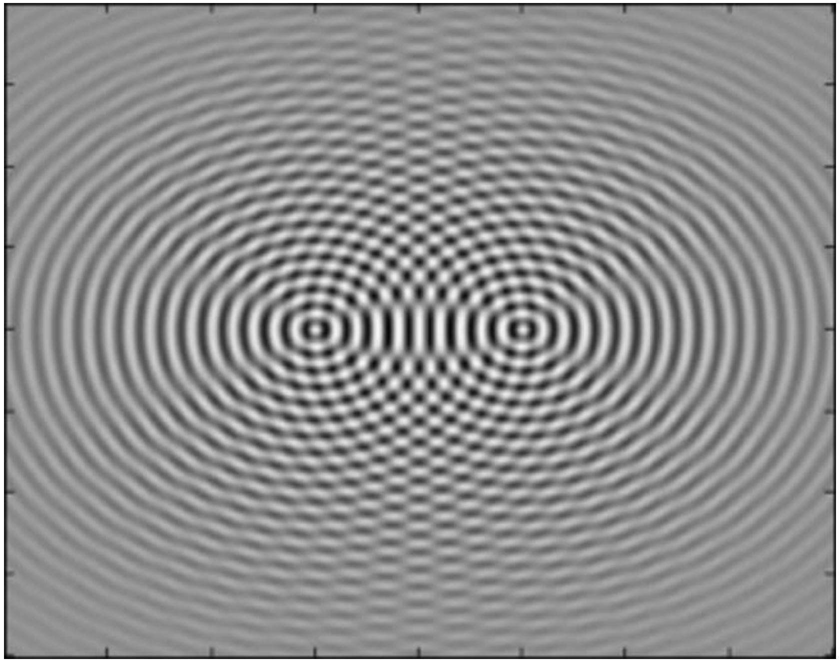
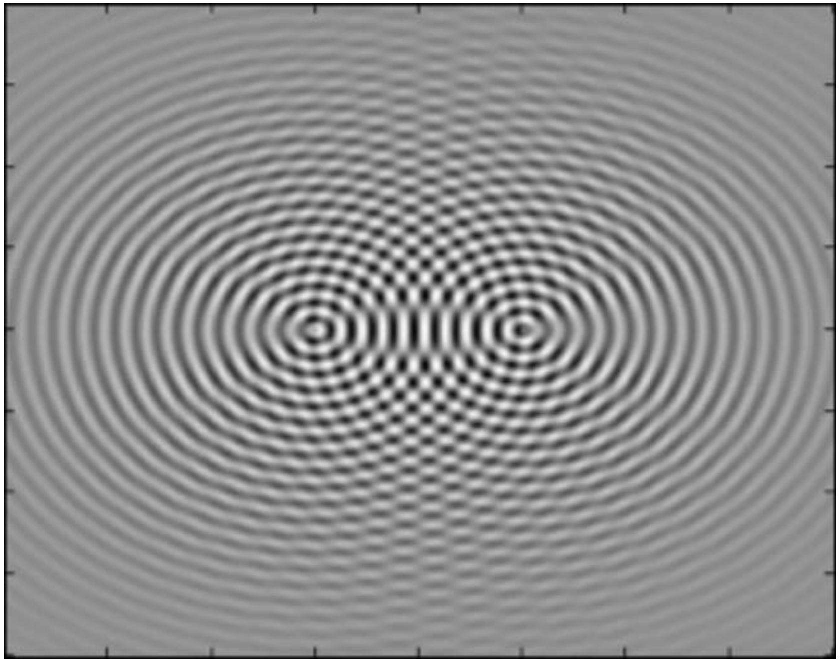
Phase delays and interference



