125)$, then

$$m\Big|_{u=50\text{mm}^{-1}} = (1-0.6250) = 0.3750 = \boxed{37.5\ \%}$$

(b) The cut-off frequency for incoherent imaging is $u_0 = 80\text{mm}^{-1}$. The cut-off frequency of the coherently illuminated system is 40mm^{-1}. Hence 50mm^{-1} frequencies do NOT go through if it is coherently illuminated.

2.

$$I(x) = \frac{1}{2}\left[1+\frac{1}{2}\cos\left(2\pi\frac{x}{40\mu\text{m}}\right)+\frac{1}{2}\cos\left(2\pi\frac{3x}{40\mu\text{m}}\right)\right]$$

(a) The contrast m is given by:

$$m = \frac{I_{\max}-I_{\min}}{I_{\max}+I_{\min}}$$

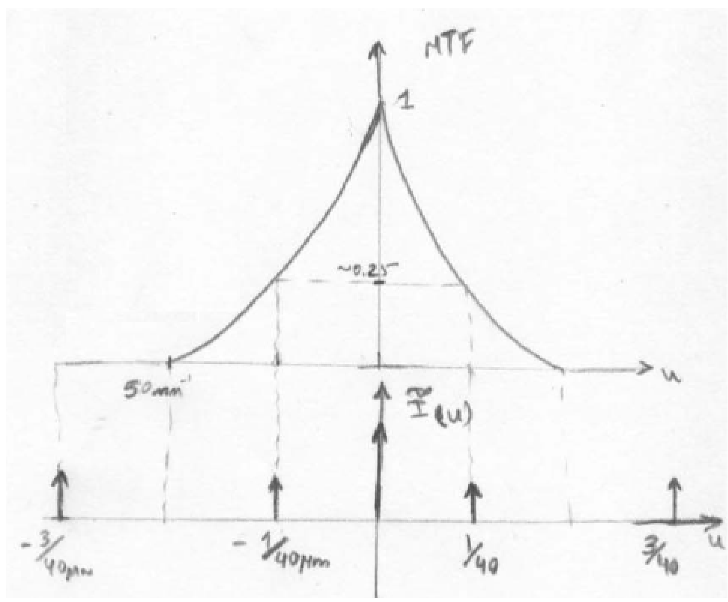
At the input, $m = \frac{1-0}{1+0} = 1$.

(b) The Fourier transform of $I(x)$ is:

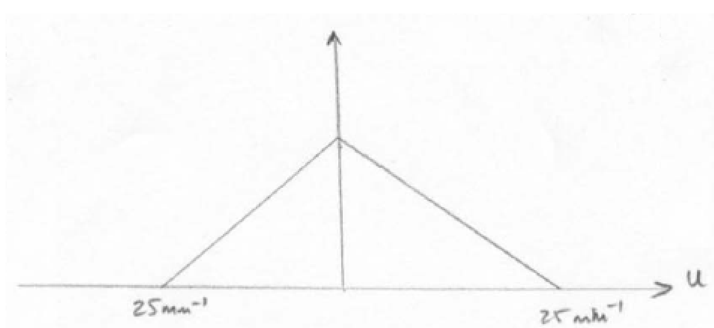
$$\tilde{I}(u) = \frac{1}{8}\left[\delta\left(u - \frac{1}{40}\right) + \delta\left(u + \frac{1}{40}\right) + \delta\left(u - \frac{3}{40}\right) + \delta\left(u + \frac{3}{40}\right)\right] + \frac{1}{2}\delta(u)$$

The Fourier transform of the output intensity is:

$$\begin{aligned}
\tilde{I}_0(u) &= (\text{MTF}) \cdot \tilde{I}(u) \\
&= \frac{1}{2}\delta(u) + (0.25)\frac{1}{8}\left[\delta\left(u - \frac{1}{40}\right) + \delta\left(u + \frac{1}{40}\right)\right] \\
&= \frac{1}{2}\delta(u) + \frac{1}{16}\left[\frac{1}{2}\delta\left(u - \frac{1}{40}\right) + \frac{1}{2}\delta\left(u + \frac{1}{40}\right)\right] \\
I_0(x') &= \frac{1}{2} + \frac{1}{16}\cos\left(2\pi\frac{x'}{40}\right) \\
m_{\text{out}} &= \frac{(\frac{1}{2} + \frac{1}{16}) - (\frac{1}{2} - \frac{1}{16})}{(\frac{1}{2} + \frac{1}{16}) + (\frac{1}{2} - \frac{1}{16})} = \frac{1}{8} = 0.125
\end{aligned}$$



(c) The incoherent transfer function is an autocorrelation of the coherent transfer function. The coherent transfer function in this case is probably a triangle function with half the cut-off frequency.



$$H(u) = \mathcal{F}\{h(x)\}$$

3.

