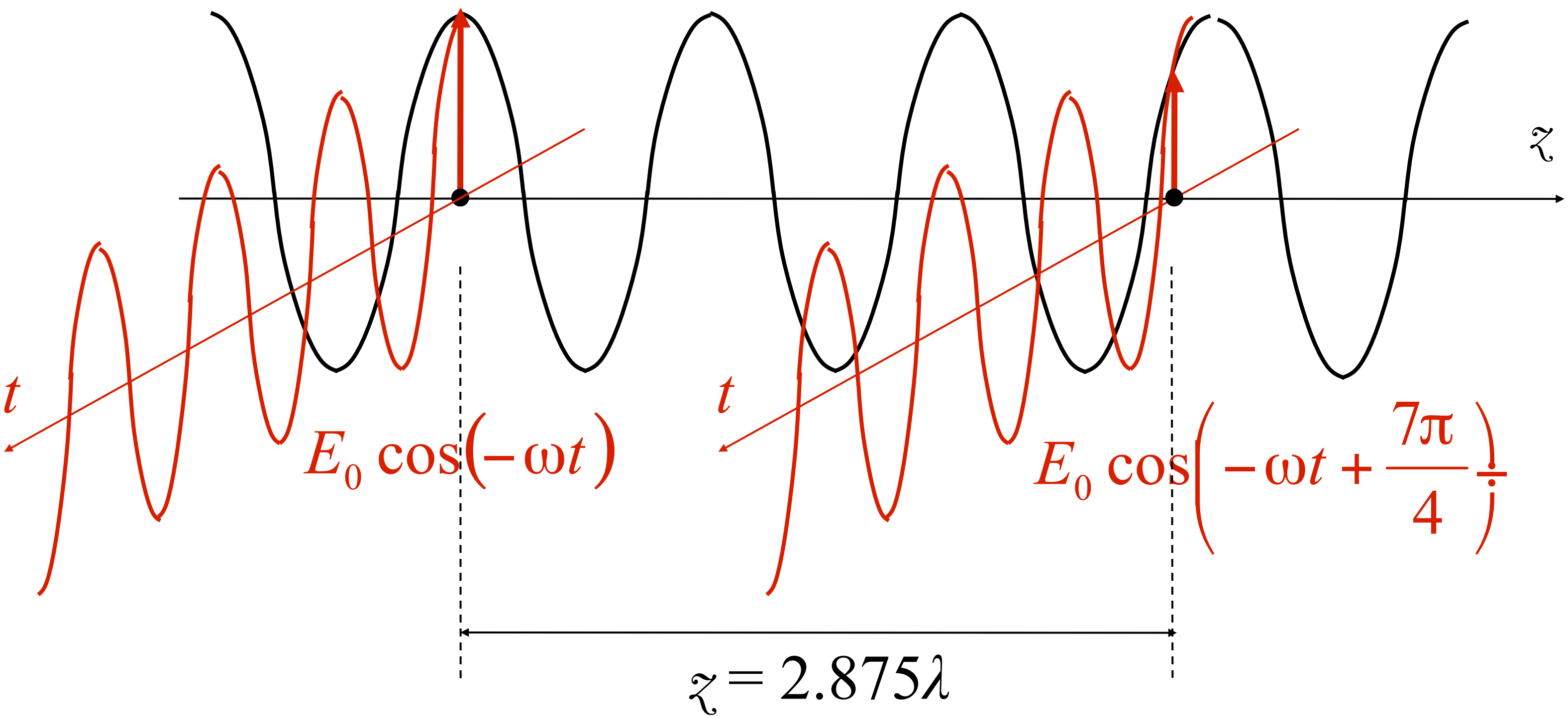
Today

- Interference
  - inadequacy of a single intensity measurement to determine the optical field
  - Michelson interferometer
    - measuring
      - distance
      - index of refraction
  - Mach-Zehnder interferometer
    - measuring
      - wavefront

A reminder: phase delay in wave propagation



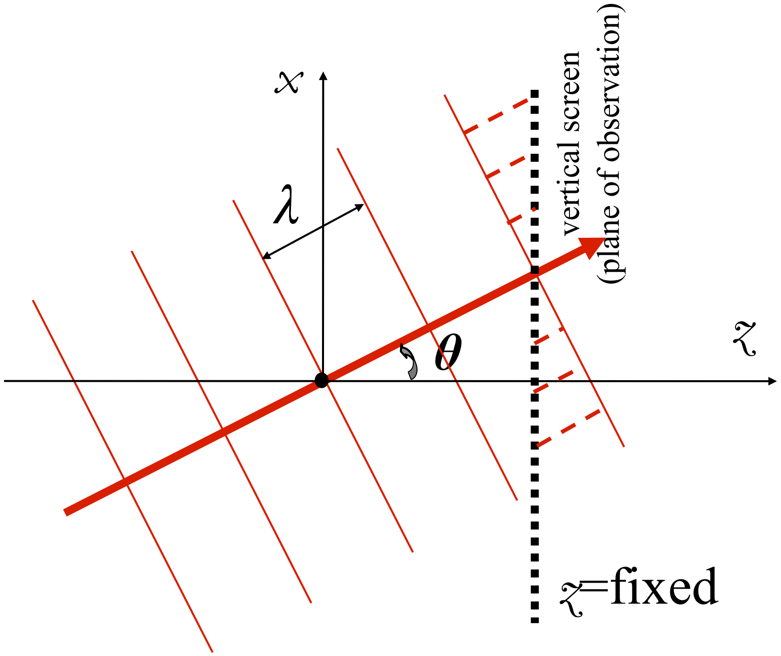
In general, $E_0 \cos(kz - \omega t) \Rightarrow E_0 \exp\left(i2\pi \frac{z}{\lambda}\right)$

phasor due to propagation (path delay)

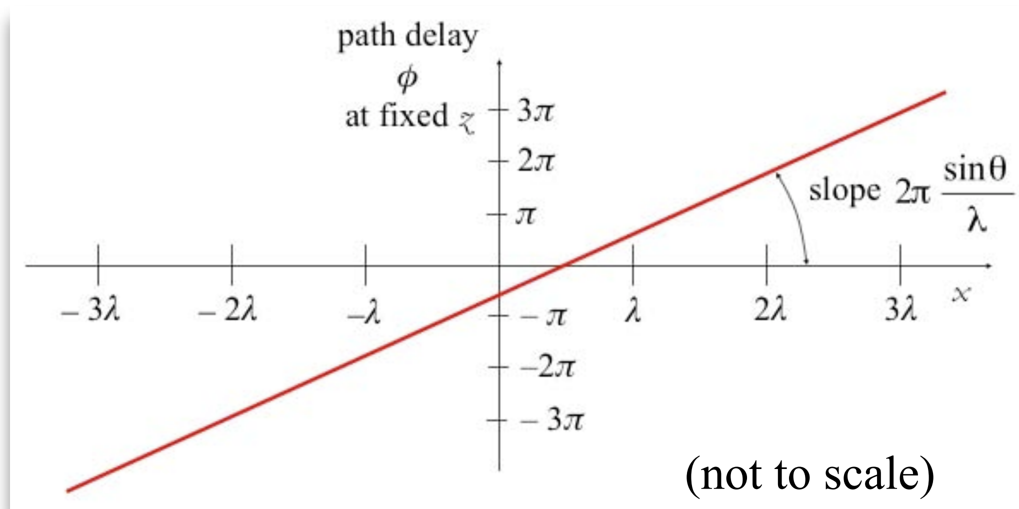
real representation

phasor representation

Phase delay from a plane wave propagating at angle θ towards a vertical screen



path delay increases **linearly** with x



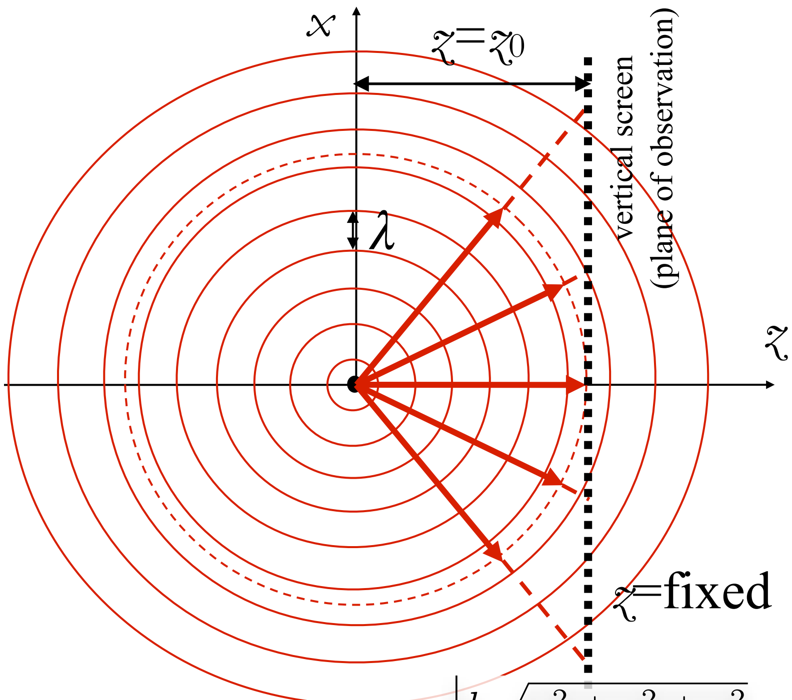
$$E(x,z,t) = E_0 \cos\left[k\left(x\sin\theta + z\cos\theta\right) - \omega t\right]$$

Phasor representation:

