MASSACHUSETTS INSTITUTE OF TECHNOLOGY

2.71 Optics **Spring '09**

Solutions to Problem Set #6 Due Wednesday, Apr. 15, 2009

Problem 1: Grating with tilted plane wave illumination

1. a) In this problem, one–dimensional geometry along the x–axis is considered. The Fresnel diffraction pattern, the field just behind the grating illuminated by the plane wave, is

$$g_{+}(x, z=0) = g_t(x) g_{-}(x, z=0) = \exp\left\{i\frac{m}{2}\sin\left(2\pi\frac{x}{\Lambda}\right)\right\}\exp\left\{i\frac{2\pi}{\lambda}\theta x\right\}. \tag{1}$$

Note that the transmission function can be expanded as

$$g_t(x) = \exp\left\{i\frac{m}{2}\sin\left(2\pi\frac{x}{\Lambda}\right)\right\} = \sum_{q=-\infty}^{\infty} J_q\left(\frac{m}{2}\right)\exp\left\{iq\frac{2\pi}{\Lambda}x\right\}. \tag{2}$$

Using eq. (2), we can rewrite eq. (1) as

$$g_{+}(x, z=0) = \sum_{q=-\infty}^{\infty} J_q\left(\frac{m}{2}\right)\exp\left\{i\frac{2\pi}{\lambda}\left(\theta + \frac{q\lambda}{\Lambda}\right)x\right\}. \tag{3}$$

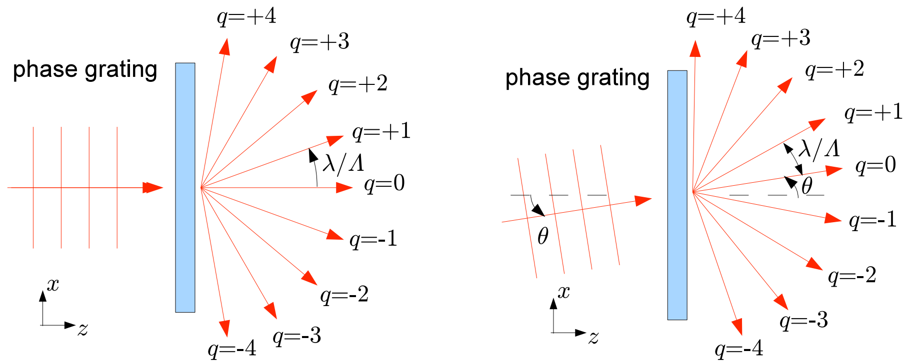
Since $\exp\left\{i\frac{2\pi}{\lambda}\left(\theta+\frac{q\lambda}{\Lambda}\right)x\right\}$ represents a tilted plane wave whose propagation angle is $\theta + q\lambda/\Lambda$, eq. (3) implies that the transmitted field just behind the grating is consisted of a infinite number of plane waves, where q denotes diffraction order and the amplitude of the diffraction order q is $J_q(m/2)$. The propagation direction of the zero–order is identical as one of the incident tilted plane wave.

1.b) The field behind the grating is identical to eq. (1). When the observation plane is in the far–zone, the Fraunhofer diffraction pattern is

$$g(x', z) = \int g_{+}(x, z=0)\exp\left\{-i\frac{2\pi}{\lambda z}(x'x)\right\}\mathrm{d}x. \tag{4}$$

Note that we neglected the scaling factor and phase term because the scaling factor change overall magnitude of diffraction pattern and the phase term does not contribute to intensity. Substituting eq. (2) into (4), we obtain the field distribution of the Fraunhofer diffraction as

$$\begin{aligned} g(x', z) &= \int \left[\sum_{q=-\infty}^{\infty} J_q\left(\frac{m}{2}\right)\exp\left\{i\frac{2\pi}{\lambda}\left(\theta+\frac{q\lambda}{\Lambda}\right)\right\}\right]\exp\left\{-i\frac{2\pi}{\lambda z}(xx')\right\}\mathrm{d}x \\ &= \sum_{q=-\infty}^{\infty} J_q\left(\frac{m}{2}\right)\left[\int \exp\left\{i2\pi\left(\frac{q}{\Lambda}+\frac{\theta}{\lambda}\right)x\right\}\exp\left\{-i2\pi\frac{x'}{\lambda z}x\right\}\mathrm{d}x\right] \\ &= \sum_{q=-\infty}^{\infty} J_q\left(\frac{m}{2}\right)\delta\left(\frac{x'}{\lambda z} - \left[\frac{q}{\Lambda}+\frac{\theta}{\lambda}\right]\right). \end{aligned} \tag{5}$$