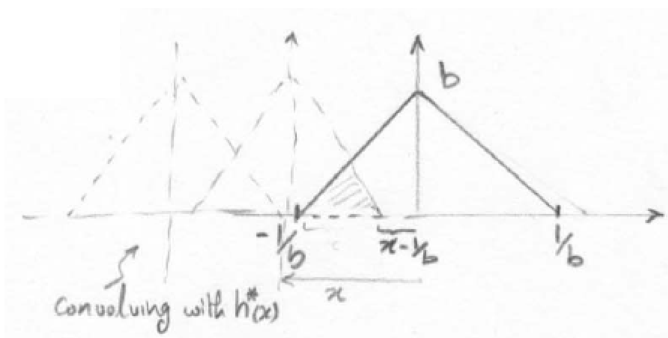
$$h(x) = \mathrm{sinc}^2\left(\frac{x}{b}\right)$$

(a) Incoherent iPSF

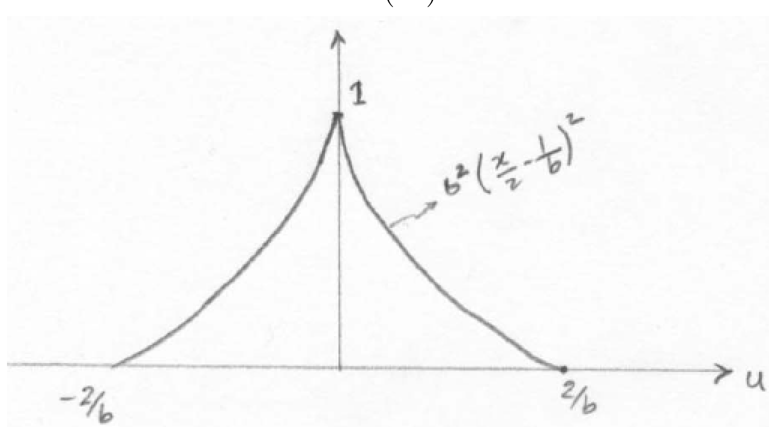
$$\tilde{h}(x) = |h(x)|^2 = \mathrm{sinc}^4\left(\frac{x}{b}\right)$$

(b) MTF $= \left|\tilde{H}(u)\right|$

$$\begin{aligned}\tilde{H}(u) &= \mathcal{F}\{\tilde{h}(x)\} = \mathcal{F}\left\{\mathrm{sinc}^2\left(\frac{x}{b}\right)\cdot \mathrm{sinc}^2\left(\frac{x}{b}\right)\right\}\\ &= \mathcal{F}\left\{\mathrm{sinc}^2\left(\frac{x}{b}\right)\right\} \otimes \mathcal{F}\left\{\mathrm{sinc}^2\left(\frac{x}{b}\right)\right\}\\ &= b\Lambda(bu) \otimes b\Lambda(bu)\end{aligned}$$



$b\Lambda(bu)$



MTF

2.710 Optics **Spring '09**

Solutions to Problem Set #8

Due Wednesday, May 13, 2009

Problem 4:

a) Consider the system shown in Figure 1. At the input we place a sinusoidal amplitude grating of perfect contrast,

$$g_t(x) = \frac{1}{2}\left[1 + \cos\left(2\pi\frac{x}{\Lambda}\right)\right], \tag{1}$$

where $\lambda = 0.5\mu$m, $f_1 = f_2 = f = 20$cm, $\Lambda = 10\mu$m, and the two apertures at the pupil plane have a diameter of 1cm and are separated by 1cm from the optical axis.

