Maxwell's Equations (free space)

Integral form

$$\oiint_A \mathbf{E} \cdot \mathrm{d}\mathbf{a} = \frac{1}{\epsilon_0} \iiint_V \rho \mathrm{d}V$$

$$\oiint_A \mathbf{B} \cdot \mathrm{d}\mathbf{a} = 0$$

$$\oint_C \mathbf{E} \cdot \mathrm{d}l = -\frac{\mathrm{d}}{\mathrm{d}t} \iint_A \mathbf{B} \cdot \mathrm{d}\mathbf{a}$$

$$\oint_C \mathbf{B} \cdot \mathrm{d}l = \mu_0 \left(\iint_A \mathbf{J} \cdot \mathrm{d}\mathbf{a} + \epsilon_0 \iint_A \frac{\partial \mathbf{E}}{\partial t} \cdot \mathrm{d}\mathbf{a} \right)$$

Differential form

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{B} = \mu_0 \left(\mathbf{J} + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right)$$

Wave Equation for electromagnetic waves

We will derive the Wave Equation
from Maxwell's electromagnetic equations
in free space and in the absence of charges and currents.
Starting from Faraday's equation,

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \Rightarrow \nabla \times \left(\nabla \times \mathbf{E}\right) = -\frac{\partial \left(\nabla \times \mathbf{B}\right)}{\partial t}$$

Now we substitute Ampère–Maxwell's Law

$$\nabla \times \mathbf{B} = \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t},$$

the following identity from vector calculus

$$\nabla \times \left(\nabla \times \mathbf{E}\right) = \nabla \left(\nabla \cdot \mathbf{E}\right) - \nabla^2 \mathbf{E},$$

and Gauss' Law for electric fields,

$$\nabla \cdot \mathbf{E} = 0.$$

Collecting all these results, we obtain

$$\nabla^2 \mathbf{E} - \mu_o \epsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0.$$

Comparing with the 3D Wave Equation,

$$\nabla^2 f - \frac{1}{c^2} \frac{\partial^2 f}{\partial t^2} = 0,$$

we see that *each component* of the vector $\mathbf{E}$ satisfies the Wave Equation with velocity

$$\frac{1}{c^2} = \mu_o \epsilon_0 \Rightarrow c = \frac{1}{\sqrt{\mu_o \epsilon_0}}$$

Since $\epsilon_0 = 8.8542 \times 10^{-12} \mathrm{Cb}^2/\mathrm{N} \cdot \mathrm{m}^2$,
$\mu_0 = 4\pi \times 10^{-7} \mathrm{N} \cdot \mathrm{sec}^2/\mathrm{Cb}^2$, we obtain
the speed of electromagnetic waves in vacuum