(a) on–axis plane wave (b) tilted plane wave

Figure 1: The whole diffraction patterns rotate by θ as the incident plane wave rotates

The intensity of the Fraunhofer diffraction pattern is

$$I(x', z) = |g(x', z)|^2 = \sum_{q=-\infty}^{\infty} J_q^2\left(\frac{m}{2}\right)\delta\left(\frac{x'}{\lambda z} - \left[\frac{q}{\Lambda} + \frac{\theta}{\lambda}\right]\right). \tag{6}$$

In the far–region, we should observe a infinite number of diffraction orders. The intensity of the diffraction order is proportional to $J_q^2(m/2)$ and the offset between two neighboring diffraction orders is $(\lambda z)/\Lambda$. The zeroth order is located at $x' = z\theta$.

1.c) In both cases (Fresnel and Fraunhofer diffraction), the diffraction patterns of the grating probed by a on-axis and tilted plane waves are identical except the angular shift by the incident angle θ, as shown in Fig. 1.

Problem 2: Grating spherical wave illumination

2.a) Using the same approach as in Prob. 1, we obtain

$$g_+(x,y,z=0) = g_t(x,y)g_-(x,y,z=0) = \frac{1}{2}\left[1+m\cos\left(2\pi\frac{x}{\Lambda}\right)\right]\frac{e^{i2\pi\frac{z_0}{\lambda}}}{i\lambda z_0}\exp\left\{i\pi\frac{x^2+y^2}{\lambda z_0}\right\}. \tag{7}$$

2.b) Since both the cosine term and exponential terms in eq. (8) vary with x, we use following relation to understand eq. (8);

$$1+m\cos\left(2\pi\frac{x}{\Lambda}\right) = 1+\frac{m}{2}\left(\exp\left\{i2\pi\frac{x}{\Lambda}\right\}+\exp\left\{-i2\pi\frac{x}{\Lambda}\right\}\right). \tag{8}$$

Hence, eq. (8) can be rewritten as superposition of three spherical waves;

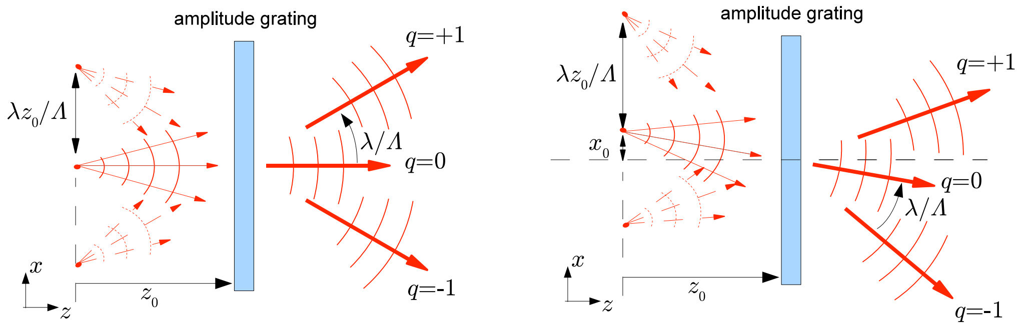
$$\begin{aligned} g_+(x,y,z=0) = {} & \frac{e^{i2\pi\frac{z_0}{\lambda}}}{i\lambda z_0}\left[\frac{1}{2}\exp\left\{i\pi\frac{x^2+y^2}{\lambda z_0}\right\}+\right. \\ & \left.\frac{m}{4}\exp\left\{i\pi\frac{x^2+y^2}{\lambda z_0}+i2\pi\frac{x}{\Lambda}\right\}+\frac{m}{4}\exp\left\{i\pi\frac{x^2+y^2}{\lambda z_0}-i2\pi\frac{x}{\Lambda}\right\}\right] \\ = {} & \frac{e^{i2\pi\frac{z_0}{\lambda}}}{i\lambda z_0}\left[\frac{1}{2}\exp\left\{i\pi\frac{x^2+y^2}{\lambda z_0}\right\}+\frac{m}{4}\exp\left\{i\pi\frac{(x+\lambda z_0/\Lambda)^2+y^2}{\lambda z_0}\right\}\exp\left\{-i\pi\frac{\lambda z_0}{\Lambda^2}\right\}+\right. \\ & \left.\frac{m}{4}\exp\left\{i\pi\frac{(x-\lambda z_0/\Lambda)^2+y^2}{\lambda z_0}\right\}\exp\left\{-i\pi\frac{\lambda z_0}{\Lambda^2}\right\}\right]. \end{aligned} \tag{9}$$