We begin by computing the input intensity (assuming uniform intensity illumination),

$$\begin{aligned} I_{in} &= |g_t(x)|^2 = \frac{1}{4}\left[1 + \cos\left(2\pi\frac{x}{\Lambda}\right)\right]^2 \\ &= \frac{1}{4}\left[1 + \cos^2\left(2\pi\frac{x}{\Lambda}\right) + 2\cos\left(2\pi\frac{x}{\Lambda}\right)\right] \\ &= \frac{1}{4}\left[1 + \frac{1}{2} + \frac{1}{2}\cos\left(4\pi\frac{x}{\Lambda}\right) + 2\cos\left(2\pi\frac{x}{\Lambda}\right)\right] \\ &= \frac{3}{8} + \frac{1}{8}\cos\left(4\pi\frac{x}{\Lambda}\right) + \frac{1}{2}\cos\left(2\pi\frac{x}{\Lambda}\right), \end{aligned} \tag{2}$$

where we used the following trigonometric identity,

$$\cos^2(\alpha) = \frac{1}{2} + \frac{1}{2}\cos(2\alpha). \tag{3}$$

Now we compute the frequency spectra of the input signal,

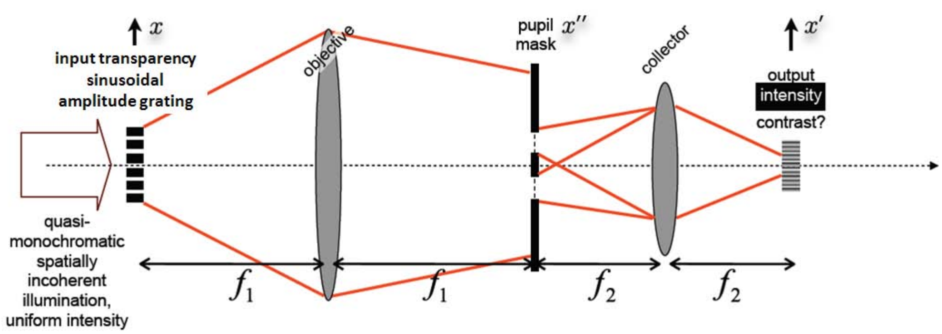


Figure 1: Optical system for problem 4.

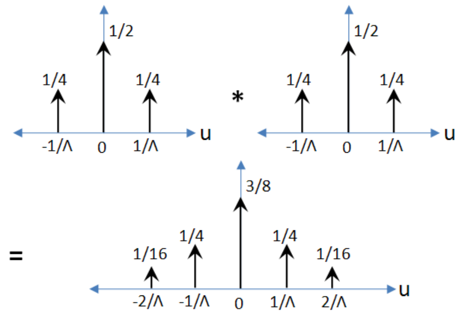


Figure 2: Input signal spectrum.

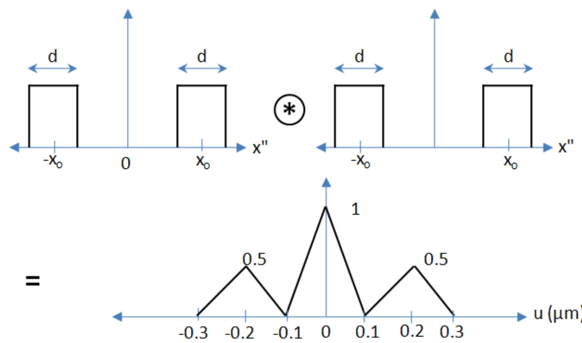


Figure 3: Graphical calculation of OTF.

$$\begin{aligned} G_{in} &= \mathcal{F}\{I_{in}\} \\ &= \frac{3}{8}\delta(u) + \frac{1}{4}\left[\delta(u-\frac{1}{\Lambda}) + \delta(u+\frac{1}{\Lambda})\right] + \frac{1}{16}\left[\delta(u-\frac{2}{\Lambda}) + \delta(u+\frac{2}{\Lambda})\right], \end{aligned} \tag{4}$$

which can be represented graphically as shown in Figure 2.

The ATF of the system is,

$$H = \text{rect}\left(\frac{x'' - x_o}{d}\right) + \text{rect}\left(\frac{x'' + x_o}{d}\right), \tag{5}$$

where x_o is the lateral shift (1cm) and d is the aperture diamter (also 1cm). We compute the OTF of the optical system graphically as shown in Figure 3. The OTF is the normalized cross-correlation of the ATF of the system.

