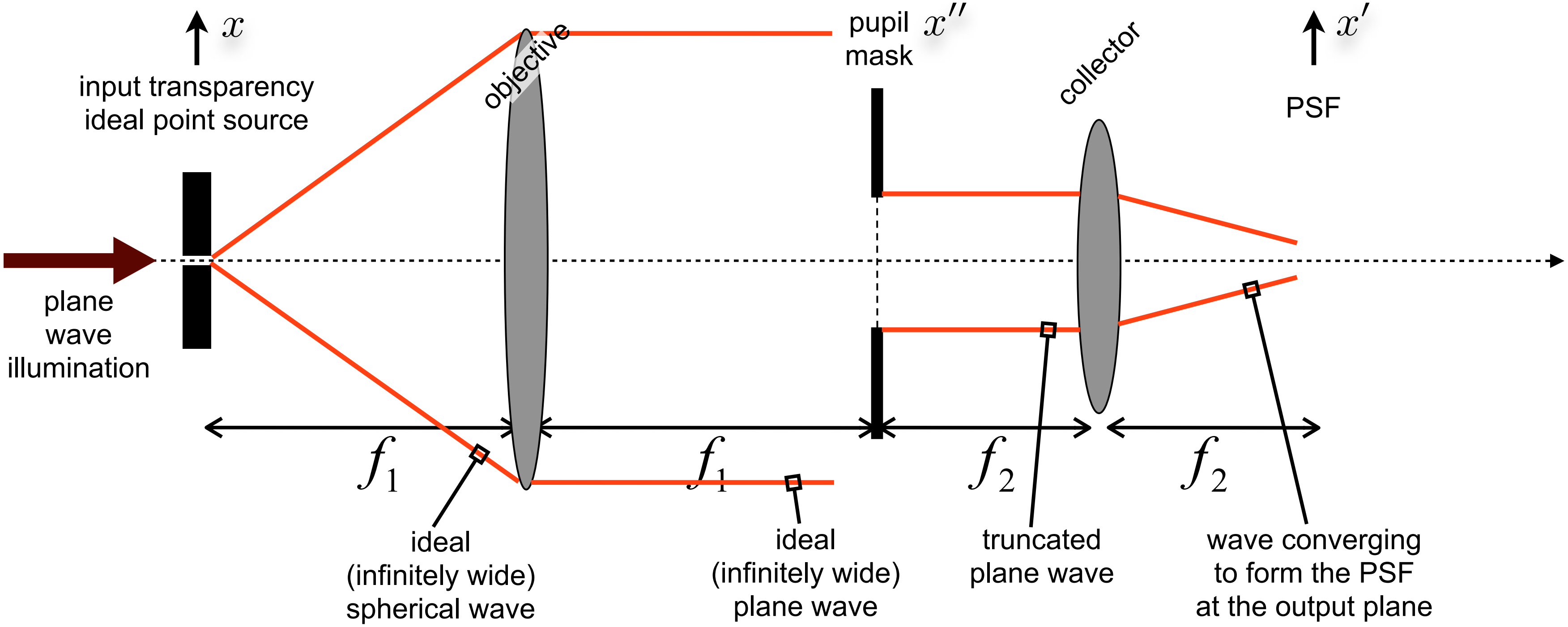
The Point-Spread Function (PSF) of a low-pass filter



Now consider the same 4F system but replace the input transparency with an ideal point source, implemented as an opaque sheet with an infinitesimally small transparent hole and illuminated with a plane wave on axis (actually, any illumination will result in a point source in this case, according to Huygens.)

In Systems terminology, we are exciting this linear system with an impulse (delta-function); therefore, the response is known as **Impulse Response**.
In Optics terminology, we use instead the term **Point-Spread Function (PSF)** and we denote it as $h(x', y')$.

