1.

$$|\vec{k}_1| = |\vec{k}_2| = \frac{2\pi}{\lambda}$$

$$\vec{k}_1 = \frac{2\pi}{\lambda}(\sin 30^\circ \hat{x} + \cos 30^\circ \hat{z}) = \frac{2\pi}{\lambda}\left(\frac{1}{2}\hat{x} + \frac{\sqrt{3}}{2}\hat{z}\right)$$

$$\begin{aligned}\vec{k}_2 &= \frac{2\pi}{\lambda}(\cos 45^\circ \hat{x} + \sin 45^\circ \sin 30^\circ \hat{y} + \sin 45^\circ \cos 30^\circ \hat{z}) \\ &= \frac{2\pi}{\lambda}\left(\frac{\sqrt{2}}{2}\hat{x} + \frac{\sqrt{2}}{4}\hat{y} + \frac{\sqrt{6}}{4}\hat{z}\right)\end{aligned}$$

Assuming $|E_1| = |E_2| = 1$,

$$\left.\begin{aligned}E_1(x,y,z) &= e^{i\vec{k}_1\cdot\vec{r}} = e^{i\frac{2\pi}{\lambda}(\frac{x}{2}+\frac{\sqrt{3}}{2}z)} \equiv e^{i\phi_1} \\ E_2(x,y,z) &= e^{i\vec{k}_2\cdot\vec{r}} = e^{i\frac{2\pi}{\lambda}(\frac{\sqrt{2}}{2}x+\frac{\sqrt{2}}{4}y+\frac{\sqrt{6}}{4}z)} \equiv e^{i\phi_2}\end{aligned}\right\}\text{interference pattern I}$$

$$\begin{aligned}I &= |E_1 + E_2|^2 = |E_1|^2 + |E_2|^2 + 2|E_1||E_2|\cos(\phi_1 - \phi_2) \\ &= 2[1 + \cos(\phi_1 - \phi_2)]\end{aligned}$$

(a) In the xy-plane, $z = 0$

$$\left.\begin{aligned}\phi_1 &= \tfrac{2\pi}{\lambda}(x/2) \\ \phi_2 &= \tfrac{2\pi}{\lambda}(\tfrac{\sqrt{2}}{2}x + \tfrac{\sqrt{2}}{4}y)\end{aligned}\right\}\phi_1 - \phi_2 = \frac{2\pi}{\lambda}\left[\left(\frac{1}{2} - \frac{\sqrt{2}}{2}\right)x - \frac{\sqrt{2}}{4}y\right] = \Delta\phi$$

$I = 2[1 + \cos\Delta\phi]$ so the profile is a sinusoidal profile. The maxima are along the lines whose equation is:

$$\frac{2\pi}{\lambda}\left[\left(\frac{1}{2} - \frac{\sqrt{2}}{2}\right)x - \frac{\sqrt{2}}{4}y\right] = 2m\pi, \text{ where } m \in \mathbb{Z}$$

$$\frac{1-\sqrt{2}}{2}x - \frac{\sqrt{2}}{4}y = m\lambda$$

(b) For the plane $z = \lambda$

$$\left.\begin{aligned}\phi_1 &= \tfrac{2\pi}{\lambda}\left(\tfrac{1}{2}x + \tfrac{\sqrt{3}}{2}\lambda\right) \\ \phi_2 &= \tfrac{2\pi}{\lambda}\left(\tfrac{\sqrt{2}}{2}x + \tfrac{\sqrt{2}}{4}y + \tfrac{\sqrt{6}}{4}\lambda\right)\end{aligned}\right\}\Delta\phi = \frac{2\pi}{\lambda}\left[\frac{1-\sqrt{2}}{2}x + \frac{\sqrt{2}}{4}y + \frac{2\sqrt{3}-\sqrt{6}}{4}\lambda\right]$$

$I = 2[1+\cos\Delta\phi]$, so the interference pattern is still a sinusoid (i.e. a set of linear fringes). The maxima occur when $\Delta\phi = 2\pi m, m \in \mathbb{Z}$. The equation of the fringe lines are:

$$\frac{2\pi}{\lambda}\left(\frac{1-\sqrt{2}}{2}x + \frac{\sqrt{2}}{4}y + \frac{2\sqrt{3}-\sqrt{6}}{4}\lambda\right) = 2\pi m, m \in \mathbb{Z}$$

$$\frac{1-\sqrt{2}}{2}x + \frac{\sqrt{2}}{4}y = \left(m - \frac{2\sqrt{3}-\sqrt{6}}{4}\right)\lambda$$

Note that the slopes are the same as 1a, but the maxima are shifted.