To compute the spectrum of the output signal we multiply the OTF by the input spectra,

$$\begin{aligned} G_{out} &= \hat{H} G_{in} \\ &= \frac{3}{8}\delta(u) + \frac{1}{32}\left[\delta(u - \frac{2}{\Lambda}) + \delta(u + \frac{2}{\Lambda})\right]_{u=\frac{x''}{\lambda f}}. \end{aligned} \tag{6}$$

The output intensity is,

$$\begin{aligned} I_{out} &= \mathcal{F}\{G_{out}\} \\ &= \frac{3}{8} + \frac{1}{16}\left[\frac{e^{-i2\pi\frac{2x''}{\Lambda}} + e^{i2\pi\frac{2x''}{\Lambda}}}{2}\right] \\ &= \frac{3}{8} + \frac{1}{16}\cos\left(2\pi\frac{2x''}{\Lambda}\right). \end{aligned} \tag{7}$$

(b) The constrast or visibility is given by,

$$V = \frac{I_{\max} - I_{\min}}{I_{\max} + I_{\min}}. \tag{8}$$

For the output intensity of equation 7, the contrast is $V = 0.167$.

(c) Finally we compare our results with the coherent case where the spectrum of the input field is,

$$G_{in} = \mathcal{F}\{g_t(x)\} = \frac{1}{2}\delta(u) + \frac{1}{4}\left[\delta(u - \frac{1}{\Lambda}) + \delta(u + \frac{1}{\Lambda})\right]. \tag{9}$$

The ATF of the system acts as a pass-band filter as shown in Figure 4 and only allows the first diffraction order to pass,

$$G_{out} = \frac{1}{4}\left[\delta(u - \frac{1}{\Lambda}) + \delta(u + \frac{1}{\Lambda})\right]_{u=\frac{x''}{\lambda f}}. \tag{10}$$

The output field is,

$$\begin{aligned} g_{out} &= \mathcal{F}\{G_{out}\} \\ &= \frac{1}{2}\left[\frac{e^{-i2\pi\frac{x'}{\Lambda}} + e^{i2\pi\frac{x'}{\Lambda}}}{2}\right] \\ &= \frac{1}{2}\cos\left(2\pi\frac{x'}{\Lambda}\right), \end{aligned} \tag{11}$$

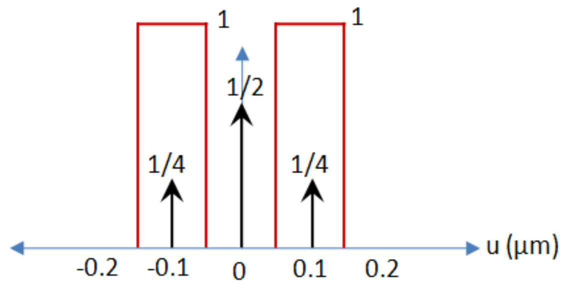


Figure 4: Band-pass filter - coherent system.

and the output intensity is,

$$
\begin{aligned}
I_{out} &= |g_{out}|^2 \qquad (12)\\
&= \frac{1}{4}\cos^2\left(2\pi\frac{x'}{\Lambda}\right)\\
&= \frac{1}{4}\left[\frac{1}{2}+\frac{1}{2}\cos\left(2\pi\frac{2x'}{\Lambda}\right)\right]\\
&= \frac{1}{8}+\frac{1}{8}\cos\left(2\pi\frac{2x'}{\Lambda}\right).
\end{aligned}
$$

For the coherent case we note the the contrast is $V = 1$.

2.710 Optics **Spring '09**

Solutions to Problem Set #8

Due Wednesday, May 13, 2009

Problem 5:

a) Consider the system shown in Figure 1. The input transparency is a binary amplitude grating with the following parameters: $m = 1$, duty cycle $= \alpha = 1/3$, $\Lambda = 10\mu$m. The operating wavelength is $\lambda = 0.5\mu$m, and the focal lengths are $f_1 = f_2 = f = 20$cm. At the Fourier plane, the pupil mask has two apertures with a diameter of 1cm shifted 2cm from the optical axis.