The sequence to compute the PSF *of a 4F system* is:

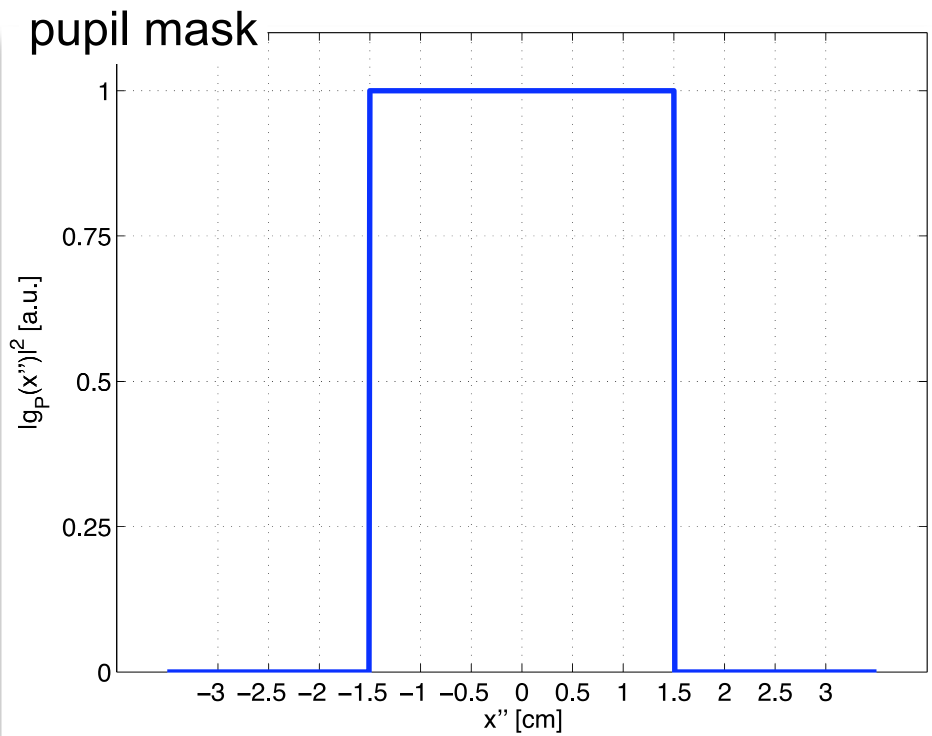
➡ observe that the Fourier transform of the input transparency $\delta(x)$ is simply 1 everywhere at the pupil plane

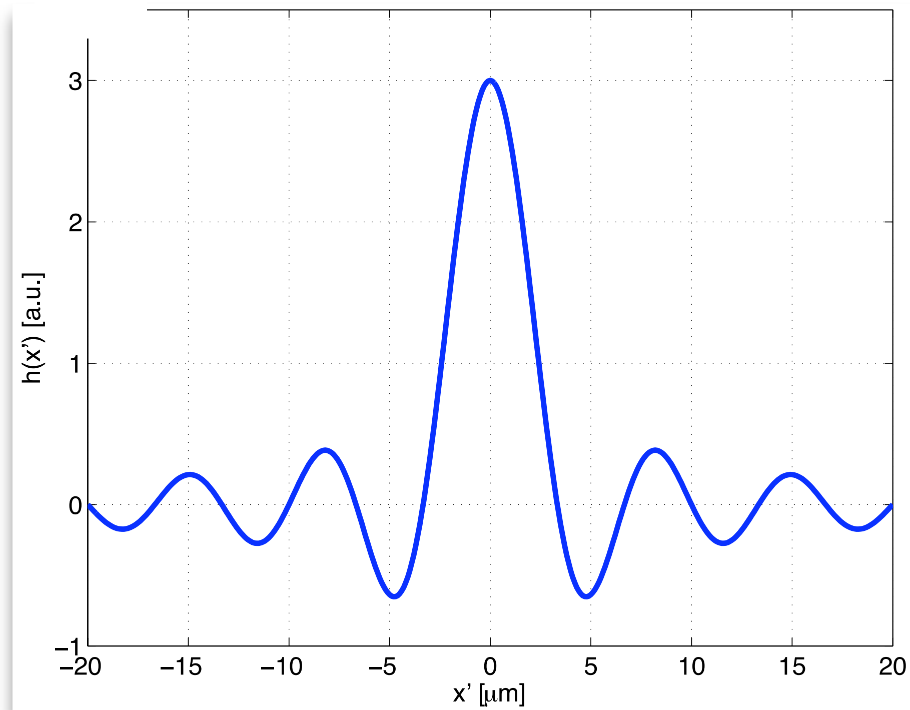
➡ multiply 1 by the complex amplitude transmittance of the pupil mask

➡ Fourier transform the product and scale to the output plane coordinates $x' = u\lambda f_2$.