$$c = 3 \times 10^8 \, \frac{\mathrm{m}}{\mathrm{sec}}.$$

Electric fields in matter

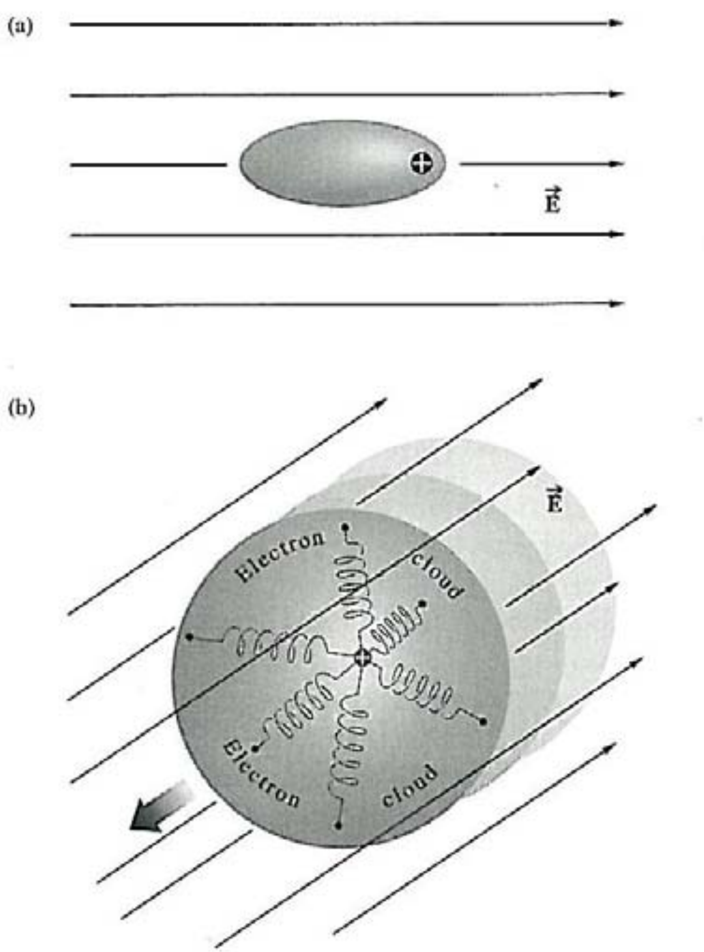


Figure 3.38 (a) Distortion of the electron cloud in response to an applied $\vec{E}$-field. (b) The mechanical oscillator model for an isotropic medium—all the springs are the same, and the oscillator can vibrate equally in all directions.

atom under electric field:
- charge neutrality is preserved
- spatial distribution of charges becomes assymetric

$$\mathbf{p} = q_+\mathbf{r}_+ - q_-\mathbf{r}$$
Dipole moment

$$\mathbf{P} = \sum \mathbf{p}$$
Polarization

Spatially variant polarization induces *local* charge imbalances (bound charges)

$$\rho_{\text{bound}} = -\nabla\cdot\mathbf{P}$$

$$\nabla\cdot\mathbf{E} = \frac{\rho_{\text{total}}}{\epsilon_0} = \frac{\rho_{\text{free}}+\rho_{\text{bound}}}{\epsilon_0}$$
$$= \frac{\rho_{\text{free}}-\nabla\cdot\mathbf{P}}{\epsilon_0} \Rightarrow$$
$$\nabla\cdot\left(\epsilon_0\mathbf{E}+\mathbf{P}\right) = \rho_{\text{free}}$$

Constitutive relationships

E: electric field

D: electric displacement

$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P}$$

polarization

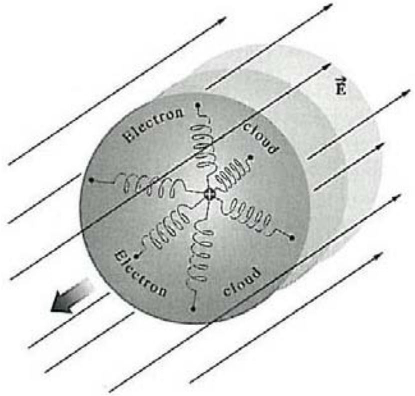
Simplest case: *Linear* polarizability

$$\mathbf{P} = \chi \mathbf{E} \Rightarrow \mathbf{D} = \epsilon_0 (1+\chi)\mathbf{E} \equiv \epsilon \mathbf{E} \equiv n^2 \epsilon_0 \mathbf{E}$$

where χ is the *linear dielectric susceptibility*,

$\epsilon \equiv \epsilon_0(1+\chi)$ is the *dielectric permittivity*,

and $n = \sqrt{1+\chi}$ is the *index of refraction.*



"Spring" binding the electron to the nucleus is linear: *i.e.*, electron displacement as function of electric force is given by Hooke's law with spring constant

$$k = \frac{q^2}{\chi}$$

B: magnetic induction

H: magnetic field

$$\mathbf{H} = \frac{\mathbf{B}}{\mu_0} - \mathbf{M}$$

magnetization

Most optical materials are non magnetic at optical frequencies, i.e. $\mathbf{M}=0$; therefore, $\mathbf{B}=\mu_0\mathbf{H}$