We begin by expressing the binary amplitude grating in a Fourier Series,

$$g_t(x) = \alpha \sum_{q=-\infty}^{\infty} \text{sinc}\left(\alpha q\right) e^{i2\pi q \frac{x}{\Lambda}}. \tag{1}$$

Since equation 1 has a binary amplitude dependence that goes from 0 to 1, the intensity is also binary, $I_{in} = |g(t)|^2$. The spectrum of the input signal is given by,

$$\begin{aligned} G_{in}(u) &= \left[\alpha \sum_{q=-\infty}^{\infty} \text{sinc}\left(\alpha q\right) \delta\left(u - \frac{q}{\Lambda}\right)\right]_{u=\frac{x''}{\lambda f}} \qquad (2) \\ &\rightarrow \quad G_{in}(x'') = \alpha \sum_{q=-\infty}^{\infty} \text{sinc}\left(\alpha q\right) \delta\left(x'' - \frac{\lambda f q}{\Lambda}\right), \end{aligned}$$

and is shown in Figure 2.

The OTF, $\hat{H}$, of the optical system is computed in the same way as in problem 4 and it is shown in Figure 3.

The output field spectrum is given by,

$$\begin{aligned} G_{out} &= \hat{H} G_{in} \qquad (3) \\ &= \frac{1}{3}\delta(x'') - \frac{\sqrt{3}}{\pi 16}\left[\delta\left(x'' - \frac{\lambda f 4}{\Lambda}\right) + \delta\left(x'' + \frac{\lambda f 4}{\Lambda}\right)\right]. \end{aligned}$$

The output intensity is,

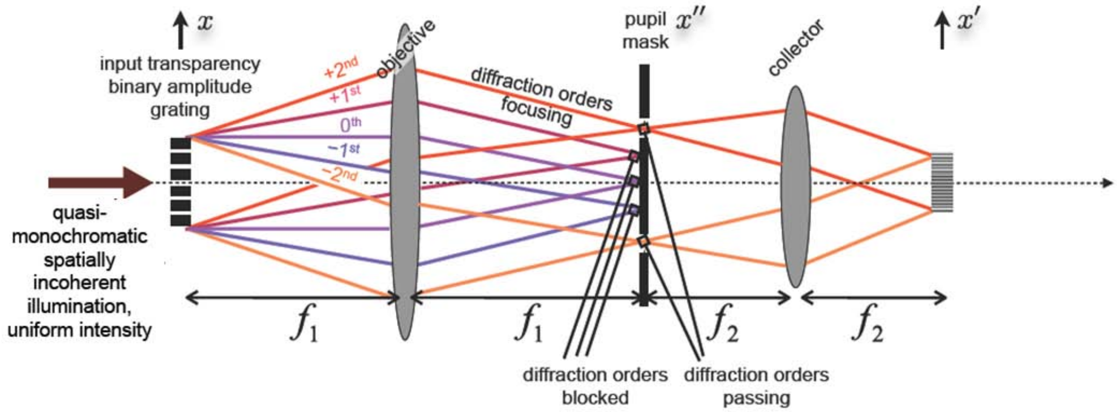


Figure 1: Optical system for problem 5.

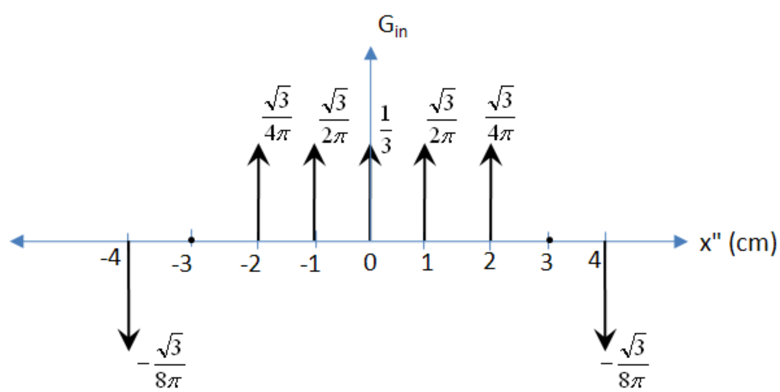


Figure 2: Input signal spectrum.