in phase



out of phase

The 1D wave equation

Consider a moving disturbance with envelope ψ of the form

$$f(z,t) = \psi\left(z \pm ct\right).$$

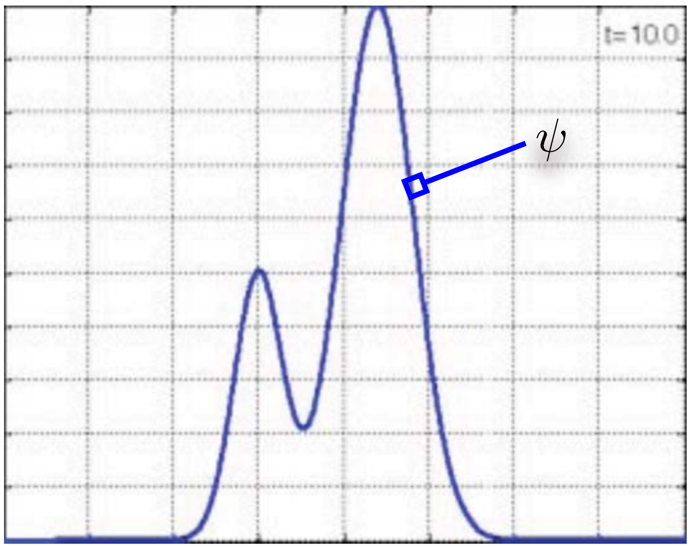
The "−" sign denotes a wave moving to the right (forward,) whereas "+" denotes a wave moving to the left (backwards.) Denoting by ψ' the envelope's derivative with respect to its argument, and taking the first partial derivatives with respect to z and t,

$$\frac{\partial f}{\partial z} = \psi' \qquad \frac{\partial f}{\partial t} = \pm c\psi' \Rightarrow \frac{\partial f}{\partial z} \mp \frac{1}{c}\frac{\partial f}{\partial t} = 0.$$

Taking second derivatives,

$$\frac{\partial^2 f}{\partial z^2} = \psi'' \qquad \frac{\partial^2 f}{\partial t^2} = c^2\psi'' \Rightarrow \frac{\partial^2 f}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 f}{\partial t^2} = 0.$$

This is referred to as the **Helmholtz Wave Equation**, or simply the **Wave Equation.**



We can solve the wave equation for specific envelopes with proper initial and boundary conditions, e.g.

$$f(z, t=0) = \psi\left(z\right) = a_0 \cos\left(kz\right),$$

where k is a shorthand for $2\pi/\lambda$ and λ is the spatial period (wavelength.) We can easily see that

$$f(z,t) = a_0 \cos\left(k(z \pm ct)\right)$$

satisfy the Wave Equation and the initial condition, so they are solutions propagating forward and backward, respectively.

The solution is often written in equivalent forms, as

$$f(z,t) = a_0 \cos\left(k(z \pm ct)\right) = a_0 \cos\left(kz \pm \omega t\right) = a_0 \cos\left(\frac{2\pi}{\lambda}(z \pm ct)\right).$$

These are consistent given the definitions

$$\omega = 2\pi\nu, \qquad k = \frac{2\pi}{\lambda}, \qquad \text{and the dispersion relation} \quad c = \lambda\nu.$$

Linear superposition of waves

$$\frac{\partial^2 f}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2 f}{\partial t^2}.$$

This is a *linear* partial differential equation. We can see easily that if f_1 is a solution and f_2 is another solution, then $a_1 f_1 + a_2 f_2$ is also a solution for arbitrary constants a_1, a_2.

For example, $a_-\psi\left(z-ct\right) + a_+\psi\left(z+ct\right)$,

a superposition of a forward and backward propagating waves, is also a solution to the Wave Equation. It is known as a **standing wave.**

Let us derive the standing wave for the case of forward and backward waves of equal amplitudes. The superposition is

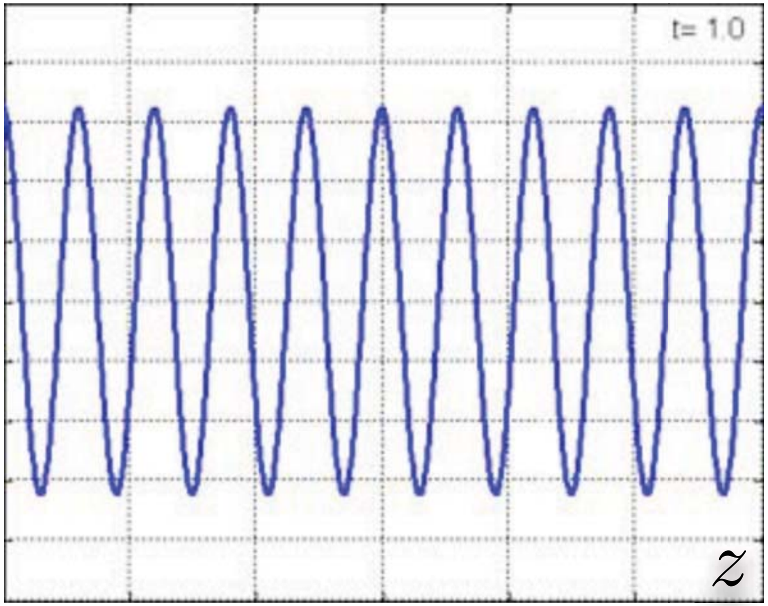
$$f(z,t) = a\cos\left(kz - \omega t + \phi\right) + a\cos\left(kz + \omega t + \phi\right).$$