(c) In the yz-plane, $x = 0$

$$\left.\begin{aligned} \phi_1 &= \tfrac{2\pi}{\lambda}\left(\tfrac{\sqrt{3}}{2}z\right) \\ \phi_2 &= \tfrac{2\pi}{\lambda}\left(\tfrac{\sqrt{2}}{4}y + \tfrac{\sqrt{6}}{4}z\right) \end{aligned}\right\} \Delta\phi = \frac{2\pi}{\lambda}\left[-\frac{\sqrt{2}}{4}y + \left(\frac{\sqrt{3}}{2} - \frac{\sqrt{6}}{4}\right)z\right]$$

We also observe a set of fringes along the lines where $\Delta\phi = 2m\pi$, i.e. $-\frac{\sqrt{2}}{4}y + \frac{2\sqrt{3}-\sqrt{6}}{4}z = m\lambda, m \in \mathbb{Z}$

2.

$$\text{Plane wave: } E_{\text{pl}} = |E_{\text{pl}}|e^{i\frac{2\pi}{\lambda}z}$$
$$\text{Spherical wave: } E_{\text{sp}} = \frac{|E_{\text{sp}}|}{\alpha z}e^{i\frac{2\pi}{\lambda}z}e^{i\pi\frac{(x^2+y^2)}{\lambda z}}$$

(a) At $z = 1000\lambda$, assuming the amplitudes of the two waves are equal:

$$E_{\text{pl}} = e^{i\phi_{\text{pl}}} \text{ where } \phi_{\text{pl}} = \frac{2\pi}{\lambda}z$$
$$E_{\text{sp}} = e^{i\phi_{\text{sp}}} \text{ where } \phi_{\text{sp}} = \frac{2\pi}{\lambda}z + \frac{\pi}{\lambda z}(x^2+y^2)$$
$$\Delta\phi = \phi_{\text{sp}} - \phi_{\text{pl}} = \frac{\pi}{\lambda z}(x^2+y^2)$$

We have bright fringes where $\Delta\phi = 2\pi m$, $m = 0, 1, 2\ldots$, so $\frac{x^2+y^2}{2z} = m\lambda$.
At $z = 1000\lambda \rightarrow x^2+y^2 = 2000\lambda^2 m$, $m = 0, 1, 2, 3\ldots$, which is a set of concentric rings of radii $R = \lambda\sqrt{2000m}$, $m = 0, 1, 2, 3\ldots$

(b) At $z = 2000\lambda$, the amplitude of the spherical wave decreases by a factor of 1/2 (energy conservation).

$$\left.\begin{aligned} E_{\text{pl}} &= e^{i\phi_{\text{pl}}} \\ E_{\text{sp}} &= \tfrac{1}{2}e^{i\phi_{\text{sp}}} \end{aligned}\right\} I = 1 + \left(\frac{1}{2}\right)^2 + 2(1)\left(\frac{1}{2}\right)\cos\Delta\phi = \frac{5}{4} + \cos\Delta\phi$$

The maxima are given by $\Delta\phi = 2\pi m$.

$$\frac{x^2+y^2}{2z} = m\lambda \Rightarrow x^2+y^2 = 4000\lambda^2 m$$

Therefore the maxima are concentric circles of radii $R = 20\lambda\sqrt{10m}$, $m = 0, 1, 2, \ldots$

(c) Observations:

i. The interference pattern is a set of concentric circles whose radii are given by

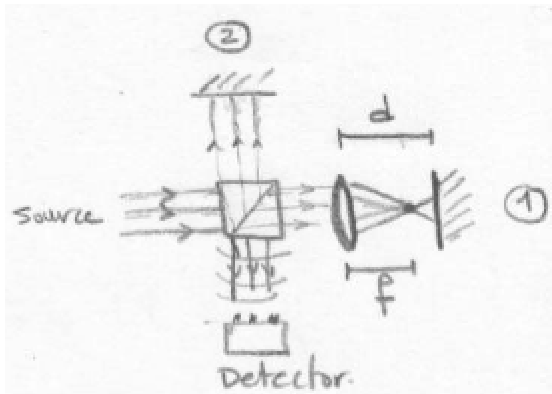
$$R_m = \sqrt{2z\lambda m}$$

ii. The radius of the first fringe (R_1) increases with both λ and z

iii. At a certain distance z, the spacing between the fringes decreases as we go radially outwards.

$$\Delta R_m = \sqrt{2z\lambda}(\sqrt{m} - \sqrt{m-1})$$

(d) If we insert a lens in branch 1 of a Michelson interferometer, the lens focuses the plane wave to a point at its back focal plane.