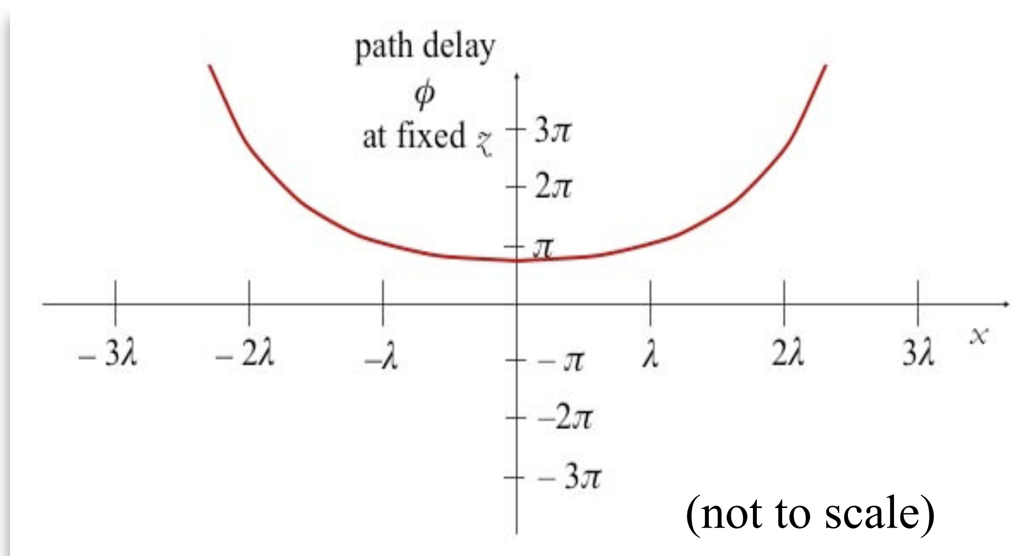
$$E(x,z) = E_0 \exp\left\{ik\left(x\sin\theta + z\cos\theta\right)\right\} = E_0 \exp\left\{i\frac{2\pi}{\lambda}\left(x\sin\theta + z\cos\theta\right)\right\}$$

may also be written as: $E_0\, \mathrm{e}^{i\phi}$, where $\phi(x) = \dfrac{2\pi}{\lambda} x\sin\theta + \text{const.}$

Phase delay from a spherical wave propagating from distance z_0 towards a vertical screen



path delay increases **quadratically** with x



$$E(x,y,z,t) = A_0 \frac{\cos\left[k\sqrt{x^2+y^2+z^2} - \omega t - \frac{\pi}{2}\right]}{\lambda\sqrt{x^2+y^2+z^2}} \approx \frac{A_0}{\lambda z}\cos\left[kz + k\frac{x^2+y^2}{2z} - \omega t - \frac{\pi}{2}\right]$$

Phasor representation:

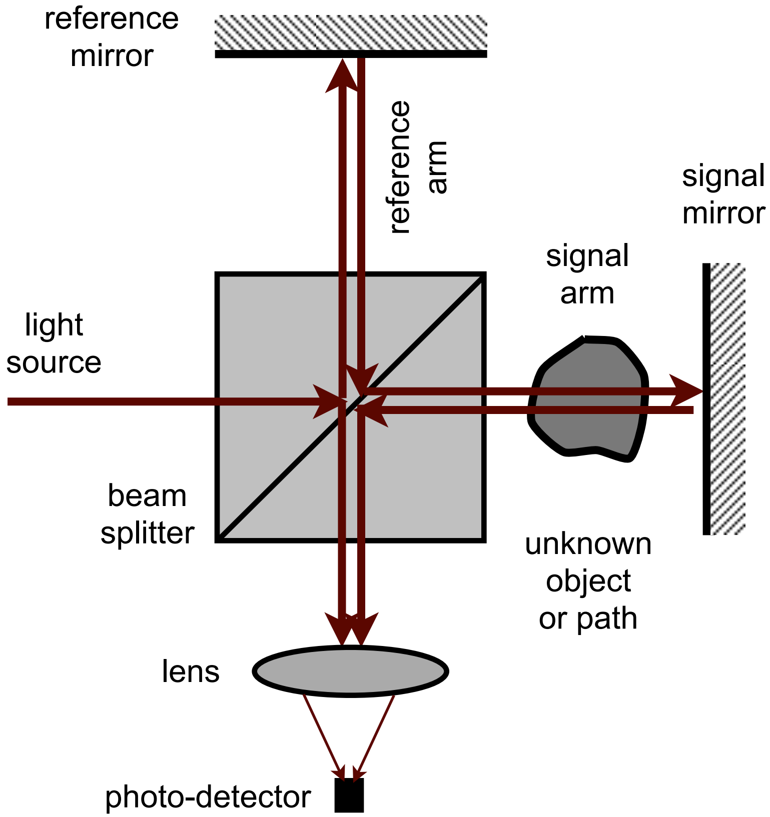
$$E(x,y,z) = A_0 \frac{\exp\left\{ik\sqrt{x^2+y^2+z^2} - \frac{\pi}{2}\right\}}{\lambda\sqrt{x^2+y^2+z^2}} \approx \frac{A_0}{i\lambda z}\exp\left\{ikz + ik\frac{x^2+y^2}{2z}\right\} = \frac{A_0}{i\lambda z}\exp\left\{i\frac{2\pi}{\lambda}z + i\pi\frac{x^2+y^2}{\lambda z}\right\}$$

may also be written as: $E_0\, e^{i\phi}$, where $\phi(x,y) = \pi\dfrac{x^2+y^2}{\lambda z} + \text{const.}$

The significance of phase delays

- The intensity of plane and spherical waves, observed on a screen, is very similar so they cannot be reliably discriminated
  - note that the $1/(x^2+y^2+z^2)$ intensity variation in the case of the spherical wave is too weak in the paraxial case $z>>|x|, |y|$ so in practice it cannot be measured reliably
- In many other cases, the phase of the field carries important information, for example
  - the "history" of where the field has been through
    - distance traveled
    - materials in the path
    - generally, the "optical path length" is inscribed in the phase
  - the evolution of the field beyond its present position ("diffraction")
- However, phase cannot be measured directly (e.g., with an oscillo-scope, because the optical field varies too rapidly, $f\sim10^{14}$-10^{15} Hz
- Interferometry measures the phase by comparing two fields: the unknown, or "signal" field with a known "reference" field

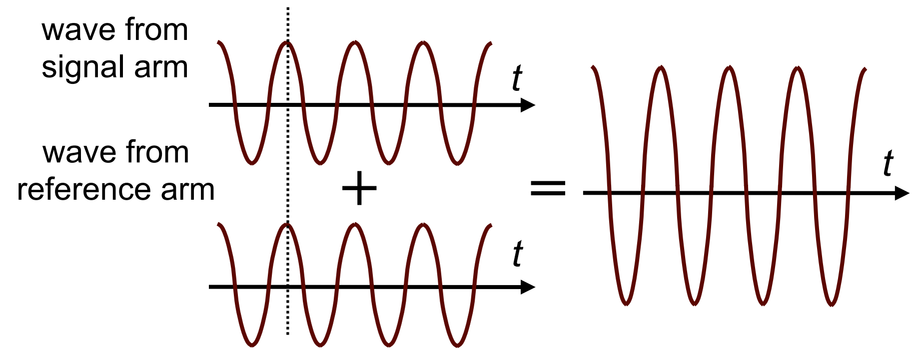
Michelson interferometer



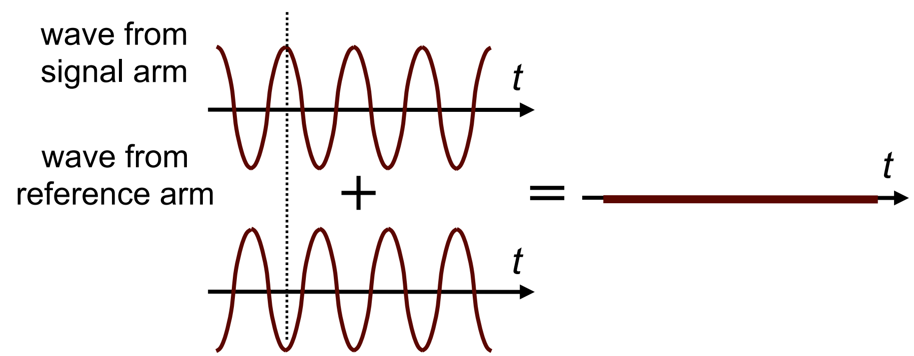
$$I_{\mathrm{d}} \propto \left| A_{\mathrm{r}} \exp\left\{ i\phi_{\mathrm{r}} \right\} + A_{\mathrm{s}} \exp\left\{ i\phi_{\mathrm{s}} \right\} \right|^2$$

where A_r, A_s are the amplitudes and φ_r, φ_s the phase delays introduced after one round trip in the reference and signal arms, respectively

The photo-detector receives the field sum of the waves arriving from the reference and signal arms:

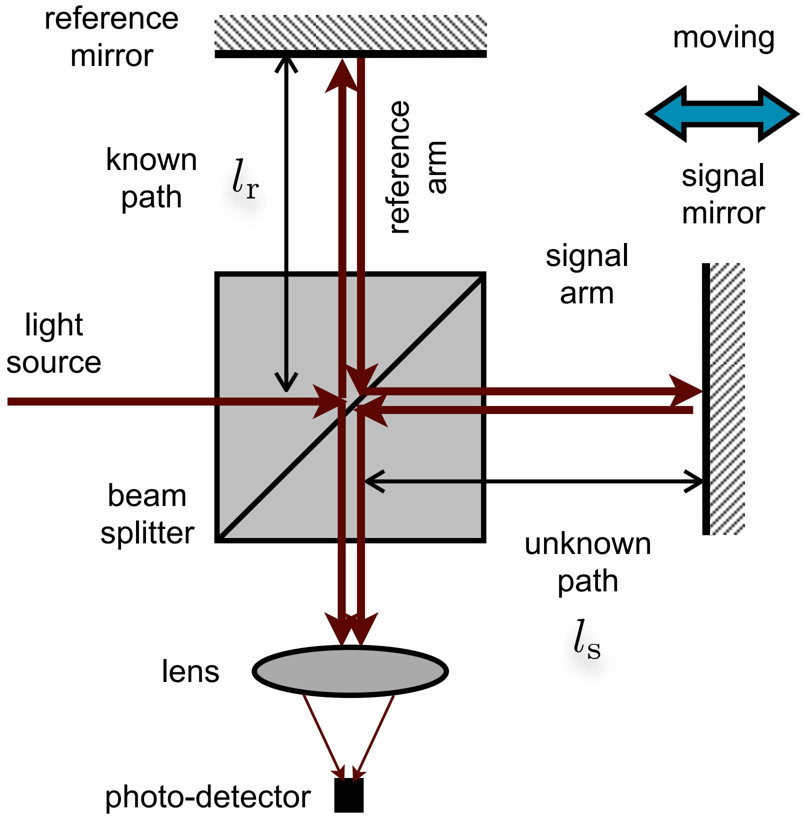


reference and signal waves in phase: **constructive** interference



reference and signal waves out of phase: **destructive** interference

Example: measuring distance



$$I_d \propto \left| A_r \exp\{i\phi_r\} + A_s \exp\{i\phi_s\} \right|^2$$

where A_r, A_s are the amplitudes and φ_r, φ_s the phase delays introduced after one round trip in the reference and signal arms, respectively