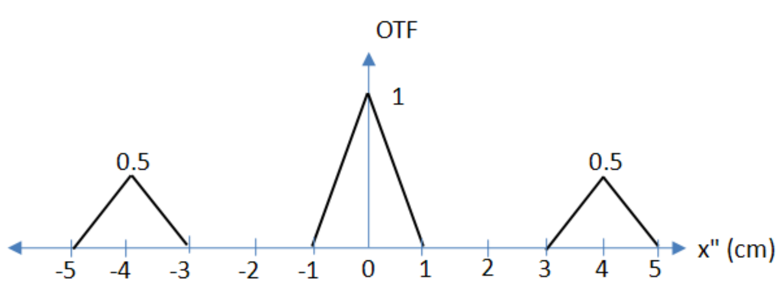


Figure 3: System's OTF.

$$
\begin{aligned}
I_{out} &= \mathcal{F}\{G_{out}\} \\
&= \frac{1}{3} - \frac{\sqrt{3}}{\pi 8}\left[\frac{e^{-i2\pi\frac{x'4}{\Lambda}} + e^{i2\pi\frac{x'4}{\Lambda}}}{2}\right] \\
&= \frac{1}{3} - \frac{\sqrt{3}}{\pi 8}\cos\left(2\pi\frac{x'4}{\Lambda}\right).
\end{aligned}
\tag{4}
$$

(b) The resulting contrast is,

$$
V = \frac{I_{\max} - I_{\min}}{I_{\max} + I_{\min}} = 0.2067. \tag{5}
$$

(c) Comparying with the results from Lecture 19, p. 24 , for the coherent case, the output intensity is,

$$
I_{out} = \frac{3}{8\pi^2} + \frac{3}{8\pi^2}\cos\left(2\pi\frac{x'4}{\Lambda}\right),
$$

and the contrast $V = 1$.

2.710 Optics **Spring '09**

Solutions to Problem Set #8

Due Wednesday, May 13, 2009

Problem 6:

Consider the optical system shown in Figure 1. The input transparency is a binary amplitude grating with the same parameters as in problem 5. The operating wavelength is $\lambda = 0.5\mu$m, and the focal lengths are $f_1 = f_2 = f = 20$cm. The pupil mask is given by,

$$\begin{aligned} g_{PM}(x'') &= \left[\mathrm{rect}\left(\frac{x''}{a}\right) + (i-1)\,\mathrm{rect}\left(\frac{x''}{b}\right)\right]_{x''=u\lambda f} \qquad (1)\\ &\rightarrow \; H(u) = g_{PM}(u) \\ &= \mathrm{rect}\left(\frac{u}{\alpha}\right) + (i-1)\,\mathrm{rect}\left(\frac{u}{\beta}\right), \end{aligned}$$

where $\alpha = a/\lambda f$, $\beta = b/\lambda f$, $a = 3$cm and $b = 1$cm. The magnitude and phase of the pupil mask are shown in Figure 2.

We now compute the coherent PSF,

$$\begin{aligned} h(x) &= \mathcal{F}^{-1}\{H(u)\} \qquad (2)\\ &= \alpha \mathrm{sinc}\,(\alpha x) + (i-1)\,\beta \mathrm{sinc}\,(\beta x) \\ &= \alpha \mathrm{sinc}\,(\alpha x) - \beta \mathrm{sinc}\,(\beta x) + i\beta \mathrm{sinc}\,(\beta x)\,. \end{aligned}$$

The incoherent PSF is given by,

$$\begin{aligned} \hat{h}(x) &= |h(x)|^2 = h \cdot h^* \qquad (3)\\ &= [\alpha \mathrm{sinc}\,(\alpha x) - \beta \mathrm{sinc}\,(\beta x)]^2 + \beta^2 \mathrm{sinc}^2\,(\beta x) \\ & \alpha^2 \mathrm{sinc}^2\,(\alpha x) + 2\beta^2 \mathrm{sinc}^2\,(\beta x) - 2\alpha\beta \mathrm{sinc}\,(\alpha x)\,\mathrm{sinc}\,(\beta x)\,. \end{aligned}$$

The OTF of the system is found by computing the Fourier transform of equation 3,

$$\hat{H}(u) = \alpha \mathrm{triag}\left(\frac{u}{\alpha}\right) + 2\beta \mathrm{triag}\left(\frac{u}{\beta}\right) - 2\left[\mathrm{rect}\left(\frac{u}{\alpha}\right) * \mathrm{rect}\left(\frac{u}{\beta}\right)\right], \qquad (4)$$

and is shown in Figure 3.