➡Therefore, the PSF is simply the Fourier transform of the pupil mask, scaled to the output coordinates $x' = u\lambda f_2$

Example: PSF of a low-pass filter





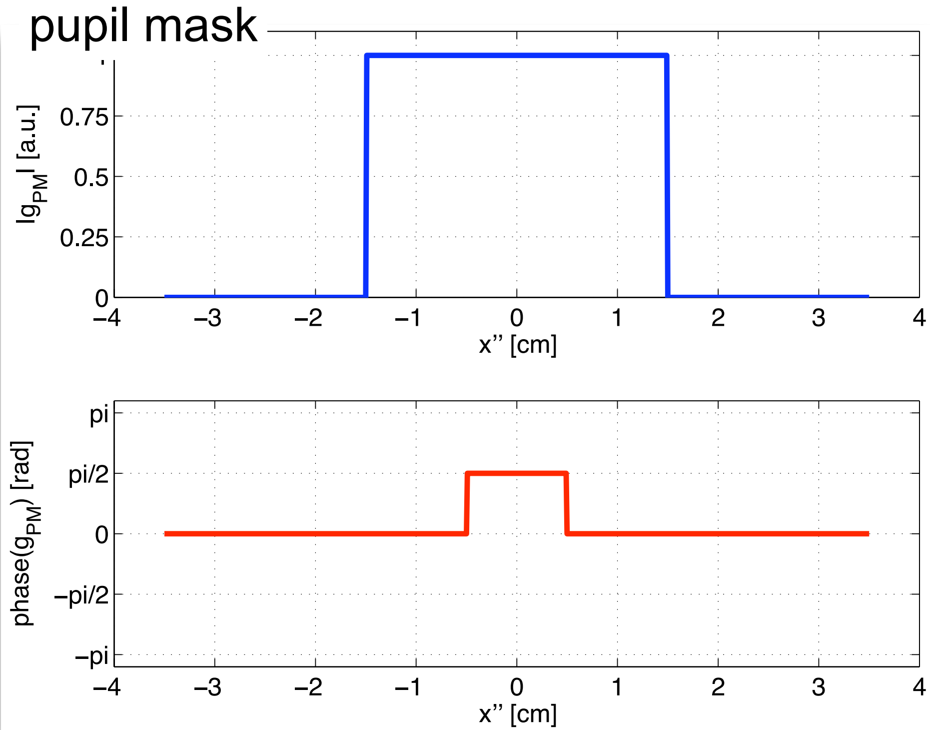
The pupil mask is $g_{\mathrm{PM}}(x'') = \mathrm{rect}\left(\frac{x''}{3\mathrm{cm}}\right)$. If the input transparency is δ(x), the field at the pupil plane to the right of the pupil mask is $g_{\mathrm{PP}+}(x'') = g_{\mathrm{PP}-}(x'')\times g_{\mathrm{PM}}(x'') = 1\times\mathrm{rect}\left(\frac{x''}{3\mathrm{cm}}\right) \Rightarrow G_{\mathrm{PP}+}(u) = (3\ \mathrm{cm})\,\mathrm{sinc}\,(u\times 3\ \mathrm{cm})$.