The interference is re-written as

$$\begin{aligned} I_d &\propto \left(A_r \exp\{i\phi_r\} + A_s \exp\{i\phi_s\}\right) \times \\ &\quad \times \left(A_r \exp\{i\phi_r\} + A_s \exp\{i\phi_s\}\right)^* \\ &= A_r^2 + A_s^2 + 2A_rA_s \cos(\phi_r - \phi_s) \\ &= I_0 \left(1 + m\cos\Delta\phi\right), \qquad \text{where} \end{aligned}$$

$$\begin{aligned} I_0 &= A_r^2 + A_s^2 && \text{mean intensity} \\ m &= 2A_rA_s/I_0 && \text{contrast (or fringe visibility)} \\ \Delta\phi &= \phi_r - \phi_s && \text{optical path difference.} \end{aligned}$$

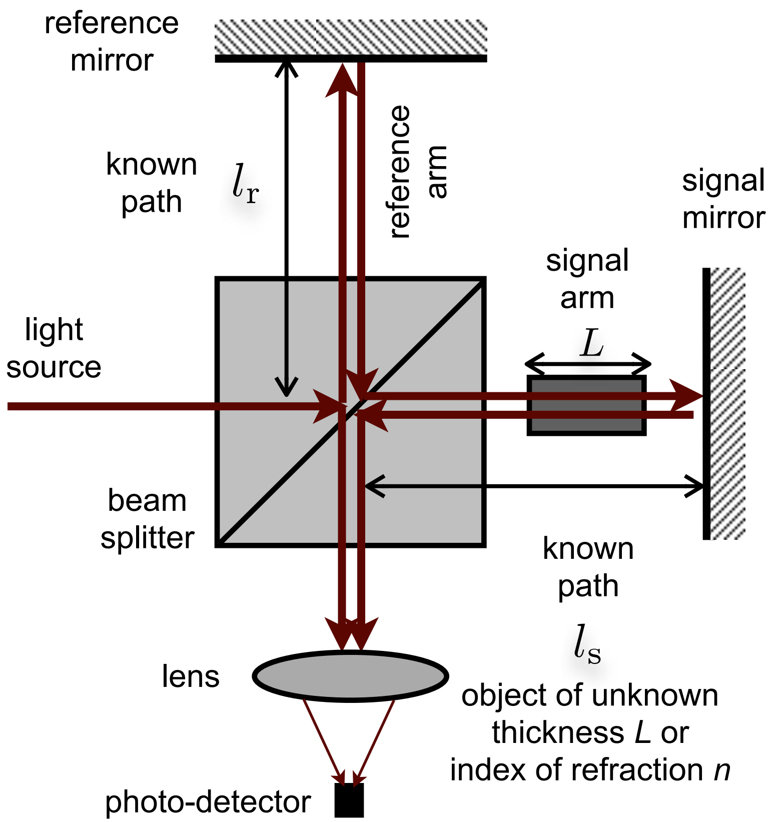
When both paths are clear, as shown on the left,

$$\phi_r = \frac{2\pi}{\lambda} 2l_r \qquad \phi_s = \frac{2\pi}{\lambda} 2l_s$$

$$\Rightarrow \Delta\phi = \frac{2\pi}{\lambda} 2(l_r - l_s)$$

The relative length of the unknown signal path compared to the reference path can be established (mod 2π).

Example: measuring optical density



$$I_d \propto \left| A_r \exp\{i\phi_r\} + A_s \exp\{i\phi_s\} \right|^2$$

where A_r, A_s are the amplitudes and φ_r, φ_s the phase delays introduced after one round trip in the reference and signal arms, respectively

The interference is again written as

$$I_d \propto I_0 \left(1 + m \cos \Delta\phi\right), \qquad \text{where}$$

$$\begin{aligned} I_0 &= A_r^2 + A_s^2 && \text{mean intensity} \\ m &= 2A_r A_s / I_0 && \text{contrast (or fringe visibility)} \\ \Delta\phi &= \phi_s - \phi_r && \text{optical path difference;} \end{aligned}$$

except now

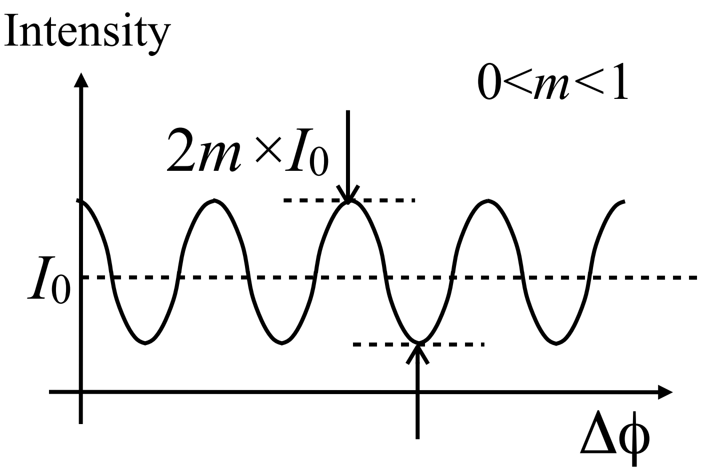
$$\phi_r = \frac{2\pi}{\lambda} 2l_r \qquad \phi_s = \frac{2\pi}{\lambda} 2\left(l_s - L\right) + \frac{2\pi n}{\lambda} 2L$$

$$\Rightarrow \Delta\phi = \frac{2\pi}{\lambda}\left(n-1\right) 2L + \frac{2\pi}{\lambda} 2\left(l_s - l_r\right)$$

Since l_s, l_r are known and constant, we may use the last equation to determine either n or L (but *not* both!)

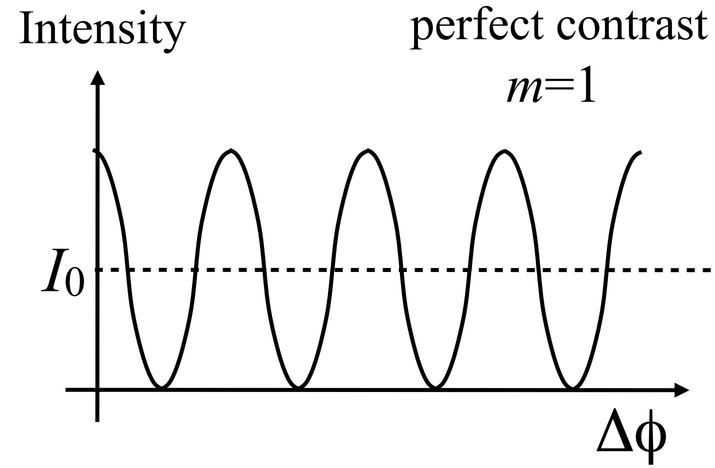
Contrast (fringe visibility) in interferometry

Highest contrast /
fringe visibility is obtained
by interfering beams of
equal amplitudes

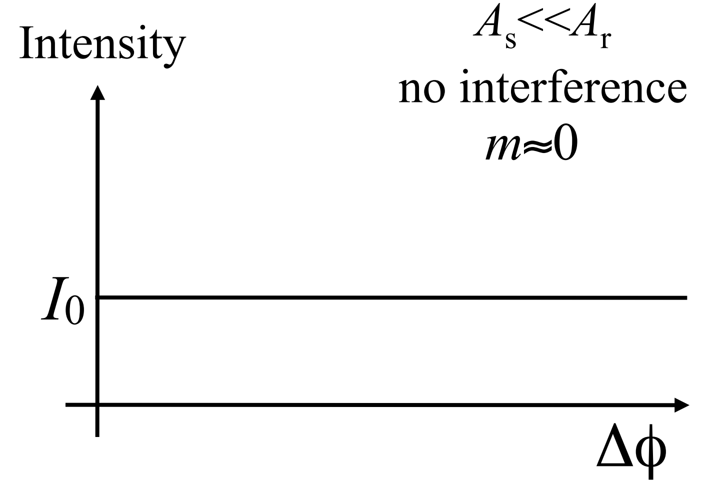


$A_r \neq A_s$
imperfect contrast

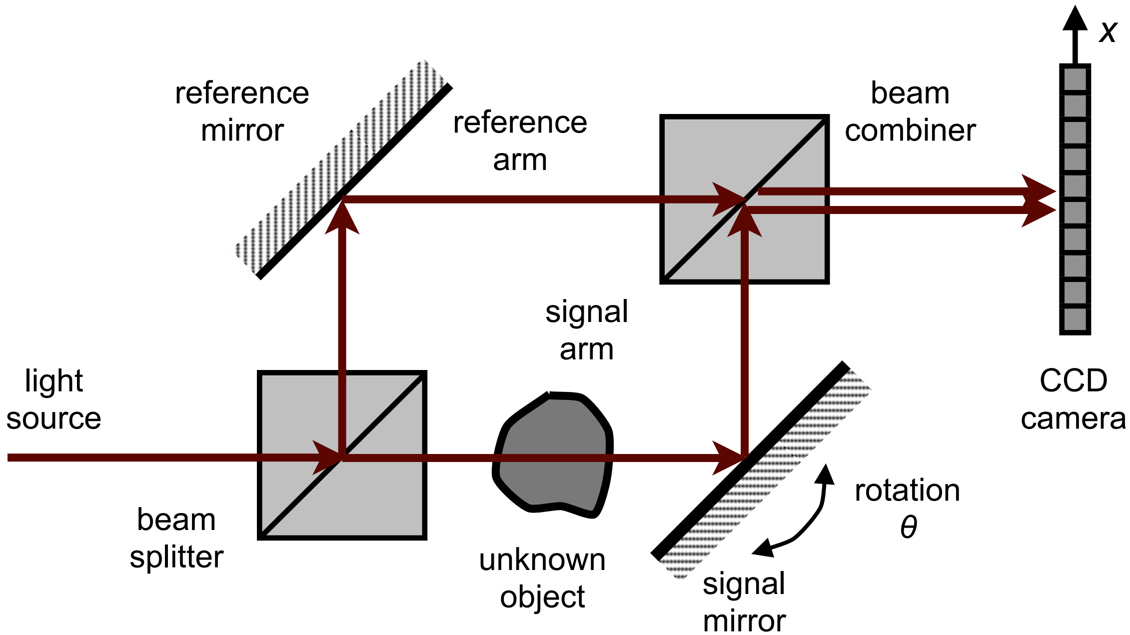
$A_r = A_s$
perfect contrast
$m=1$



$A_s << A_r$
no interference
$m \approx 0$

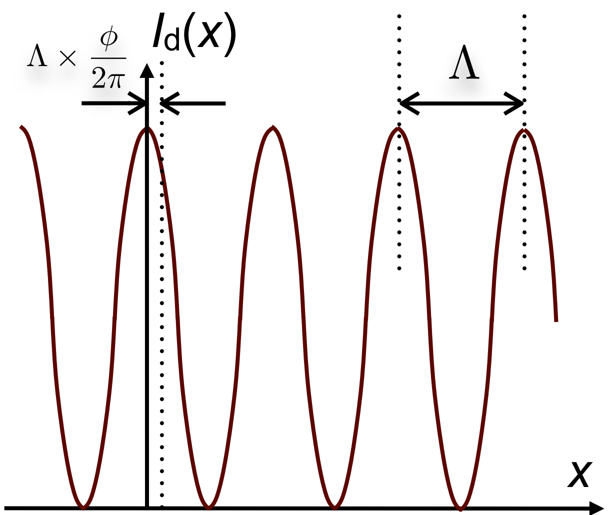


Mach-Zehnder interferometer



in this case the "interference pattern" I_d is function of position x so it is digitized by a pixelated sensor array (digital camera)