At this point we must recall the trigonometric identity

$$\cos A + \cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}.$$

Using the identity, we rewrite the superposition as

$$\begin{aligned} f(z,t) &= 2a\cos\frac{\left(kz-\omega t+\phi\right)+\left(kz+\omega t+\phi\right)}{2}\times \\ &\quad\times\cos\frac{\left(kz-\omega t+\phi\right)-\left(kz+\omega t+\phi\right)}{2} \\ &= 2a\cos\left(kz+\phi\right)\cos\left(\omega t\right). \end{aligned}$$



$$f(z,t) = 2a\cos\left(kz+\phi\right)\cos\left(\omega t\right)$$

is a spatial sinusoid of period $\lambda = 2\pi/k$, phase delay ϕ and is also oscillating with temporal period $T = 2\pi/\omega$. It is *stationary*, that is non–propagating. Hence the term **standing wave.**

Complex (phasor) representation of waves

Consider two simple wave forms with amplitude a, wave number k, angular frequency ω, and phase delay ϕ:

$$f_{\mathrm{c}}(z,t) = a\cos\left(kz - \omega t + \phi\right), \qquad f_{\mathrm{s}}(z,t) = a\sin\left(kz - \omega t + \phi\right).$$

These are both solutions to the Wave Equation (but with different initial conditions.) Hence, their superposition is also a solution to the Wave Equation. We form the following special superposition

$$\begin{aligned} \hat{f}(z,t) &= f_{\mathrm{c}}(z,t) + i f_{\mathrm{s}}(z,t) \\ &= a\exp\left\{i(kz - \omega t + \phi)\right\}. \end{aligned}$$

This is the **complex representation** of the wave. The complex exponential provides immense mathematical convenience, as we will see, but it is important to remember that *only its real part has physical significance.* Further acknowledging that in linear media the temporal frequency ω of the wave does not change, it is common to drop the $\mathrm{e}^{-i\omega t}$ term, resulting in

$$a\exp\left\{i(kz + \phi)\right\}.$$

This reduced complex representation is sometimes referred to as **phasor.**

Solution to the standing wave using phasors

Recall the superposition of forward and backward waves resulting in the standing wave,

$$f(z,t) = a\cos\left(kz - \omega t + \phi\right) + a\cos\left(kz + \omega t + \phi\right).$$

Some care needs to be taken in deriving the phasor of the backward wave. We need to get rid of a term of the form $\mathrm{e}^{-i\omega t}$ term, so we write

$$a\cos\left(kz + \omega t + \phi\right) = a\cos\left(-kz - \omega t - \phi\right) = \mathrm{Re}\left\{\mathrm{e}^{-i(kz+\omega t+\phi)}\right\}$$

Therefore, the proper phasor for the backward propagating wave is

$$a\exp\left\{-ikz - \phi\right\}.$$

The superposition is now written as

$$a\mathrm{e}^{i(kz+\phi)} + a\mathrm{e}^{-i(kz+\phi)} = 2a\cos\left(kz + \phi\right).$$