After reflecting off the mirror, the lens is effectively imaging a point source at a distance $(d - f) + d = 2d - f$; thus it forms a point source image at S_i, where

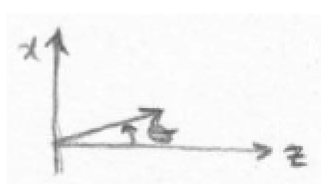
$$\frac{1}{S_i} = \frac{1}{f} - \frac{1}{S_0} = \frac{1}{f} - \frac{1}{2d-f} = \frac{2(d-f)}{f(2d-f)}$$

$$S_i = \frac{f(2d-f)}{2(d-f)}$$

If $d = f$, i.e. $S_i = \infty$, we get a plane wave back and the output is a uniform intensity, because we would be observing the interference of two on-axis plane waves.

If $d \neq f$, we get circular fringes due to the interference of a plane wave and a spherical wave.

3. The general off-axis plane wave propagates at θ with respect to the z axis.



The off-axis plane wave equation is:

$$E_{\text{pl}} = |E_{\text{pl}}| e^{i\frac{2\pi}{\lambda}(x\sin\theta + z\cos\theta)}$$

The equation of the spherical wave is:

$$E_{\text{sp}} = \frac{|E_{\text{sp}}|}{\alpha z} e^{i\frac{2\pi}{\lambda}z} e^{i\frac{\pi}{\lambda z}(x^2+y^2)}$$

(a) Assuming the amplitudes are equal at $z = 1000\lambda$, $I = 2|E_{\text{pl}}|^2(1+\cos\Delta\phi)$, where $\Delta\phi = \phi_{\text{sp}} - \phi_{\text{pl}}$:

$$\left.\begin{array}{l}\phi_{\text{sp}} = \frac{2\pi}{\lambda}z + \frac{\pi}{\lambda z}(x^2+y^2) \\ \phi_{\text{pl}} = \frac{2\pi}{\lambda}(x\sin\theta + z\cos\theta)\end{array}\right\} \Delta\phi = \frac{\pi}{\lambda z}x^2 - \frac{2\pi}{\lambda}\sin\theta x + \frac{\pi}{\lambda z}y^2 + \frac{2\pi}{\lambda}z(1-\cos\theta)$$

Bright fringes occur when $\Delta\phi = 2\pi m$:

$$\frac{1}{2z}x^2 - (\sin\theta)x + \frac{1}{2z}y^2 = m\lambda - z(1-\cos\theta)$$

$$x^2 - 2z\sin\theta x + z^2\sin^2\theta + y^2 = \underbrace{2z[m\lambda - z(1-\cos\theta)] + z^2\sin^2\theta}_{{R_m}^2}\ldots \quad \text{(Eq. A)}$$

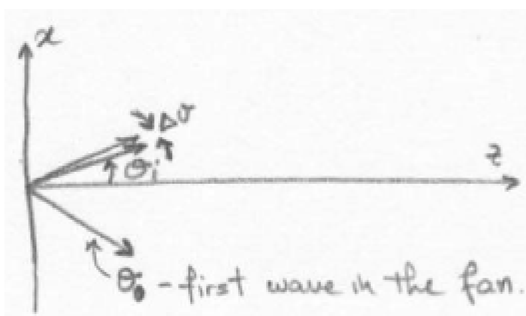
$$(x - z\sin\theta)^2 + y^2 = {R_m}^2$$

(b) At $z = 2000\lambda$, $|E_{\text{sp}}| = \frac{1}{2}|E_{\text{pl}}|$

$$I = |E_{\text{pl}}|^2\left(\frac{5}{4} + \cos\Delta\phi\right)$$

The fringes are still given by equation A where $z = 2000\lambda$. This gives a bigger shift along the x-axis and lower contrast in the fringes as well as larger spacing of peaks.