Interference pattern



For example, consider a plane wave incident on the interferometer, with no unknown object (*i.e. clear object path*) and the signal mirror tilted by θ. Assuming that the beam splitter ratio is 50% in intensity and that the incident plane wave has amplitude A and is parallel to the optical axis, the interference pattern at the camera plane is written as

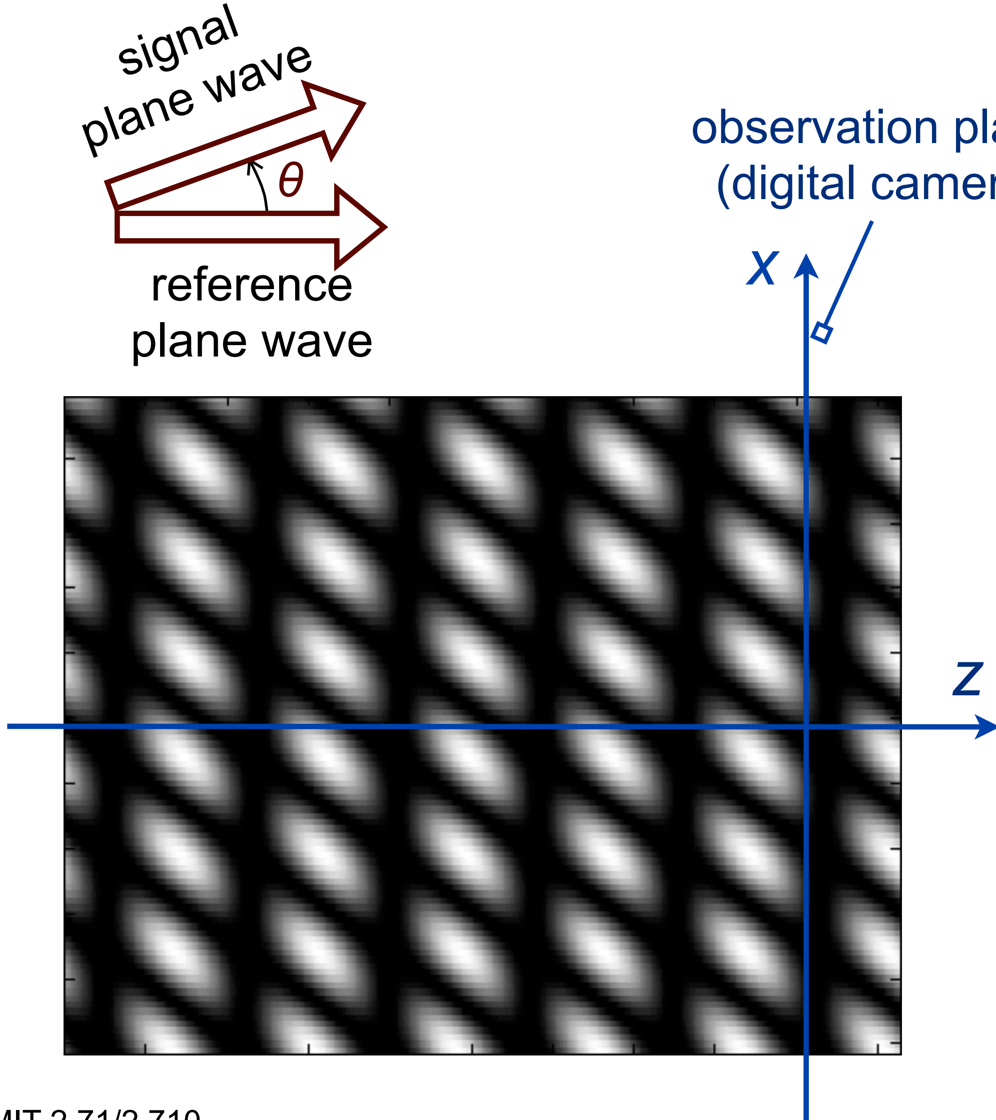
$$I_d(x) \propto \left| \frac{A}{\sqrt{2}} \exp\left\{ i\frac{2\pi}{\lambda} z_0 \right\} + \frac{A}{\sqrt{2}} \exp\left\{ i\frac{2\pi}{\lambda} \left(x \sin\theta + z_0 \cos\theta \right) \right\} \right|^2, \qquad \text{where}$$

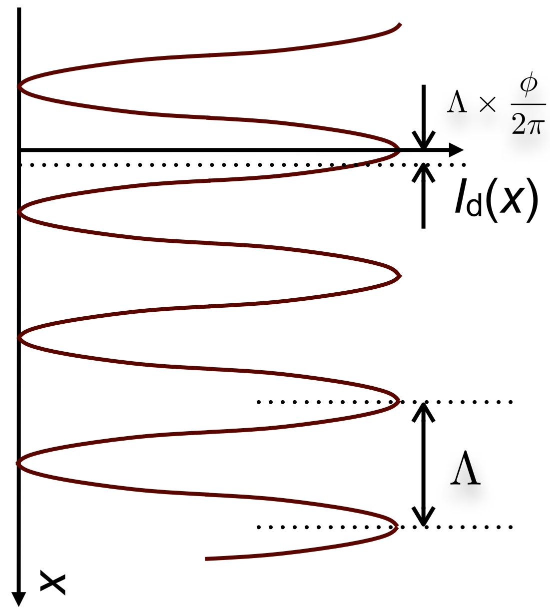
z_0 (constant) is the camera location and x the camera coordinate.

This works out to $I_d(x) \propto \frac{A}{2}\left[1 + \cos\left(\frac{2\pi}{\lambda} x \sin\theta + \phi \right) \right]$

i.e. a spatial sinusoid of period $\Lambda \equiv \frac{\lambda}{\sin\theta}$ shifted by $\phi \equiv \frac{2\pi}{\lambda} z_0 \left(\cos\theta - 1 \right)$.

Interference pattern: mapping the incident *field* ⇔ incident *wavefront*





Today

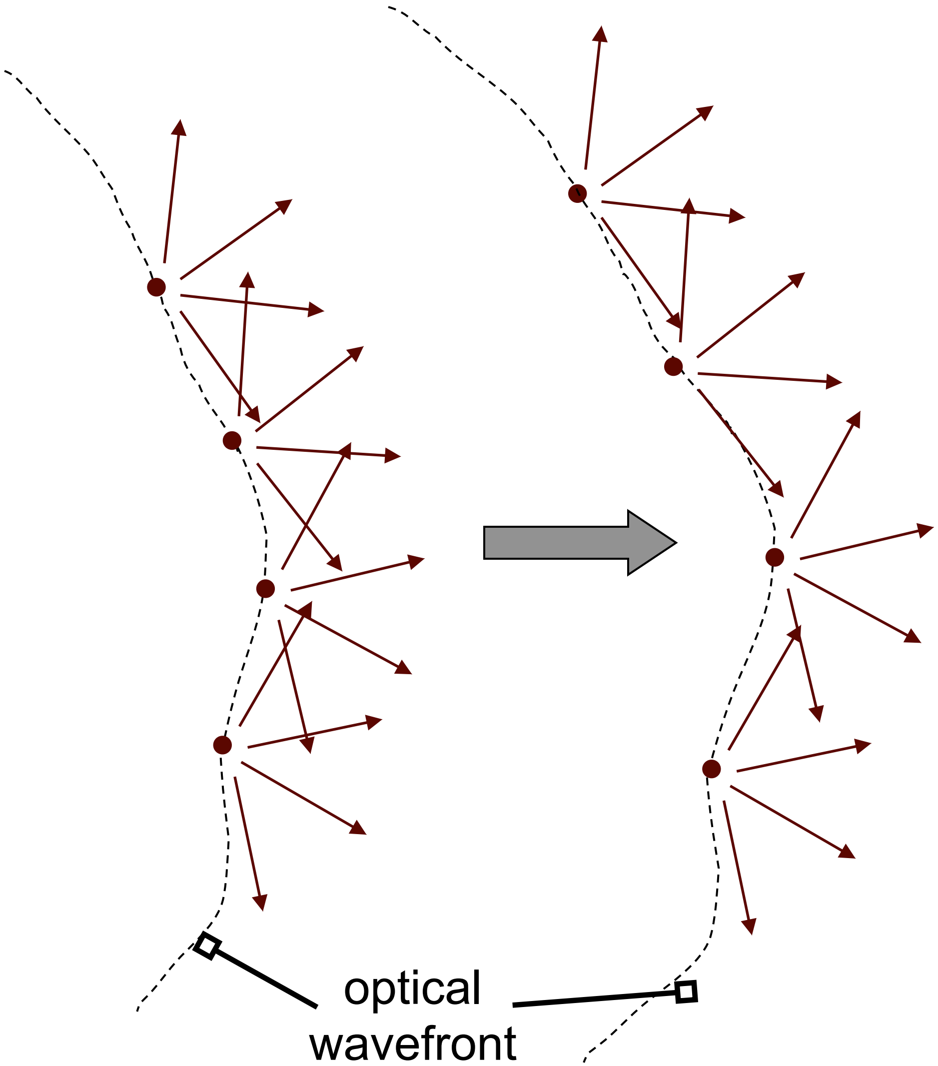
- Huygens principle
- Young's interferometer
- Generalizing Young's interferometer:
  Huygens principle and thin transparencies ⇒

  ⇒ Fresnel diffraction integral
- Diffraction
  - Fresnel regime

Next week

  - Fraunhofer regime
- Spatial frequencies and Fourier transforms
- Fraunhofer patterns of typical apertures