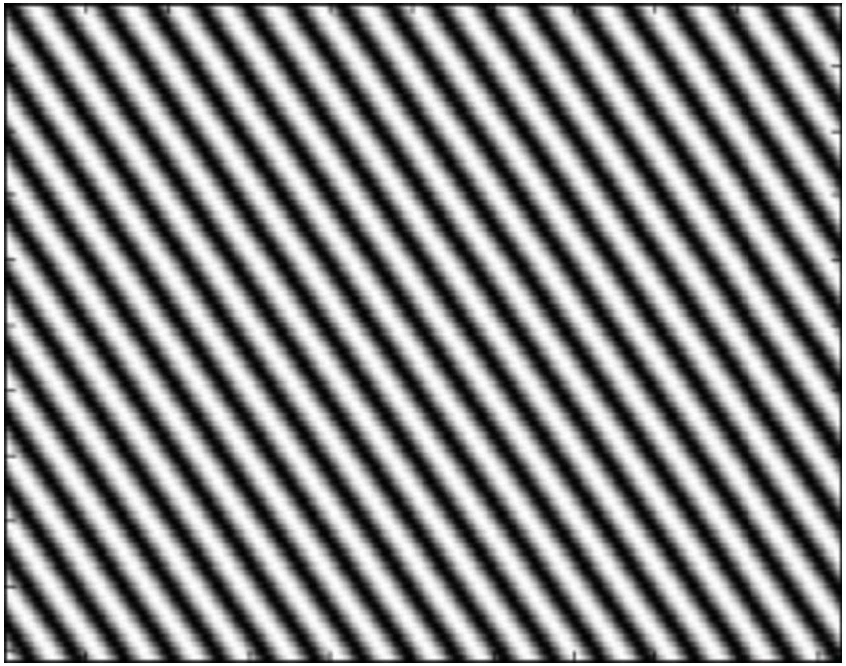
This becomes identical to our earlier result on the standing wave after we put back the $\cos(\omega t)$ term that is implicit in the phasor notation; however, the derivation was relatively painless, without need for the trigonometric formulae.

The 3D wave equation

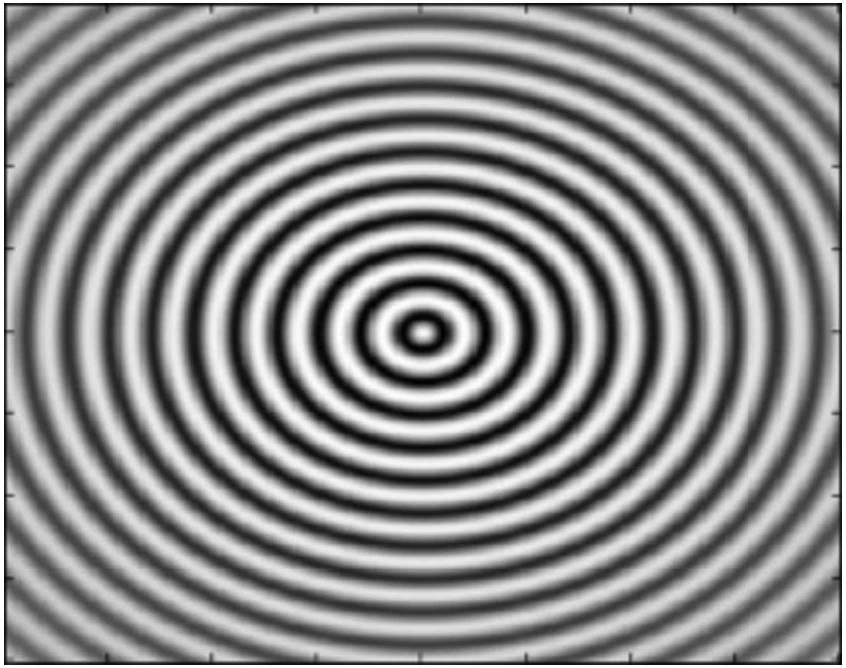
In three-dimensions, the Wave Equation is generalized as

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 f}{\partial t^2} = 0.$$

Our familiar plane and spherical waves are special solutions.