2.c) Figure 2(a) conceptually shows the diffraction pattern expressed in eq. (10). The first exponential term represents the zero–order diffraction, which is identical to the incident spherical wave originated at $(x=0, y=0, z=-z_0)$ except amplitude attenuation. The second and third exponential terms indicate two spherical waves originated at $(\pm\lambda z_0/\Lambda, 0, -z_0)$ with additional phase factor of $e^{-i\pi\lambda z_0/\Lambda^2}$, which is independent on x and y.

2.d) If the illumination is a spherical wave emitted at $(x_0, 0, -z_0)$ as shown in Fig. 2(b), then we expect that the origins of the three spherical waves will be shifted by x_0; i.e., the three origins are $(x_0, 0, -z_0)$, $(x_0-\lambda z_0/\Lambda, 0, -z_0)$, and $(x_0+\lambda z_0/\Lambda, 0, -z_0)$ if the paraxial approximation holds.

More rigorously, the Fresnel diffraction pattern is computed as

$$g_+(x,z=0) = \frac{1}{2}\left[1+m\cos\left(2\pi\frac{x}{\Lambda}\right)\right]\frac{e^{i2\pi\frac{z_0}{\lambda}}}{i\lambda z_0}\exp\left\{i\pi\frac{(x-x_0)^2+y^2}{\lambda z_0}\right\}, \tag{10}$$



(a) on–axis spherical wave (b) off–axis spherical wave

Figure 2: The diffraction patterns rotate in the same fashion as the incident spherical wave rotates

and using the same expansion we eventually obtain

$$
\begin{aligned}
g_+(x, y, z = 0) = \frac{e^{i2\pi\frac{z_0}{\lambda}}}{i\lambda z_0} \Bigg[\frac{1}{2} \exp\left\{ i\pi \frac{(x - x_0)^2 + y^2}{\lambda z_0} \right\} + \\
\frac{m}{4} \exp\left\{ i\pi \frac{(x - x_0)^2 + y^2}{\lambda z_0} + i2\pi \frac{x}{\Lambda} \right\} + \frac{m}{4} \exp\left\{ i\pi \frac{(x - x_0)^2 + y^2}{\lambda z_0} - i2\pi \frac{x}{\Lambda} \right\} \Bigg] \\
= \frac{e^{i2\pi\frac{z_0}{\lambda}}}{i\lambda z_0} \Bigg[\frac{1}{2} \exp\left\{ i\pi \frac{(x - x_0)^2 + y^2}{\lambda z_0} \right\} + \\
\frac{m}{4} \exp\left\{ i\pi \frac{(x - x_0 + \lambda z_0/\Lambda)^2 + y^2}{\lambda z_0} \right\} \exp\left\{ -i\pi \left(-\frac{2x_0}{\Lambda} + \frac{\lambda z_0}{\Lambda^2} \right) \right\} + \\
\frac{m}{4} \exp\left\{ i\pi \frac{(x - x_0 - \lambda z_0/\Lambda)^2 + y^2}{\lambda z_0} \right\} \exp\left\{ -i\pi \left(\frac{2x_0}{\Lambda} + \frac{\lambda z_0}{\Lambda^2} \right) \right\} \Bigg]. \quad (11)
\end{aligned}
$$

As expected, the diffraction pattern is consisted of three spherical waves originated at $(x_0, 0, -z_0)$, $(x_0 - \lambda z_0/\Lambda, 0, -z_0)$, and $(x_0 + \lambda z_0/\Lambda, 0, -z_0)$, respectively.