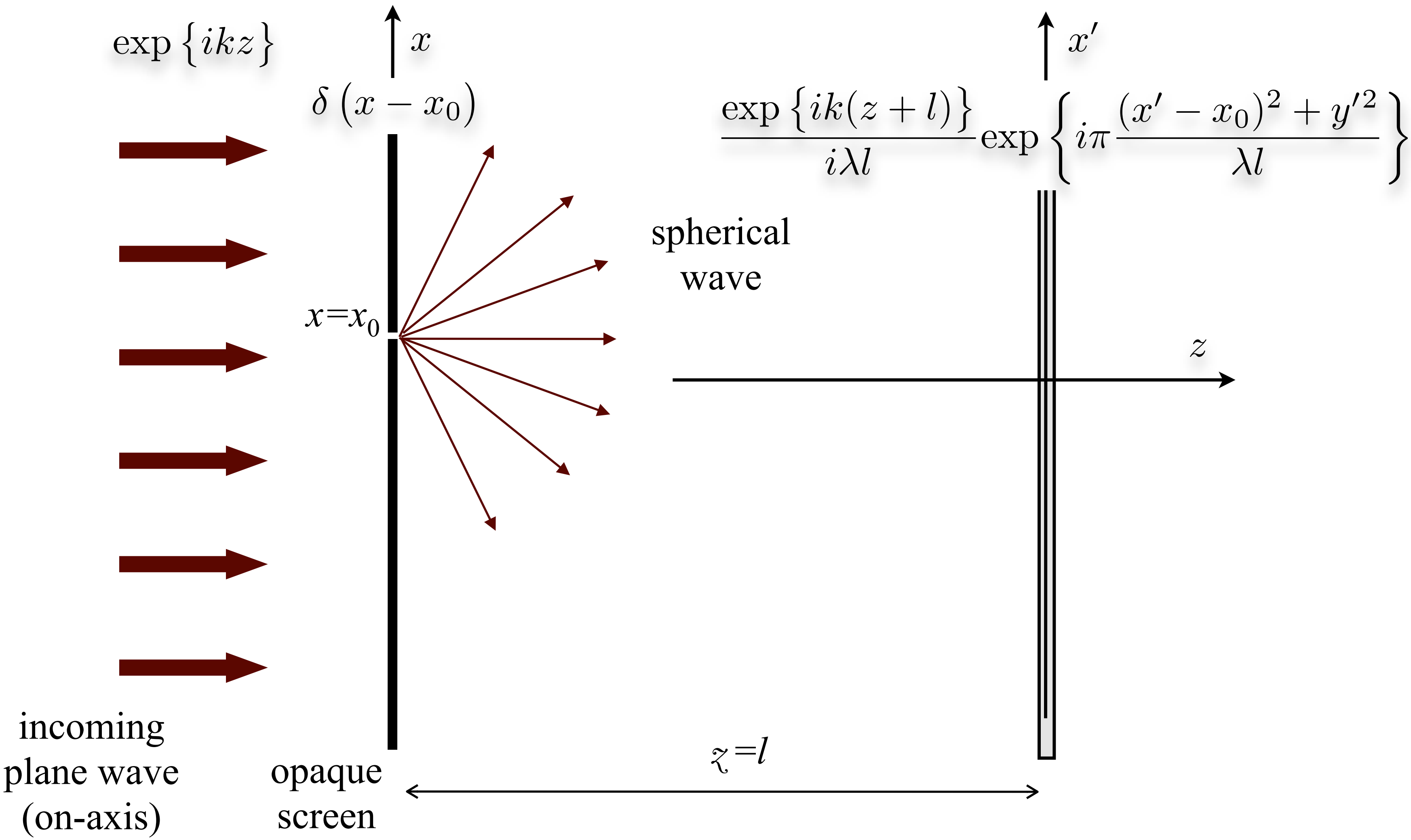
Huygens principle



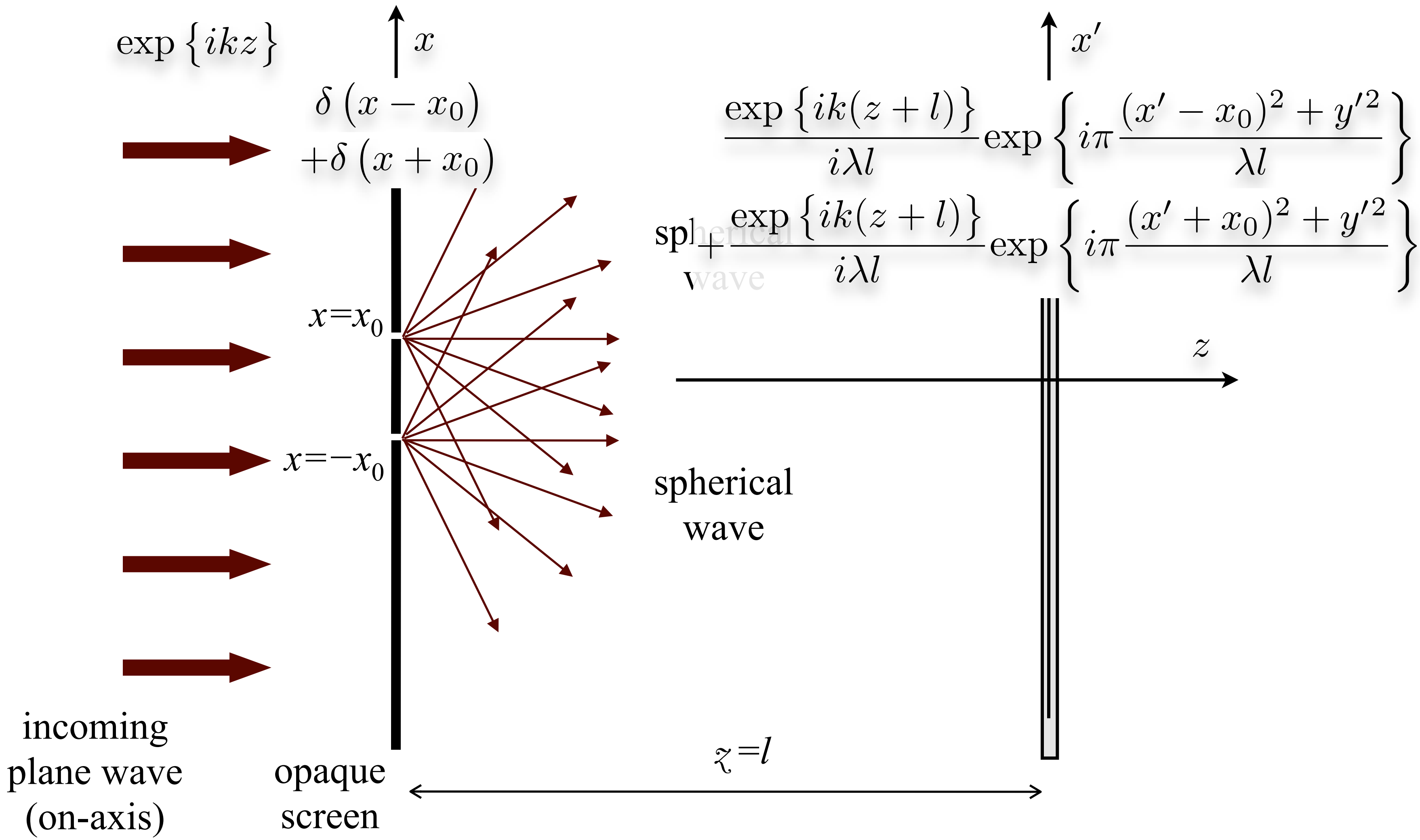
Each point on the wavefront acts as a secondary light source emitting a spherical wave

The wavefront after a short propagation distance is the result of superimposing all these spherical waves, i.e. adding the corresponding phasors including the phase delay incurred due to propagation