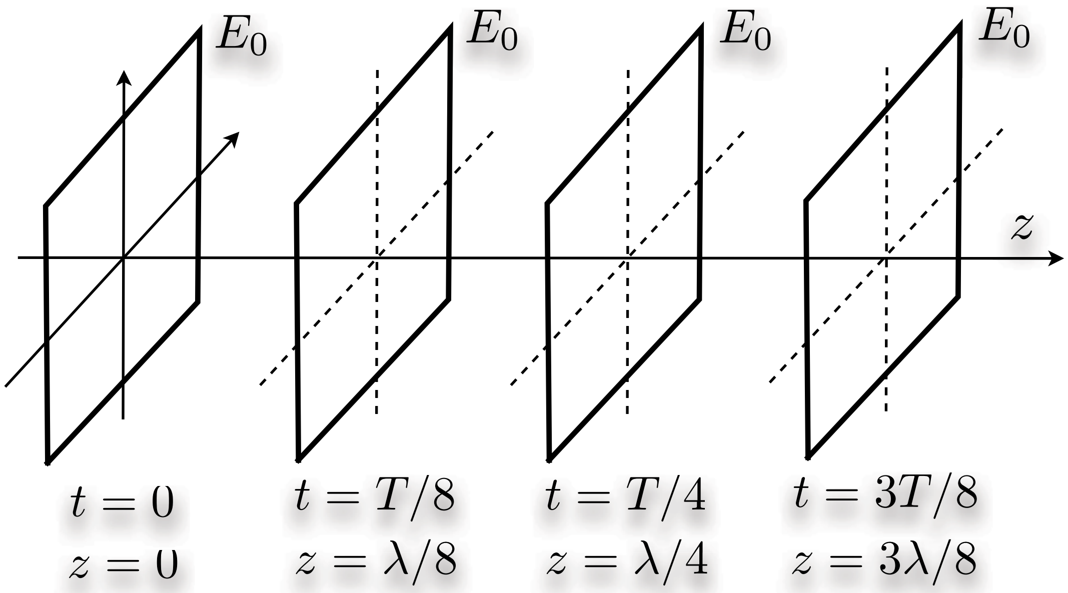


Plane wave



Spherical wave

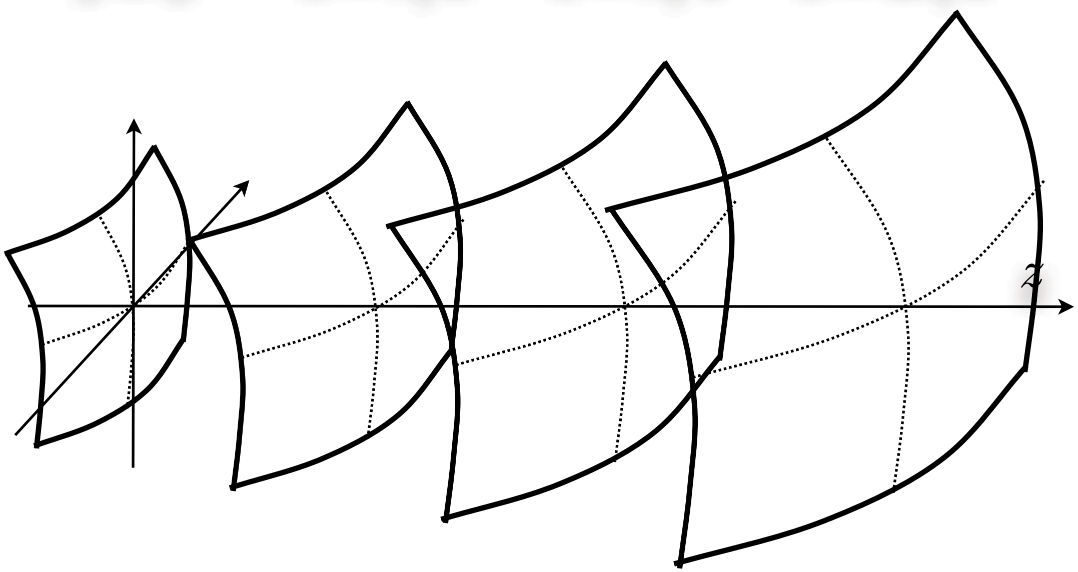
Planar and Spherical Wavefronts



Planar wavefront (plane wave):

The wave phase is constant along a planar surface (the wavefront).

As time evolves, the wavefronts propagate at the wave speed without changing; we say that the wavefronts are *invariant to propagation* in this case.