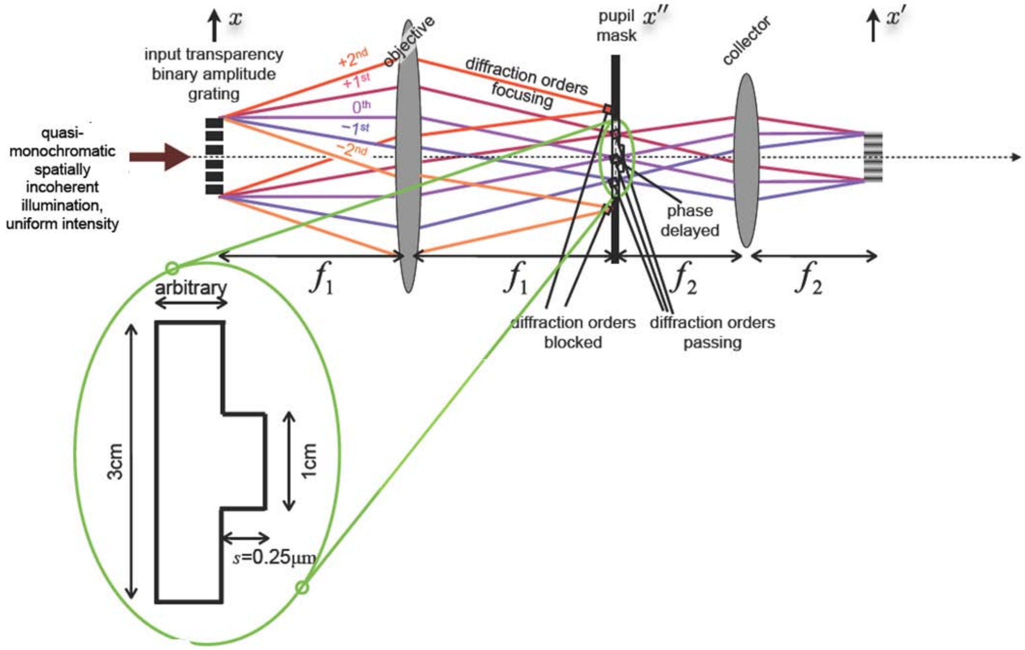


Figure 1: Optical system for problem 6.

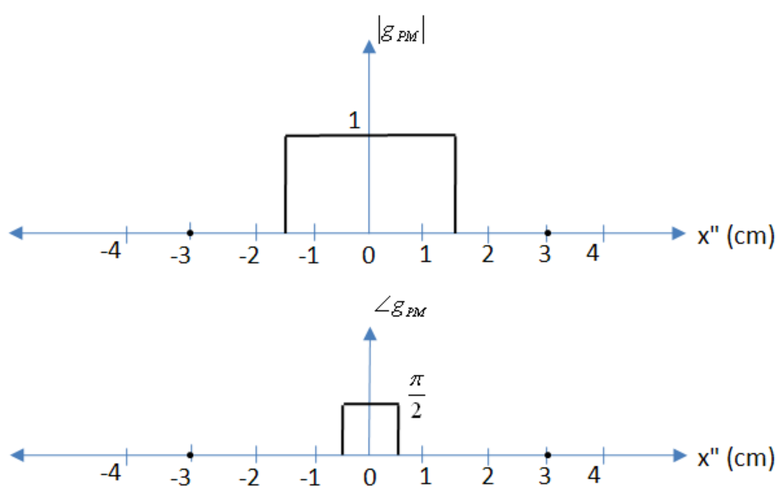


Figure 2: Magnitud and phase of the pupil mask.