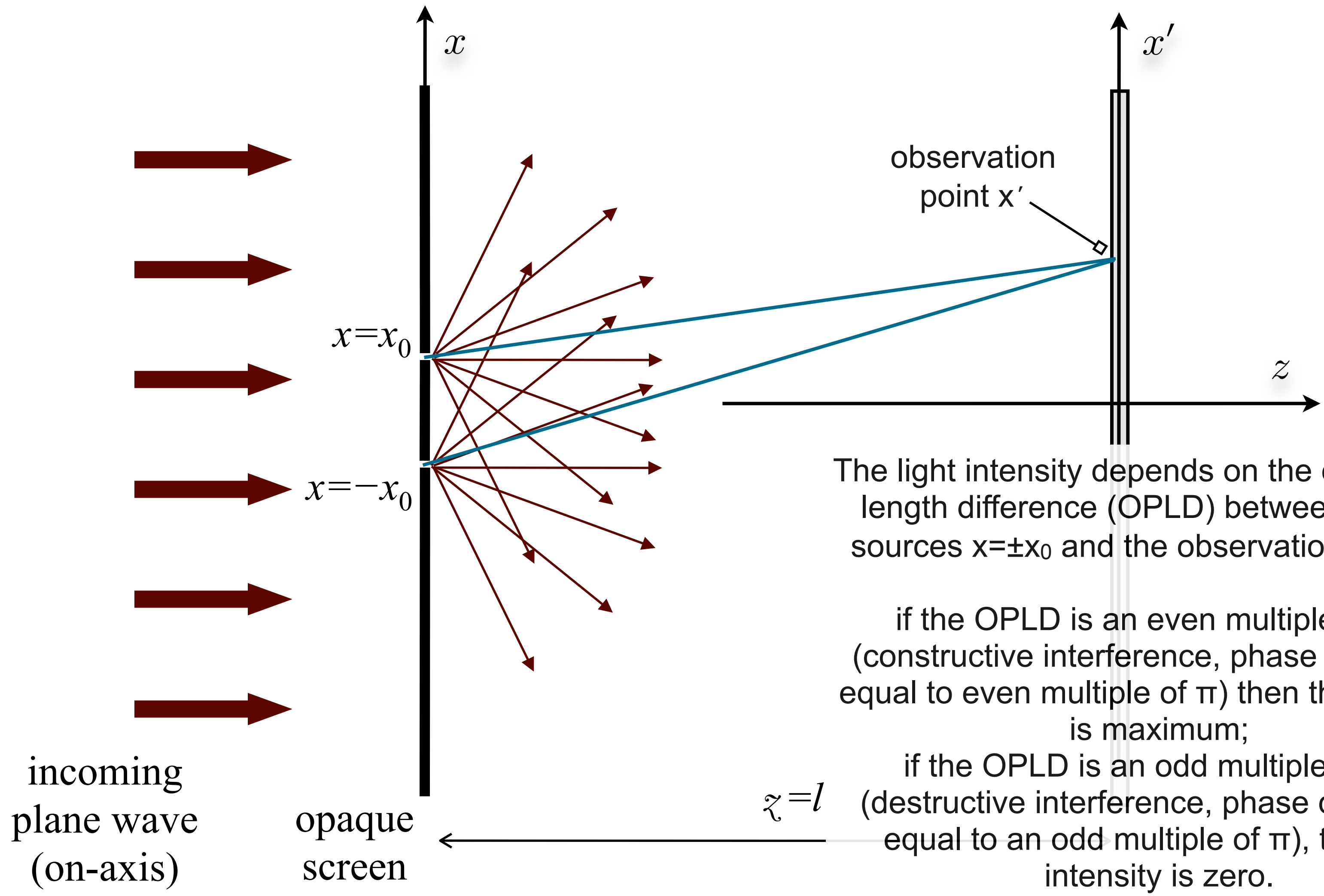
Example: hole in an opaque screen



Example: *two* holes in an opaque screen



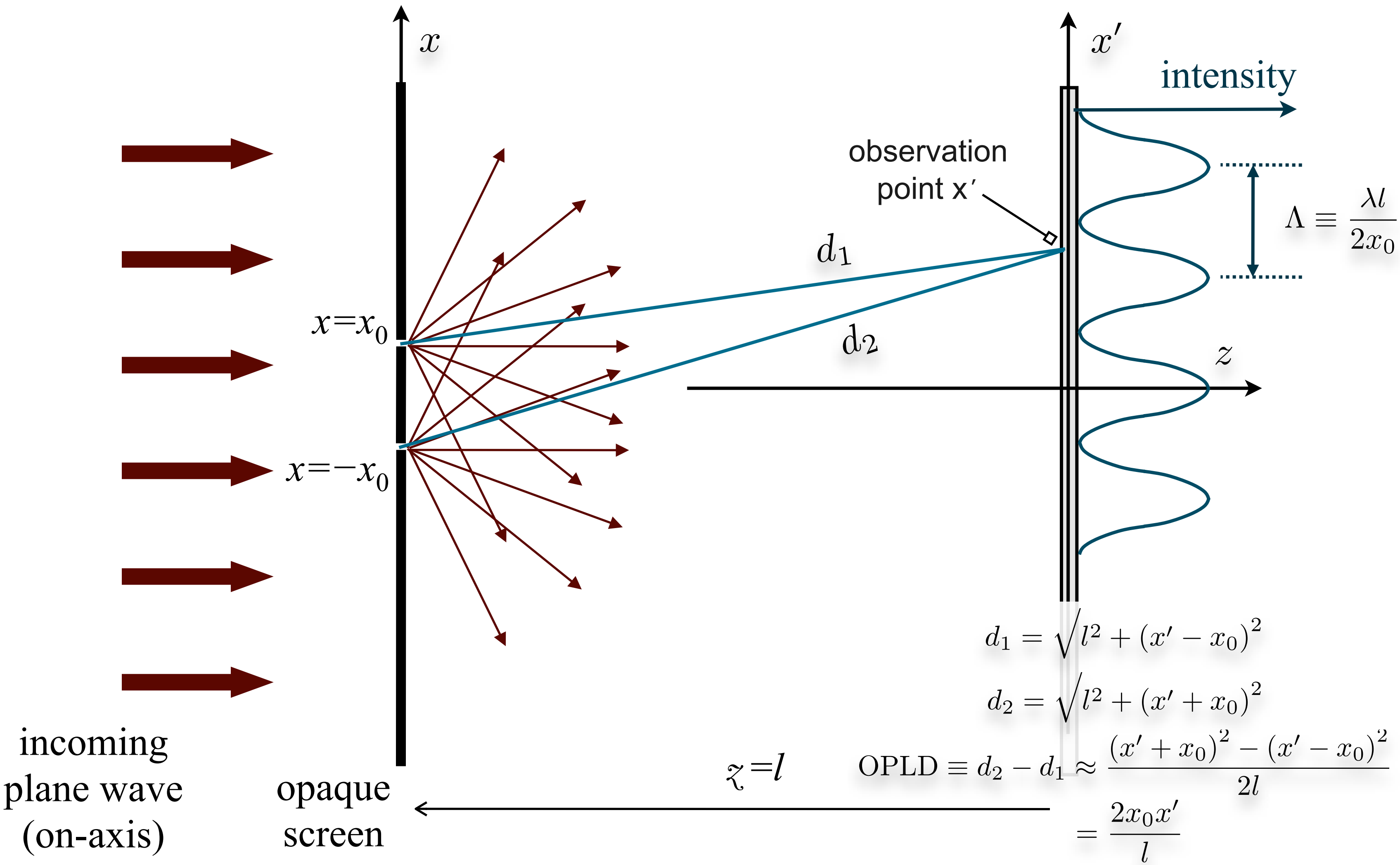
Young interferometer



The light intensity depends on the optical path length difference (OPLD) between the two sources $x=\pm x_0$ and the observation point x':

if the OPLD is an even multiple of $\lambda/2$ (constructive interference, phase difference equal to even multiple of π) then the intensity is maximum;
if the OPLD is an odd multiple of $\lambda/2$ (destructive interference, phase difference equal to an odd multiple of π), then the intensity is zero.

Young interferometer



$$d_1 = \sqrt{l^2 + (x' - x_0)^2}$$

$$d_2 = \sqrt{l^2 + (x' + x_0)^2}$$

$$\text{OPLD} \equiv d_2 - d_1 \approx \frac{(x' + x_0)^2 - (x' - x_0)^2}{2l}$$

$$= \frac{2x_0 x'}{l}$$

$$\text{OPLD} = \lambda \Leftrightarrow x' = \frac{\lambda l}{2x_0}$$

Derivation of Young's interference pattern

The superposition of the fields generated by the two holes is $g(x', y') =$

$$= \frac{\exp\{ik(z+l)\}}{i\lambda l}\exp\left\{i\pi\frac{(x'-x_0)^2+y'^2}{\lambda l}\right\}+\frac{\exp\{ik(z+l)\}}{i\lambda l}\exp\left\{i\pi\frac{(x'+x_0)^2+y'^2}{\lambda l}\right\}=$$

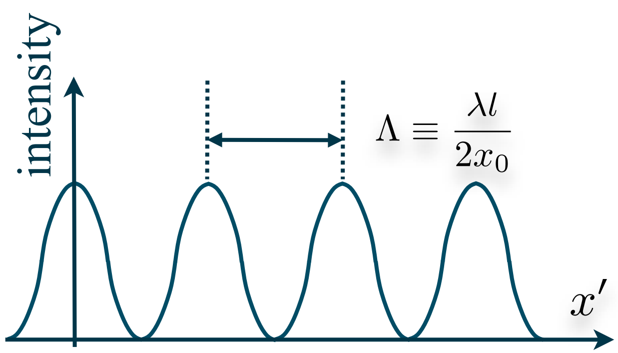
$$= \frac{\exp\{ik(z+l)\}}{i\lambda l}\exp\left\{i\pi\frac{x'^2+x_0^2+y'^2}{\lambda l}\right\}\left(\exp\left\{-i2\pi\frac{x_0x'}{\lambda l}\right\}+\exp\left\{i2\pi\frac{x_0x'}{\lambda l}\right\}\right)$$

$$= \frac{\exp\{ik(z+l)\}}{i\lambda l}\exp\left\{i\pi\frac{x'^2+x_0^2+y'^2}{\lambda l}\right\}\times 2\cos\left(2\pi\frac{x_0x'}{\lambda l}\right).$$

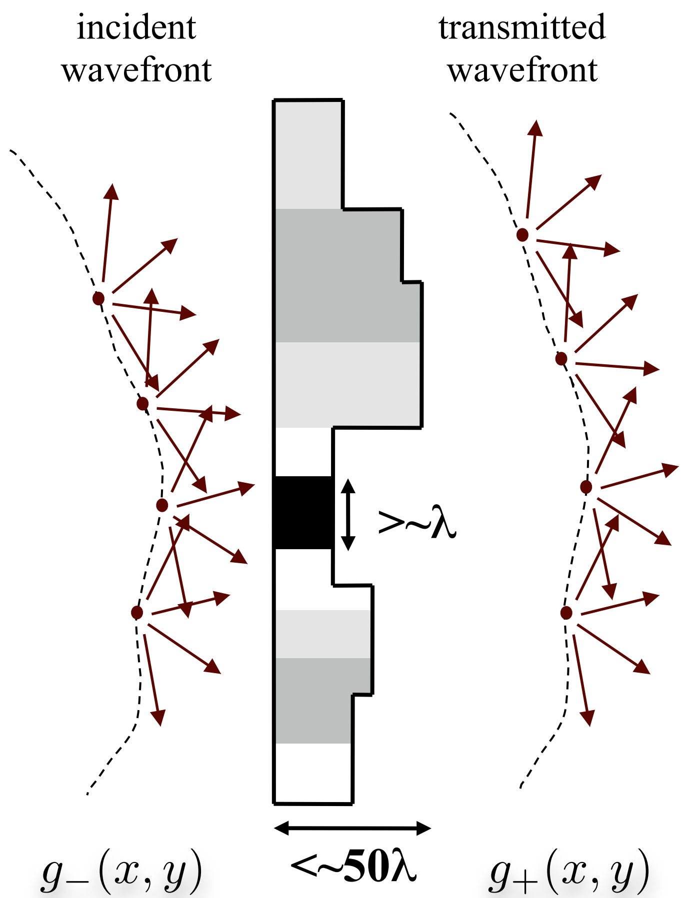
The intensity is the modulus–squared of the field phasor, *i.e.*

$$I(x',y') = |g(x',y')|^2 = 4\cos^2\left(2\pi\frac{x_0x'}{\lambda l}\right) = 2\left[1+\cos\left(2\pi\frac{2x_0x'}{\lambda l}\right)\right] \equiv 2\left[1+\cos\left(2\pi\frac{x'}{\Lambda}\right)\right],$$

where $\Lambda = \dfrac{\lambda l}{2x_0}$ is the spatial period of the sinusoidal interference pattern.