Maxwell's equations in constitutive form

Vacuum

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{B} = \mu_0 \left(\mathbf{J} + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right)$$

Matter with free charges and currents

$$\nabla \cdot \mathbf{D} = \rho_{\text{free}}$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{H} = \mathbf{J}_{\text{free}} + \frac{\partial \mathbf{D}}{\partial t}$$

Matter without free charges or currents

$$\nabla \cdot \mathbf{D} = 0$$

$$\nabla \cdot \mathbf{B} = 0$$

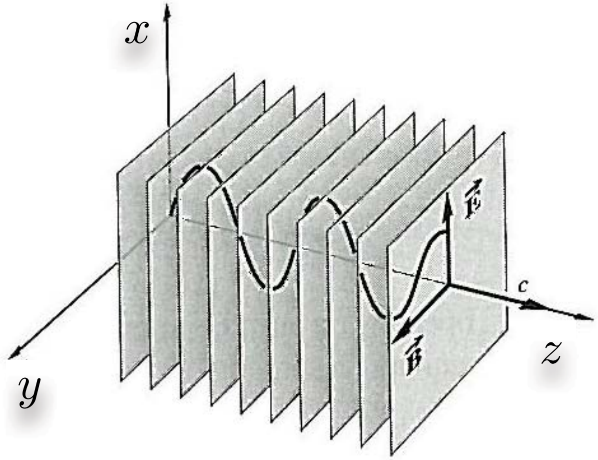
$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t}$$

Wave equation in matter but without free charges or currents becomes:

$$\nabla^2 \mathbf{E} - \mu_o \epsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0 \Rightarrow c = \frac{1}{\sqrt{\epsilon \mu_0}} = \frac{1}{n} \times \frac{1}{\sqrt{\epsilon_0 \mu_0}} = \frac{c_{\text{free}}}{n}$$

k, E, B form a right-handed triad



$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad \text{where} \quad \mathbf{E} = \hat{\mathbf{x}} E_0 \, e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}$$

$$\Rightarrow \nabla \times \equiv i\mathbf{k} \times \quad \text{and} \quad \frac{\partial}{\partial t} \equiv -i\omega$$

$$\Rightarrow \mathbf{B} = \frac{1}{\omega} \mathbf{k} \times \mathbf{E}$$

In free space or isotropic media, the vectors **k**, **E**, **B** form a right-handed triad.