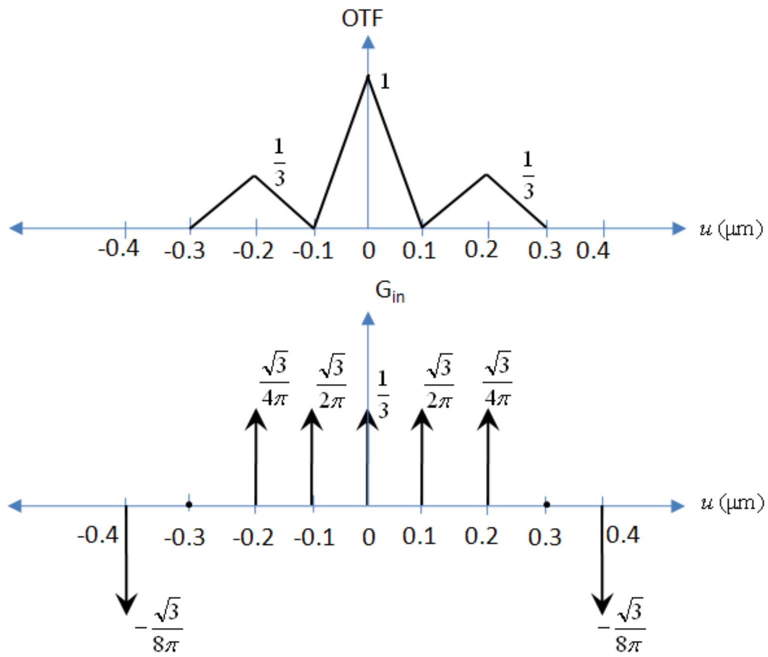


Figure 3: OTF and spectrum of the input signal.

From problem 5, the spectrum of the input signal is,

$$G_{in}(u) = \alpha \sum_{q=-\infty}^{\infty} \text{sinc}\,(\alpha q)\,\delta\left(u - \frac{q}{\Lambda}\right). \tag{5}$$

To compute the spectrum of the output signal we multiply equations 5 and 6,

$$G_{out}(u) = \frac{1}{3}\delta\,(u) + \frac{\sqrt{3}}{12\pi}\left[\delta\left(u - \frac{2}{\Lambda}\right) + \delta\left(u + \frac{2}{\Lambda}\right)\right]. \tag{6}$$

The output intensity is given by,

$$\begin{aligned} I_{out} &= \mathcal{F}\{G_{out}\} \\ &= \frac{1}{3} + \frac{\sqrt{3}}{6\pi}\left[\frac{e^{-i2\pi\frac{x'2}{\Lambda}} + e^{i2\pi\frac{x'2}{\Lambda}}}{2}\right] \\ &= \frac{1}{3} + \frac{\sqrt{3}}{6\pi}\cos\left(2\pi\frac{x'2}{\Lambda}\right). \end{aligned} \tag{7}$$

(b) The resulting contrast is,

$$V = \frac{I_{\max} - I_{\min}}{I_{\max} + I_{\min}} = 0.2757. \tag{8}$$

(c) Finally, we compare our results to those from the coherent case as discussed in Lecture 19, p. 27, where the output intensity is,

$$I_{out}(x^{'}) = \left(\frac{1}{3}\right)^2 + \frac{3}{2\pi^2} + \frac{3}{2\pi^2}\cos\left(2\pi\frac{x^{'}2}{\Lambda}\right), \tag{9}$$

and the contrast is $V = 0.2548$.

2.710 Optics **Spring '09**

Solutions to Problem Set #8

Due Wednesday, May 13, 2009

Problem 7:

In this problem we are required to show that the intensity at the Fourier plane of the system shown in Figure 1 is proportional to the autocorrelation of the input field, $g_{in} = g_{illum}(x) \cdot g_t(x)$. From the supplement to lecture 18 , we can see that the optical field at the Fourier plane for the special case of $z = f$ is,

$$g_F(x'', y'') = \frac{e^{i2\pi\frac{2f}{\lambda}}}{i\lambda f} G\left(\frac{x''}{\lambda f}, \frac{y''}{\lambda f}\right). \tag{1}$$

The intensity at that plane is,

$$I_F = g_F \cdot g_F^* \propto G \cdot G^*, \tag{2}$$

which is proportional to the autocorrelation of the input field as they are related by a Fourier transform,

$$\Im\{g_{in} \otimes g_{in}\} = G \cdot G^*. \tag{3}$$

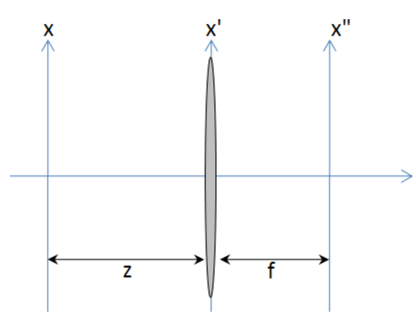


Figure 1: Fourier transforming lens.

2.71 / 2.710 Optics
Spring 2009