Thin transparency



<u>Assumptions:</u>

➡ features on the transparency are larger than ~λ
➡ thickness of the transparency may be neglected

The transparency has two effects on the incoming wavefront:

- attenuation, which is determined by the opacity of the transparency at a given location, and typically is
  - binary (more common; transmission is either completely clear or completely opaque) or
  - grayscale (at greater expense)
- phase delay, which is dependent on the optical path length (OPL) at the transmissive (or grayscale) locations, and is
  - binary (more common; phase delay is one of two values);
  - multi-level (at greater expense; phase delay is one of M values); or
  - continuous (also known as surface relief)

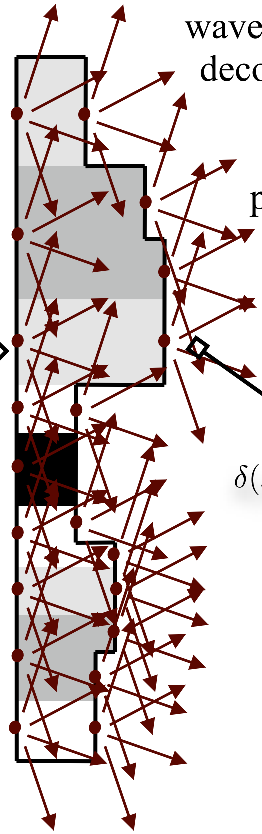
The attenuation and phase delay imposed by the thin transparency are described together as a *complex transmission function*, whose

- modulus is the attenuation; and
- phase is the phase delay

$$g_{+}(x,y) = g_{\mathrm{t}}(x,y) \times g_{-}(x,y) \qquad g_{\mathrm{t}}(x,y) = a(x,y)\exp\left\{i\phi(x,y)\right\}$$

Transmission function

Thin transparency: generalized Young interferometer



The thin transparency may also be thought of as a generalized Young's interferometer in the following sense:

We decompose the incident wavefront into Huygens point sources; Huygens principle says that the transmitted wavefront may also be decomposed into point sources.

If the transparency is sufficiently thin, the each Huygens point source in the incident wavefront is directly transmitted to a Huygens point source in the outgoing wavefront, possibly with an attenuation and phase delay given by the transparency's complex transmission function.

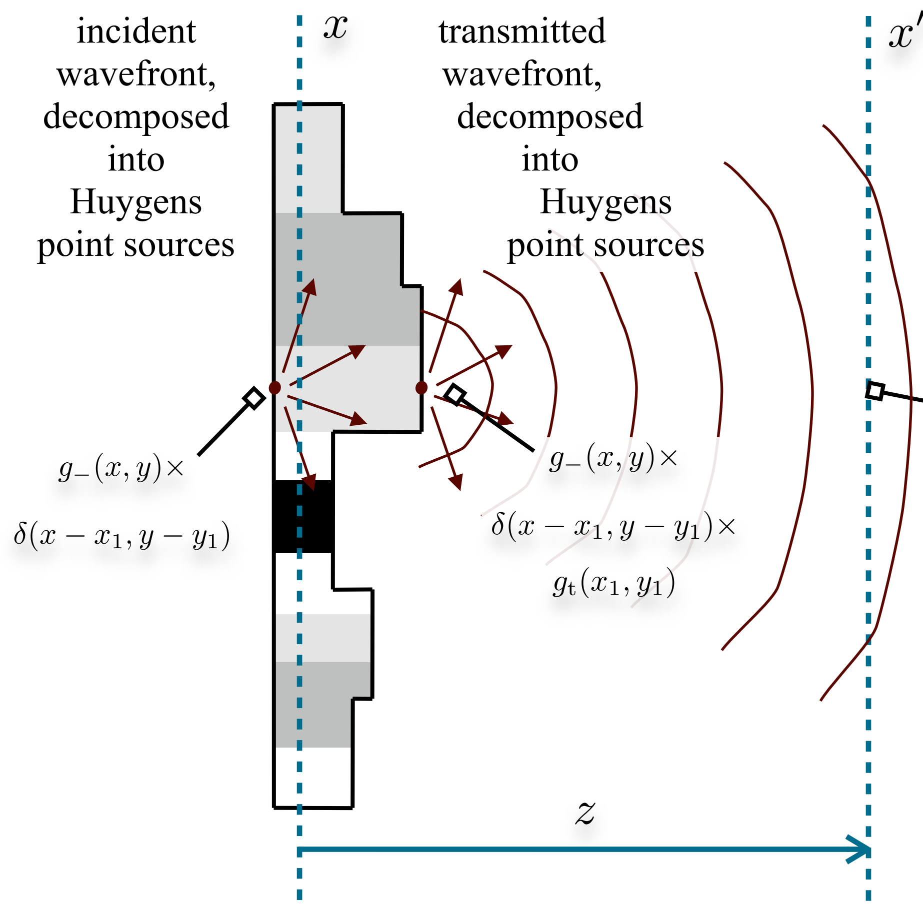
The overall transmitted wavefront is the superposition of the Huygens point sources obtained by point-by-point multiplication of the incident wavefront times the complex transmission function. Therefore, the overall transmitted wavefront is obtained as a generalized Young's interferometer with not just two, but a continuum of (infinite) point sources.

Using the sifting property of delta functions we can express the incident and transmitted wavefronts as

$$g_-(x_1,y_1)=\iint g_-(x,y)\times\delta(x-x_1,y-y_1)\,\mathrm{d}x\mathrm{d}y$$

$$g_+(x_1,y_1)=\iint g_-(x,y)\times\delta(x-x_1,y-y_1)\times g_t(x_1,y_1)\,\mathrm{d}x\mathrm{d}y$$

The Fresnel diffraction integral



Each Huygens point source is a divergent spherical wave; therefore, after propagating by distance z in free space, it can be expressed as

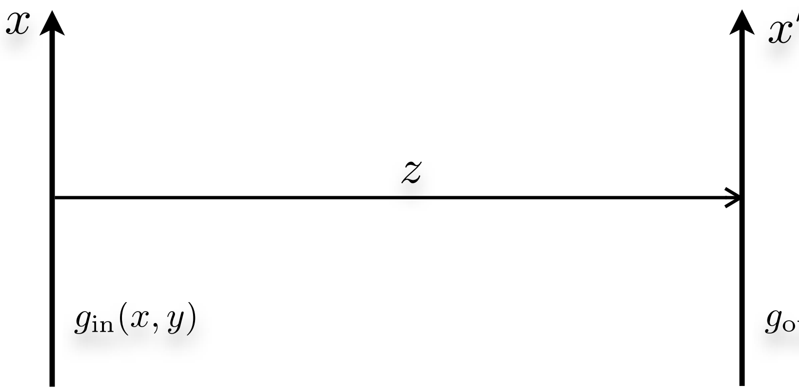
$$g_-(x_1,y_1)\times \frac{1}{i\lambda z}\exp\left\{i2\pi\frac{z}{\lambda}\right\}\exp\left\{i\pi\frac{(x'-x_1)^2+(y'-y_1)^2}{\lambda z}\right\}\times g_t(x_1,y_1)$$

The entire propagated wavefront is the superposition of the propagated Huygens point sources at the wavefront transmitted by the transparency, so it may be expressed as

$$g(x',y';z)=\iint g_-(x_1,y_1)\times g_t(x_1,y_1)\times \frac{1}{i\lambda z}\exp\left\{i2\pi\frac{z}{\lambda}\right\}\exp\left\{i\pi\frac{(x'-x_1)^2+(y'-y_1)^2}{\lambda z}\right\}\mathrm{d}x_1\mathrm{d}y_1$$

This result is known as Fresnel diffraction integral (or simply **Fresnel integral**)

Fresnel integral is a convolution



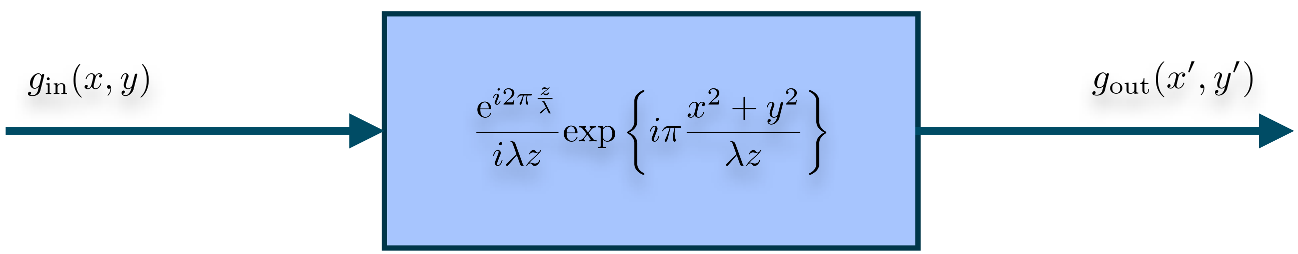
Fresnel convolution integral

$$g_{\text{out}}(x',y') = g_{\text{in}}(x,y) \star \left(\frac{e^{i2\pi\frac{z}{\lambda}}}{i\lambda z} \exp\left\{ i\pi \frac{x^2+y^2}{\lambda z} \right\} \right)$$

$$g_{\text{out}}(x',y';z) = \frac{1}{i\lambda z} \exp\left\{ i2\pi \frac{z}{\lambda} \right\} \iint g_{\text{in}}(x,y) \exp\left\{ i\pi \frac{(x'-x)^2+(y'-y)^2}{\lambda z} \right\} \mathrm{d}x\mathrm{d}y$$

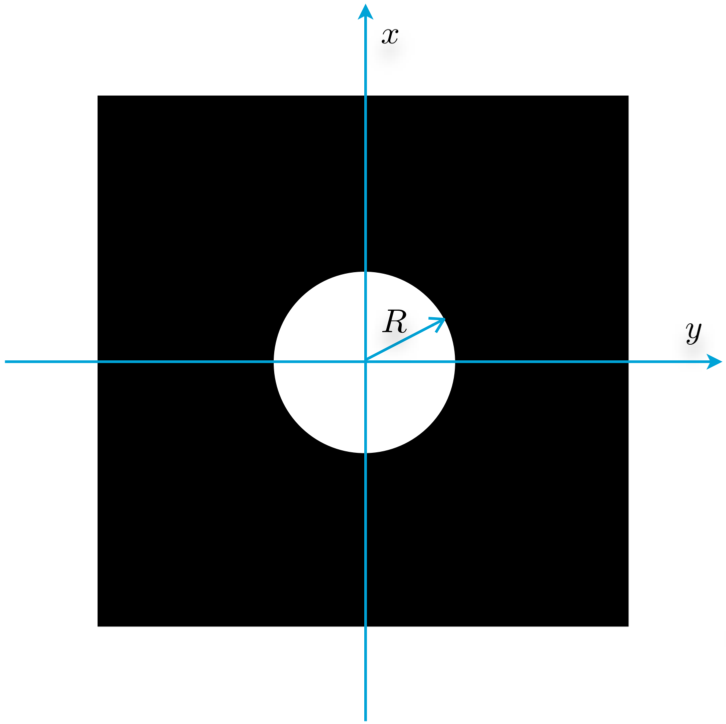
Free space propagation is expressed as a Fresnel diffraction integral, which is mathematically identical to convolution of the incoming wavefront with (the paraxial approximation expression for) a spherical wave.