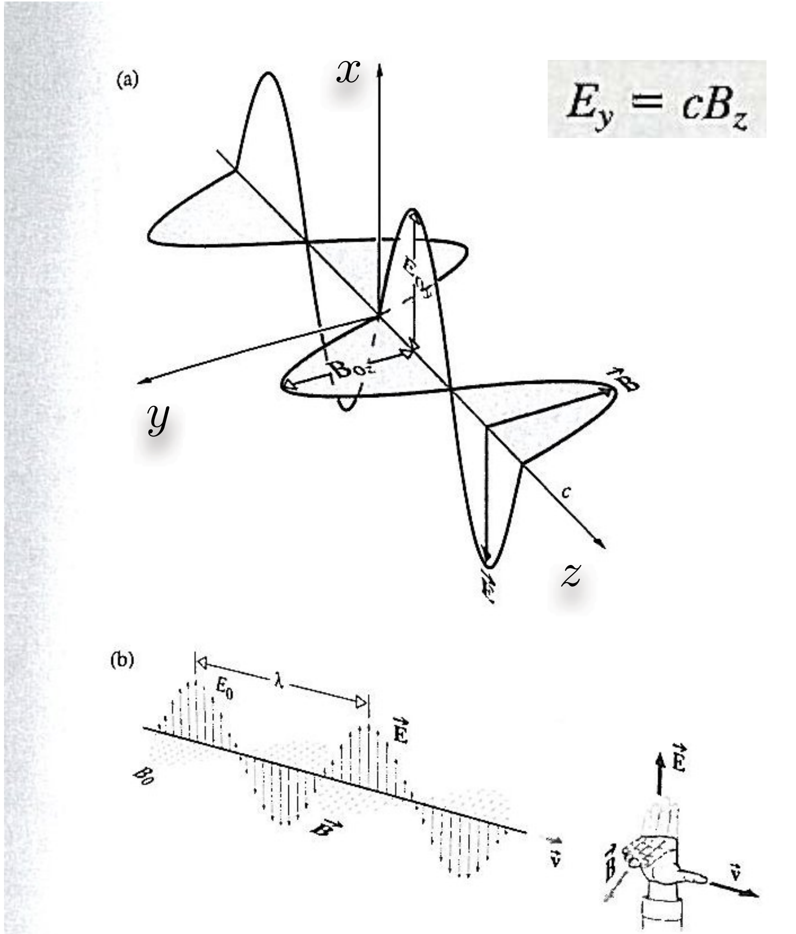
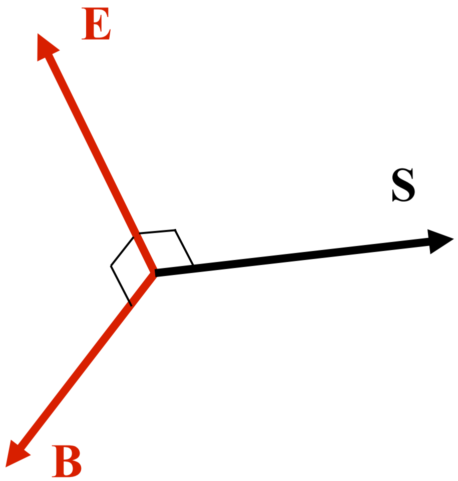


Figure 3.14 (*a*) Orthogonal harmonic $\vec{E}$- and $\vec{B}$-fields for a plane polarized wave. (*b*) The wave propagates in the direction of $\vec{E} \times \vec{B}$.

The Poynting vector



$$\vec{\mathbf{S}} = \frac{1}{\mu_0} \vec{\mathbf{E}} \times \vec{\mathbf{B}} \tag{3.39}$$

$$\Leftrightarrow \quad \vec{\mathbf{S}} = c^2 \epsilon_0 \, \vec{\mathbf{E}} \times \vec{\mathbf{B}} \tag{3.40}$$

In the case of harmonic waves,

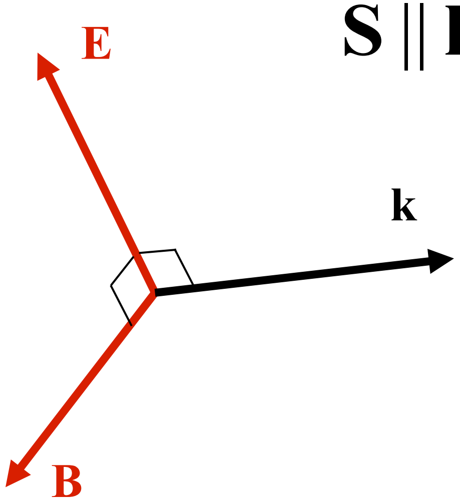
$$\vec{\mathbf{E}} = \vec{\mathbf{E}}_0 \cos(\vec{\mathbf{k}} \cdot \vec{\mathbf{r}} - \omega t) \tag{3.41}$$

$$\vec{\mathbf{B}} = \vec{\mathbf{B}}_0 \cos(\vec{\mathbf{k}} \cdot \vec{\mathbf{r}} - \omega t) \tag{3.42}$$

$$\Rightarrow$$

$$\vec{\mathbf{S}} = c^2 \epsilon_0 \vec{\mathbf{E}}_0 \times \vec{\mathbf{B}}_0 \cos^2(\vec{\mathbf{k}} \cdot \vec{\mathbf{r}} - \omega t) \tag{3.43}$$