3. (a) The Fourier series coefficients of a periodic function $t(x)$ are given by:

$$a_0 = \frac{1}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} t(x')dx'$$

$$a_n = \frac{2}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} t(x')\cos\left(\frac{n\pi x'}{L/2}\right)dx'$$

$$b_n = \frac{2}{L}\int_{-L}^{L} t(x')\sin\left(\frac{n\pi x'}{L/2}\right)dx'$$

where L is the period of $t(x)$. The function $t(x)$ can then be written as an infinite sum:

$$t(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L/2}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L/2}\right)$$

For the given function,

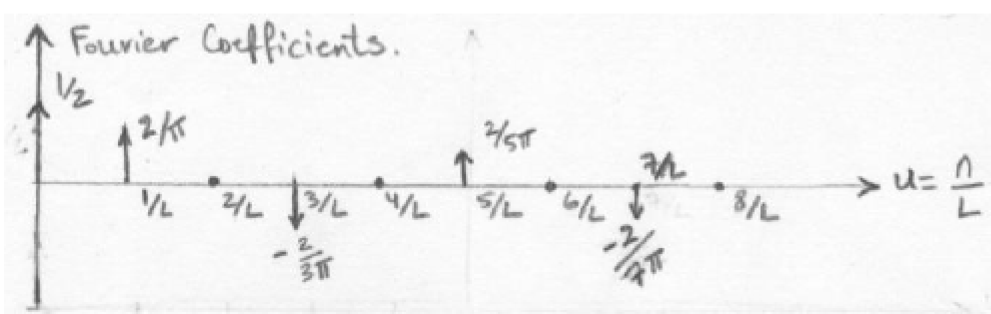
$$a_0 = \frac{1}{L}\int_{-\frac{L}{4}}^{\frac{L}{4}} dx' = \frac{1}{2}$$

$$a_n = \frac{2}{L}\int_{-\frac{L}{4}}^{\frac{L}{4}} \cos\left(\frac{2\pi n x'}{L}\right)dx' = \frac{\cancel{2}}{\cancel{L}}\cdot\frac{\cancel{L}}{\cancel{2}\pi n}\sin\left(\frac{2\pi n x'}{L}\right)\Big|_{-\frac{L}{4}}^{\frac{L}{4}} = \frac{2}{\pi n}\sin\left(\frac{\pi n}{2}\right)$$

$$b_n = \frac{2}{L}\int_{-\frac{L}{4}}^{\frac{L}{4}} \sin\left(\frac{2\pi n x'}{L}\right)dx' = \frac{\cancel{2}}{\cancel{L}}\cdot\frac{\cancel{L}}{\cancel{2}\pi n}\underbrace{\left[-\cos\left(\frac{2\pi n x'}{L}\right)\Big|_{-\frac{L}{4}}^{\frac{L}{4}}\right]}_{0} = 0$$

$$\therefore a_0 = \frac{1}{2}, \quad b_n = 0, \quad a_n = \frac{\sin\left(\frac{\pi n}{2}\right)}{\frac{\pi n}{2}} \text{ where } n = 1, 2, 3, \ldots$$

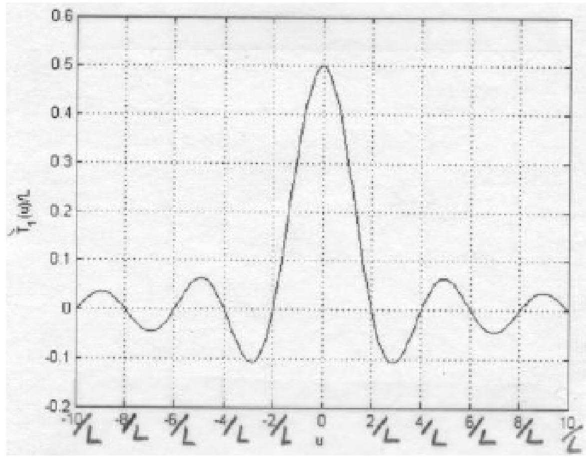
Note that when n is even, $a_n = 0$, when $n = 1 + 4m, a_n = +1$, and when $n = 3 + 4m, a_n = -1$, where m is a positive integer.