In systems language, we can express free space propagation in a block diagram as



The spherical wave is the system's impulse response; in Optics, we refer to it as the **Point Spread Function (PSF)**.

Example: circular aperture



The wavefront produced by a circular aperture of radius R is

$$g_{\text{in}}(x,y) = \begin{cases} 1, & x^2+y^2<R^2; \\ 0, & \text{otherwise.} \end{cases}$$

The Fresnel diffraction integral for the circular aperture is

$$g_{\text{out}}(x',y';z) = \frac{e^{i2\pi\frac{z}{\lambda}}}{i\lambda z}\iint_{x^2+y^2<R^2} \exp\left\{i\pi\frac{(x'-x)^2+(y'-y)^2}{\lambda z}\right\}\mathrm{d}x\mathrm{d}y.$$

For an on–axis observation point $(x'=0, y'=0)$, this becomes

$$g_{\text{out}}(0,0;z) = \frac{e^{i2\pi\frac{z}{\lambda}}}{i\lambda z}\iint_{x^2+y^2<R^2} \exp\left\{i\pi\frac{x^2+y^2}{\lambda z}\right\}\mathrm{d}x\mathrm{d}y.$$

We rewrite this integral in polar coordinates as

$$g_{\text{out}}(0,0;z) = \frac{e^{i2\pi\frac{z}{\lambda}}}{i\lambda z}\int_0^{2\pi}\mathrm{d}\theta\int_0^R \rho\,\mathrm{d}\rho\exp\left\{i\pi\frac{\rho^2}{\lambda z}\right\} = \frac{e^{i2\pi\frac{z}{\lambda}}}{i\lambda z}\times 2\pi\times\int_0^R \rho\exp\left\{i\pi\frac{\rho^2}{\lambda z}\right\}\mathrm{d}\rho.$$

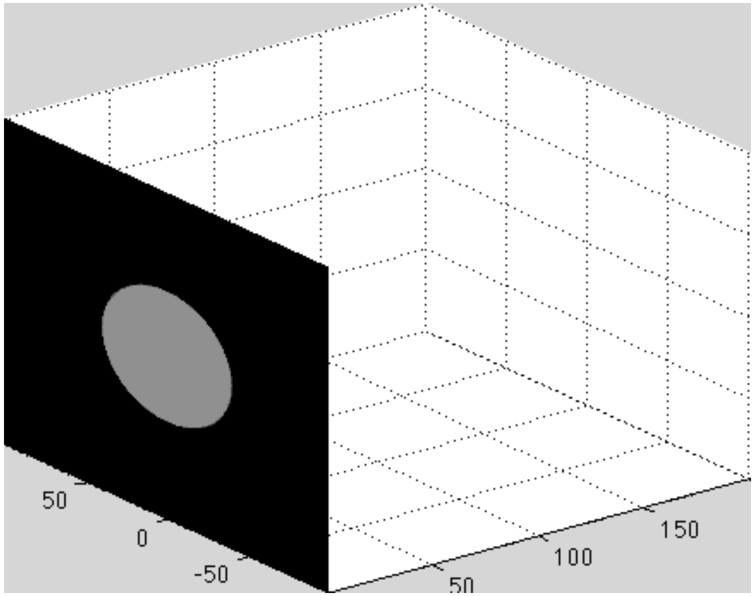
Now we make the substitution $\xi = \pi\rho^2/\lambda z$ and obtain

$$\begin{aligned} g_{\text{out}}(0,0;z) &= \frac{e^{i2\pi\frac{z}{\lambda}}}{i\lambda z}\times 2\pi\times\frac{\lambda z}{2\pi}\int_0^{\frac{\pi R^2}{\lambda z}} e^{i\xi}\mathrm{d}\xi = \frac{e^{i2\pi\frac{z}{\lambda}}}{2i}2\left(\exp\left\{i\pi\frac{R^2}{\lambda z}\right\}-1\right) \\ &= 2\exp\left\{i2\pi\frac{z}{\lambda}+i\pi\frac{R^2}{2\lambda z}\right\}\frac{1}{2i}\left(\exp\left\{i\pi\frac{R^2}{2\lambda z}\right\}-\exp\left\{-i\pi\frac{R^2}{2\lambda z}\right\}\right) \\ &= 2\exp\left\{i2\pi\frac{z}{\lambda}+i\pi\frac{R^2}{2\lambda z}\right\}\sin\left(\pi\frac{R^2}{2\lambda z}\right). \end{aligned}$$

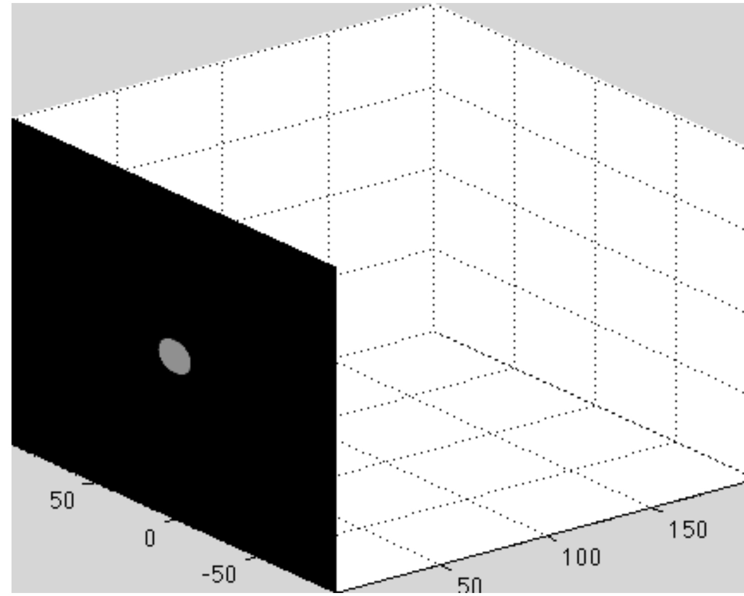
We observe that the diffracted field has harmonically spaced on–axis peaks at

$$z_{\text{p},m} = \frac{R^2}{2\lambda(m-1/2)}, \qquad \text{and nulls at} \qquad z_{\text{n},m} = \frac{R^2}{2\lambda m}, \qquad m=1,2,\ldots$$

Example: circular aperture



$R = 40\lambda$

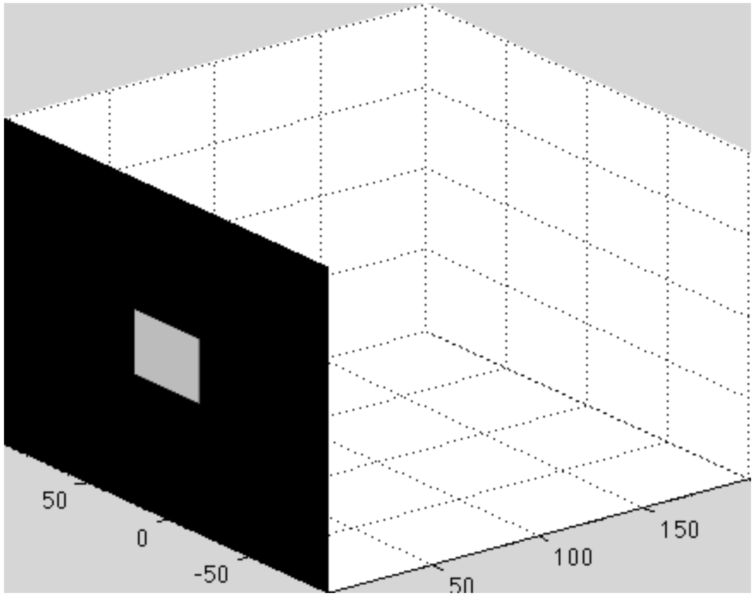


$R = 10\lambda$

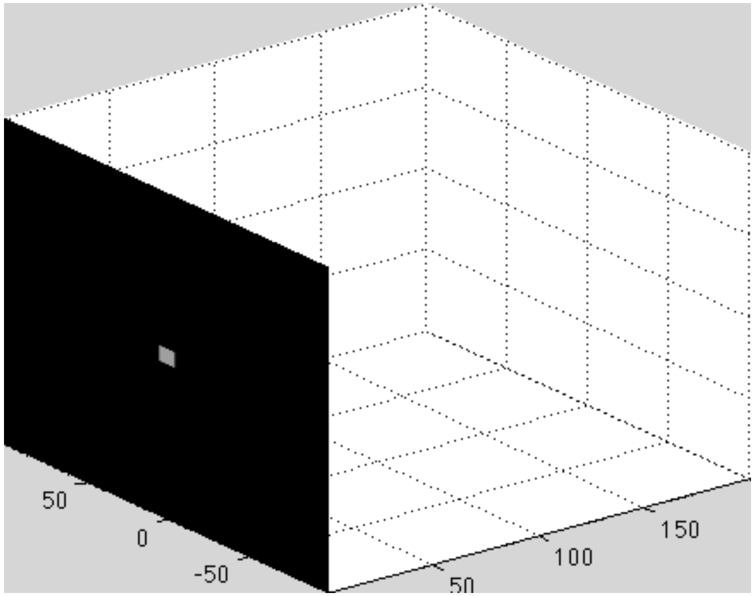
➡ The ripples that arise near the edges of the aperture after a very short propagation distance and are noticeable in the large aperture case are characteristic of diffraction with coherent illumination and are referred to as “Fresnel ripples” or “diffraction ripples.”

➡ The alternating peaks (bright) and nulls (dark) that are noticeable in the small aperture case are referred to as “Poisson spot” or “blinking spot.”

Example: rectangular aperture



$\text{Aperture size} = 40\lambda$



$\text{Aperture size} = 10\lambda$

➡ Fresnel ripples are again noticeable in the large aperture case but produce a different ripple structure because of the rectangular geometry.

➡ The diffraction pattern from the small aperture changes qualitatively after some propagation distance; it begins to look like a sinc function, the Fourier transform of the boxcar function. We will explain this phenomenon quantitatively very soon; we refer to it as the Fraunhofer diffraction regime.

➡ Fraunhofer diffraction occurs in the case of the large aperture as well, but after a longer propagation distance (we will quantify that as well.)

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