4. We sketch the system as follows:



The m^{th} plane wave is at an angle $\theta_m = \theta_0 + m\Delta\theta$

$$E_m = e^{i\frac{2\pi}{\lambda}[\cos\theta_m z + \sin\theta_m x]}$$

Assuming small angles (paraxial approximation), $\theta_m \ll 1$

$$\cos\theta_m \approx 1, \sin\theta_m \approx \theta_m = \theta_0 + m\Delta\theta$$

$$E_m \approx e^{i\frac{2\pi}{\lambda}(z+\theta_m x)} = e^{i\frac{2\pi}{\lambda}(z+\theta_0 x + m\Delta\theta x)}$$

Adding all the plane waves,

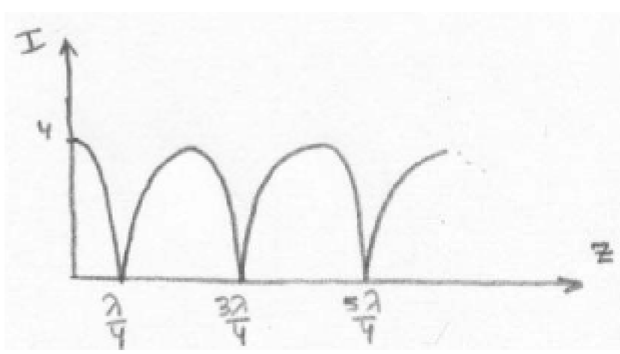
$$\begin{aligned}
E_T &= \sum_{m=0}^{N-1} E_m \\
&= \sum_{m=0}^{N-1} e^{i\frac{2\pi}{\lambda}(z+\theta_0 x+m\Delta\theta x)} \\
&= e^{i\frac{2\pi}{\lambda}z}e^{i\frac{2\pi}{\lambda}\theta_0 x}\underbrace{\sum_{m=0}^{N-1}(e^{i\frac{2\pi}{\lambda}x\Delta\theta})^m}_{\text{Geometric series: } \theta_0=1,\ r=e^{i\frac{2\pi}{\lambda}x\Delta\theta}}
\end{aligned}$$

$$\begin{aligned}
\therefore E_T &= e^{i\frac{2\pi}{\lambda}z}e^{i\frac{2\pi}{\lambda}\theta_0 x}\cdot\frac{1-e^{i(\frac{2\pi}{\lambda}Nx\Delta\theta)}}{1-e^{i(\frac{2\pi}{\lambda}x\Delta\theta)}} = e^{i\frac{2\pi}{\lambda}z}e^{i\frac{2\pi}{\lambda}\theta_0 x}\cdot\frac{1-e^{i\phi_1}}{1-e^{i\phi_2}} \\
&= e^{i\frac{2\pi}{\lambda}z}e^{i\frac{2\pi}{\lambda}\theta_0 x}\cdot\frac{e^{i\frac{\phi_1}{2}}}{e^{i\frac{\phi_2}{2}}}\cdot\frac{e^{-i\frac{\phi_1}{2}}-e^{i\frac{\phi_1}{2}}}{e^{-i\frac{\phi_2}{2}}-e^{i\frac{\phi_2}{2}}} \\
|E_T|^2 &= \left|\frac{e^{-i\frac{\phi_1}{2}}-e^{i\frac{\phi_1}{2}}}{e^{-i\frac{\phi_2}{2}}-e^{i\frac{\phi_2}{2}}}\right|^2 = \left(\frac{2i\sin(\frac{\phi_1}{2})}{2i\sin(\frac{\phi_2}{2})}\right)^2 = \frac{\sin^2(\frac{\phi_1}{2})}{\sin^2(\frac{\phi_2}{2})}
\end{aligned}$$

5. Forward propagating plane wave: $\vec{k}_1 = \frac{2\pi}{\lambda}\hat{z}$

   Backward propagating plane wave: $\vec{k}_2 = -\frac{2\pi}{\lambda}\hat{z}$

$$\begin{aligned}
E_1 &= e^{i(\frac{2\pi}{\lambda}z-\omega t)}, \quad E_2 = e^{i(-\frac{2\pi}{\lambda}z-\omega t)} \\
I &= 2(1+\cos\Delta\phi),\ \text{where } \Delta\phi = \phi_1-\phi_2 = \frac{4\pi}{\lambda}z \\
\therefore I &= 2\left[1+\cos\left(\frac{4\pi}{\lambda}z\right)\right] = 4\cos^2\left(\frac{2\pi}{\lambda}z\right)
\end{aligned}$$



Note that although we did not ignore the time dependence of each wave (ωt), the interference wave is independent of time, thus the term "standing wave."

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