(b) A single boxcar is given by

$$t_1(x) = \text{rect}\left(\frac{2x}{L}\right)$$

$$T_1(u) = \mathcal{F}(t_1(x)) = \frac{L}{2}\text{sinc}\left(\frac{L}{2}u\right)$$



(c) An infinite array of boxcars of width $\frac{L}{2}$ with a spacing of $\frac{L}{2}$ between them can be expressed as a convolution of a comb() function and a rect() function:

$$t_2(x) = \text{rect}\left(\frac{2x}{L}\right) \otimes \text{comb}\left(\frac{x}{L}\right)$$

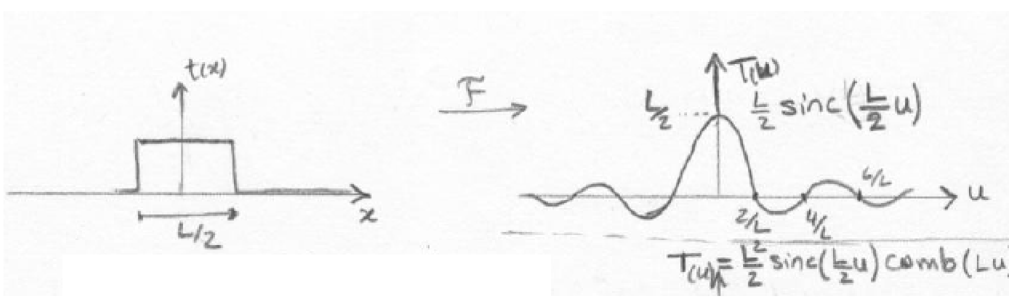
A truncated centered portion containing N boxcars is then given by

$$t(x) = t_2(x) \cdot \text{rect}\left(\frac{x}{NL}\right) = \left[\text{rect}\left(\frac{2x}{L}\right) \otimes \text{comb}\left(\frac{x}{L}\right)\right] \cdot \text{rect}\left(\frac{x}{NL}\right)$$

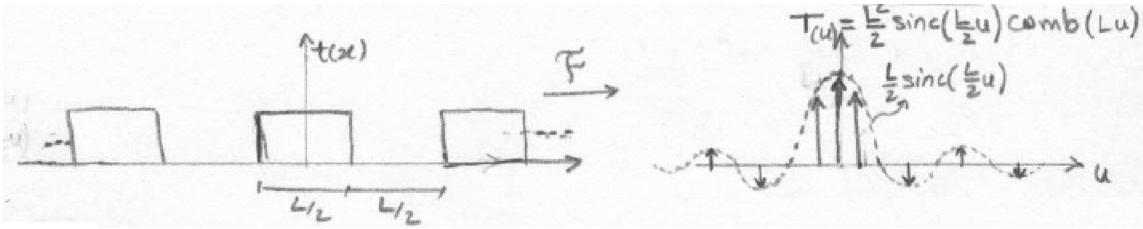
The Fourier transform of $t(x)$ becomes

$$T(u) = T_2(u) \otimes (\text{NL})\text{sinc}(\text{NL}u) = \left[\frac{L^2}{2}\text{sinc}\left(\frac{L}{2}u\right) \cdot \text{comb}(Lu)\right] \otimes (\text{NL})\text{sinc}(\text{NL}u)$$

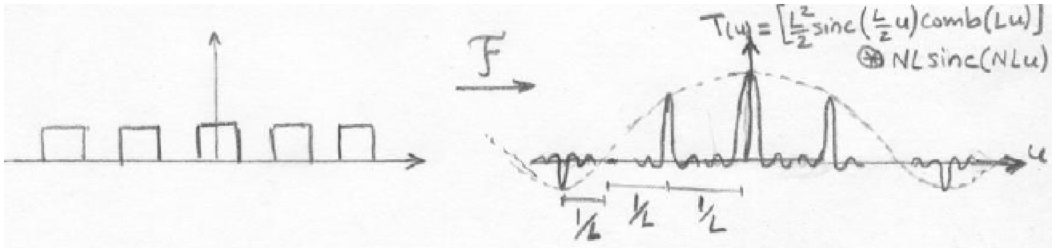
(d) Single box car: $T(u) = \frac{L}{2}\text{sinc}(\frac{L}{2}u)$



Infinite array:

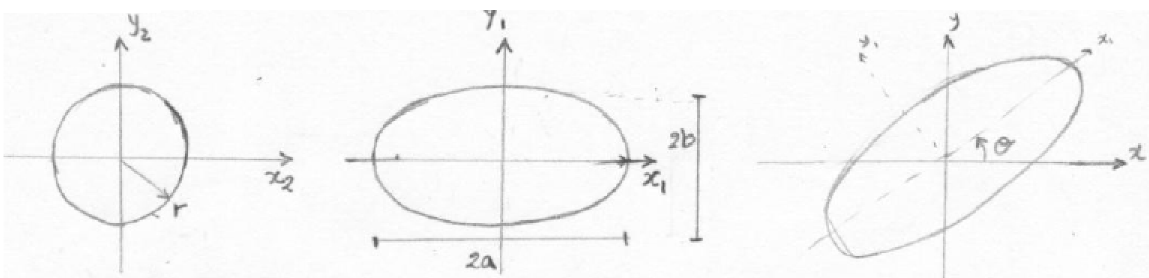


Finite array (N):



- The Fraunhofer diffraction pattern is similar to the Fourier transform of the functions (with a scaling factor $u = x/\lambda L$)
- A single box car creates a sinc() diffraction pattern. Having an infinitely long array would generate a set of $\delta()$ functions, i.e. single dots whose amplitude is modulated by a sinc() envelope profile identical to that generated by one boxcar. The spacing of the $\delta()$'s is the reciprocal of the period of the array.
- A finite array of boxcars generates a set of sinc() functions whose peaks are modulated by another sinc() function and whose spacing is the reciprocal of the period of the boxcar array. Limiting the size of the array is equivalent to imposing a window onto an infinite array. This window spreads the $\delta()$ functions into sinc() functions. The spread of each of these sinc()'s is inversely proportional to the width of the 'window.'

4. Tilted ellipse:



$$\text{Circle: } f_2(x_2, y_2) = \text{circ}\left(\frac{\sqrt{x^2+y^2}}{r}\right)$$

$$\text{Ellipse: } x_1 = \frac{a}{r}x_2;\, y_1 = \frac{b}{r}y_2$$

$$x_2 = \frac{r}{a}x_1;\, y_2 = \frac{r}{b}y_1$$

$$\text{Tilted: } x = x_1\cos\theta - y_1\sin\theta$$

$$y = x_1\sin\theta + y_1\cos\theta$$

$$f_1(x_1, y_2) = \text{circ}\left(\frac{\sqrt{(\frac{r}{a}x_1)^2 + (\frac{r}{b}y_1)^2}}{r}\right) = \text{circ}\left(\sqrt{\left(\frac{x_1}{a}\right)^2 + \left(\frac{y_1}{b}\right)^2}\right)$$

$$F_1(u_1, v_1) = \mathcal{F}(f_1) = |ab|\text{jinc}(2\pi\sqrt{(au_1)^2 + (bv_1)^2})$$

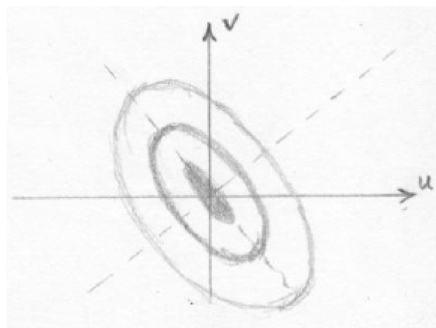
A rotation by θ in the space domain is equivalent to a rotation by θ in the frequency domain; hence,

$$u_1 = u\cos\theta + v\sin\theta, \quad v_1 = -u\sin\theta + v\cos\theta$$

$\therefore$ The Fourier transform of an ellipse tilted by an angle θ is

$$F(u, v) = |ab|\text{jinc}(2\pi\sqrt{a^2(u\cos\theta + v\sin\theta)^2 + b^2(-u\sin\theta + v\cos\theta)^2})$$

(a) Sketch of Fourier transform



(b) The Fraunhofer diffraction pattern is given by

$$F(u, v)\Big|_{(\frac{x'}{\lambda\ell}, \frac{y'}{\lambda\ell})} = |ab|\text{jinc}\left(2\pi\sqrt{a^2(\frac{x'}{\lambda\ell}\cos\theta + \frac{y'}{\lambda\ell}\sin\theta)^2 + b^2(-\frac{x'}{\lambda\ell}\sin\theta + \frac{y'}{\lambda\ell}\cos\theta)^2}\right)$$

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