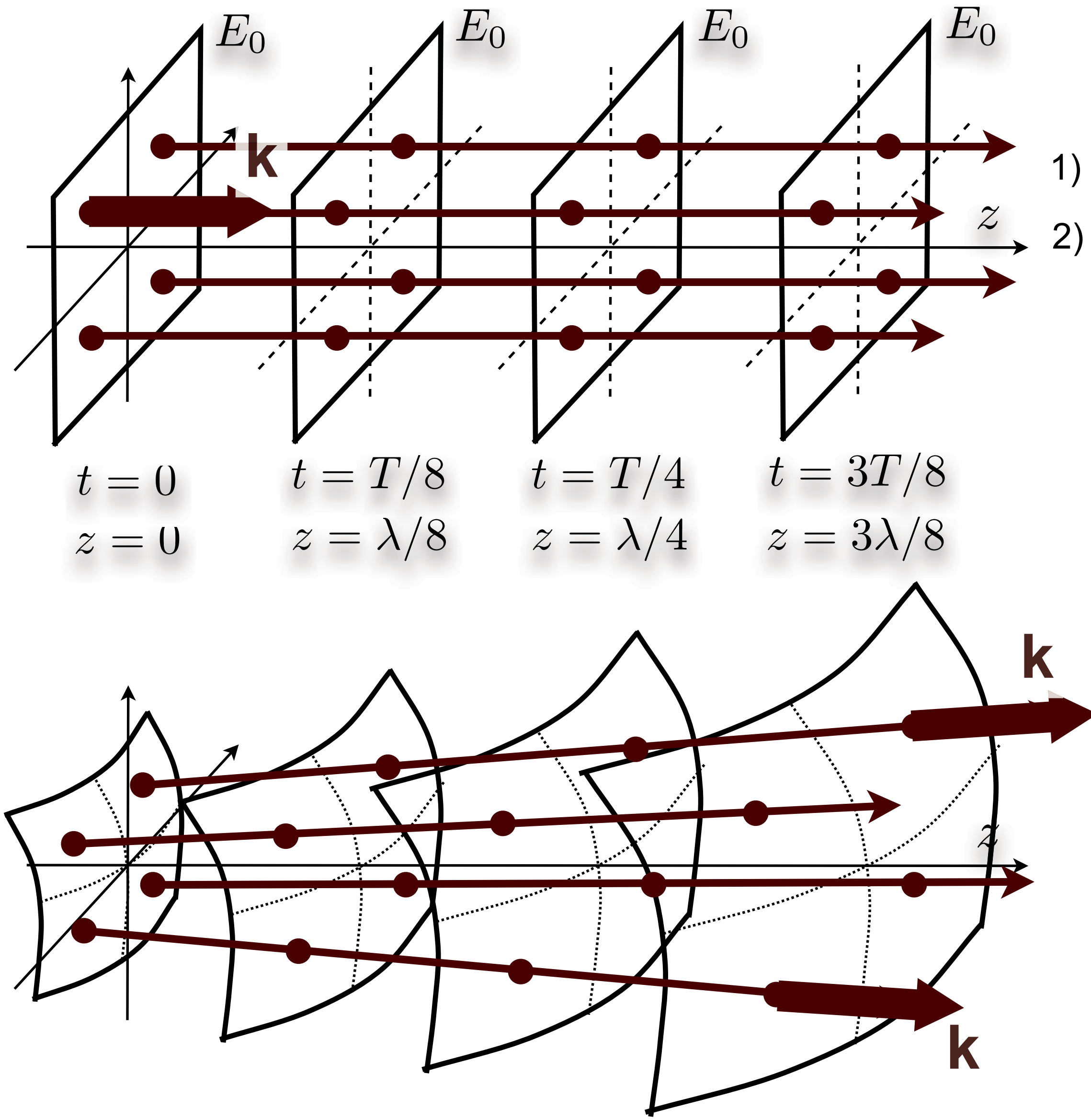


Spherical wavefront (spherical wave):

The wave phase is constant along a spherical surface (the wavefront).

As time evolves, the wavefronts propagate at the wave speed and expand outwards while preserving the wave's energy.

Wavefronts, rays, and wave vectors



Rays are:

1) normals to the wavefront surfaces

2) trajectories of "particles of light"

Wave vectors:

At each point on the wavefront, we may assign a normal vector **k**

This is known as the wave vector;
it magnitude *k* is the wave number and it is defined as

$$k \equiv |\mathbf{k}| = \frac{2\pi}{\lambda}$$

2.71 / 2.710 Optics
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