$\mathbf{S} \parallel \mathbf{k}$



Irradiance (*aka* Intensity)

From the definition of the Poynting vector and the **k**, **E**, **B** relationship,

$$\mathbf{S} = c^2\varepsilon_0 \mathbf{E}\times\mathbf{B}$$

$$\mathbf{B} = \frac{\mathbf{k}}{\omega}\times\mathbf{E} \Rightarrow \|\mathbf{B}\| = \frac{k}{\omega}\|\mathbf{E}\| = \frac{1}{c}\|\mathbf{E}\| \Rightarrow \|\mathbf{S}\| = c\varepsilon_0\|\mathbf{E}\|^2$$

For a harmonic (sinusoidal) wave propagating along z

$$\mathbf{E} = \hat{\mathbf{x}}E_0\cos(kz-\omega t) \Rightarrow \|\mathbf{S}\| = c\varepsilon_0 E_0^2\cos^2(kz-\omega t)$$

Recall at optical frequencies, the field oscillates very rapidly, $\omega \sim 10^{14} \rightarrow 10^{15}$ Hz. Therefore, opto-electronic detectors only register the *average* energy flux

$$\langle\|\mathbf{S}\|\rangle = \frac{1}{T}\int_t^{t+T}\|\mathbf{S}\|\mathrm{d}t$$

This referred to as *irradiance* or *intensity* of the optical field, and is measured in units of W/m^2 (more commonly W/cm^2)

(T=2π/ω or integral multiples)

Since $\langle\cos^2(kz-\omega t)\rangle = \int_t^{t+T}\cos^2(kz-\omega t)\mathrm{d}t = \frac{1}{2}$ for the harmonic wave the intensity is

$$I \equiv \langle S\rangle_T = \frac{c\epsilon_0}{2}E_0^2$$

Calculation of the intensity from phasors

Recall the phasor representation of a harmonic wave

$$f(z,t) = A\cos(kz - \omega t - \phi)$$
$$\hat{f}(z,t) = A\cos(kz - \omega t - \phi) + iA\sin(kz - \omega t - \phi)$$

complex amplitude or "phasor": $A\exp\{i(kz - \phi)\}$

We cannot quite represent the Poynting vector as a phasor, because the Poynting vector is obtained as a product of the E, B fields (recall we may only add or subtract phasors; we are not allowed to do nonlinear operations with them.)

Nevertheless, we can obtain the intensity (time-averaged Poynting vector) from the phasor as follows:

Consider the time-averaged superposition of two fields at the same frequency:

$$E_1(z,t) = E_{10}\cos(kz - \omega t)$$
$$E_2(z,t) = E_{20}\cos(kz - \omega t - \phi)$$
$$\langle\|\mathbf{S}\|\rangle = \frac{c\varepsilon_0}{T}\int_t^{t+T}(E_1 + E_2)^2\,\mathrm{d}t = \ldots = \frac{c\varepsilon_0}{2}\left(E_{10}^2 + E_{20}^2 + 2E_{10}E_{20}\cos\phi\right)$$

Clearly,

$$\langle\|\mathbf{S}\|\rangle = \frac{c\varepsilon_0}{2}|\text{phasor}|^2 \quad \text{or} \quad \langle\|\mathbf{S}\|\rangle \propto |\text{phasor}|^2$$

Now consider the corresponding phasors:

$$E_{10}$$
$$E_{20}\,\mathrm{e}^{-i\phi}$$

and form the quantity

$$\frac{c\varepsilon_0}{2}\left|E_{10} + E_{20}\,\mathrm{e}^{-i\phi}\right|^2 = \ldots = \frac{c\varepsilon_0}{2}\left(E_{10}^2 + E_{20}^2 + 2E_{10}E_{20}\cos\phi\right)$$

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