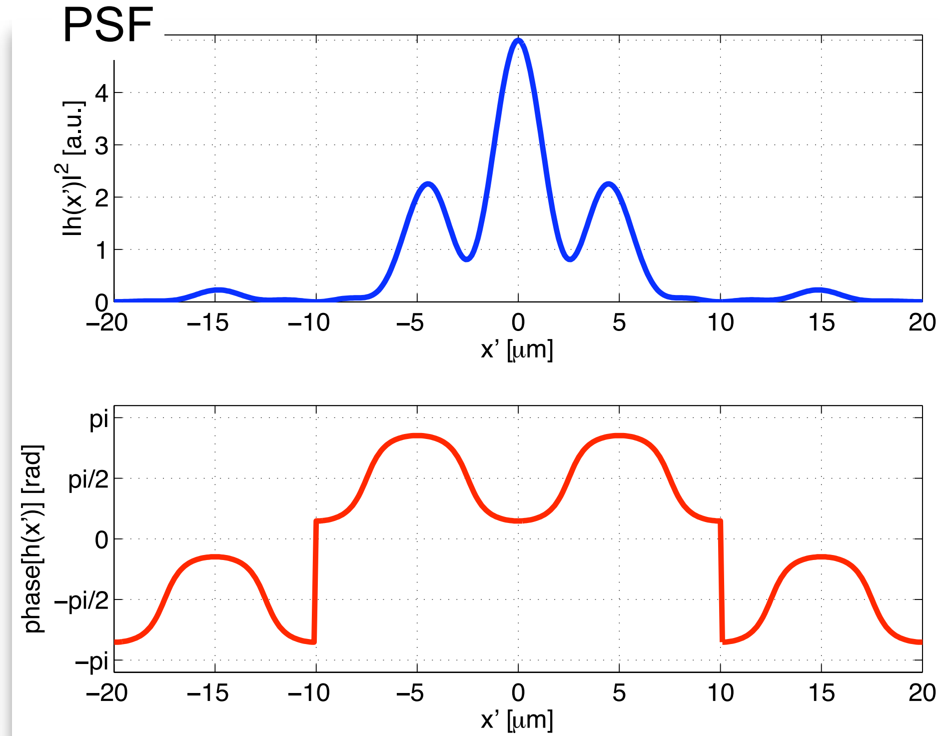
The output field, i.e. the PSF is

$$g_{\mathrm{out}}(x') \equiv h(x') = G_{\mathrm{PM}}\left(\frac{x'}{\lambda f}\right) = (3\ \mathrm{cm})\,\mathrm{sinc}\left(\frac{x'\times 3\ \mathrm{cm}}{0.5\mu\mathrm{m}\times 20\mathrm{cm}}\right) = (3\ \mathrm{cm})\,\mathrm{sinc}\left(\frac{x'}{3.33\mu\mathrm{m}}\right).$$

The scaling factor (3×) in the PSF ensures that the integral $\int |h(x')|^2 \mathrm{d}x$ equals the portion of the input energy transmitted through the system

Example: PSF of a phase pupil filter





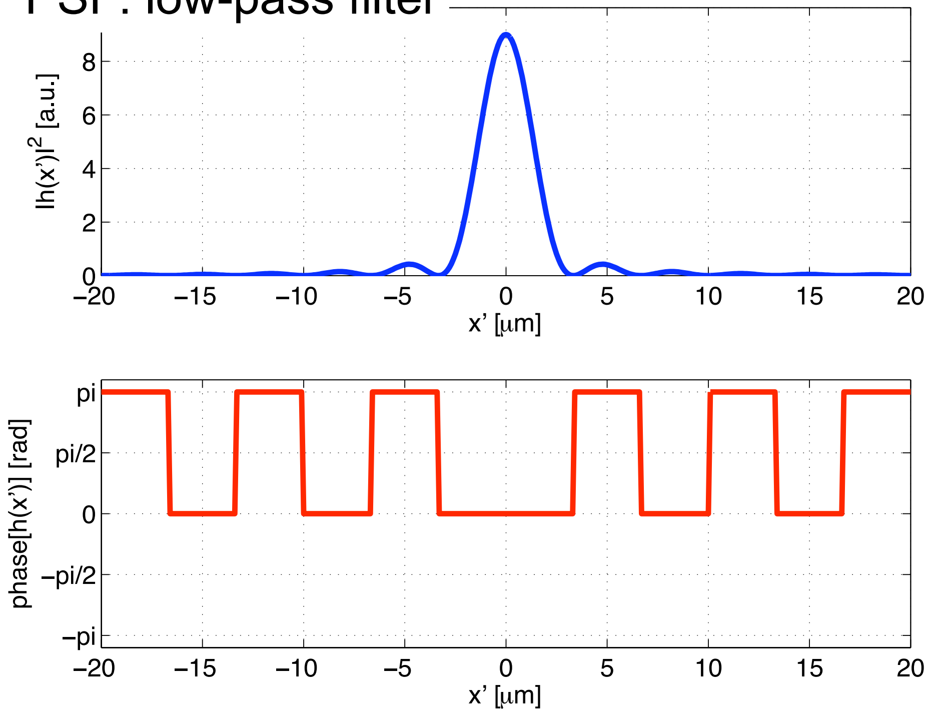
The pupil mask is $g_{\mathrm{PM}}(x'') = \mathrm{rect}\left(\frac{x''}{3\mathrm{cm}}\right) + (i-1)\,\mathrm{rect}\left(\frac{x''}{1\mathrm{cm}}\right)$.

The PSF is

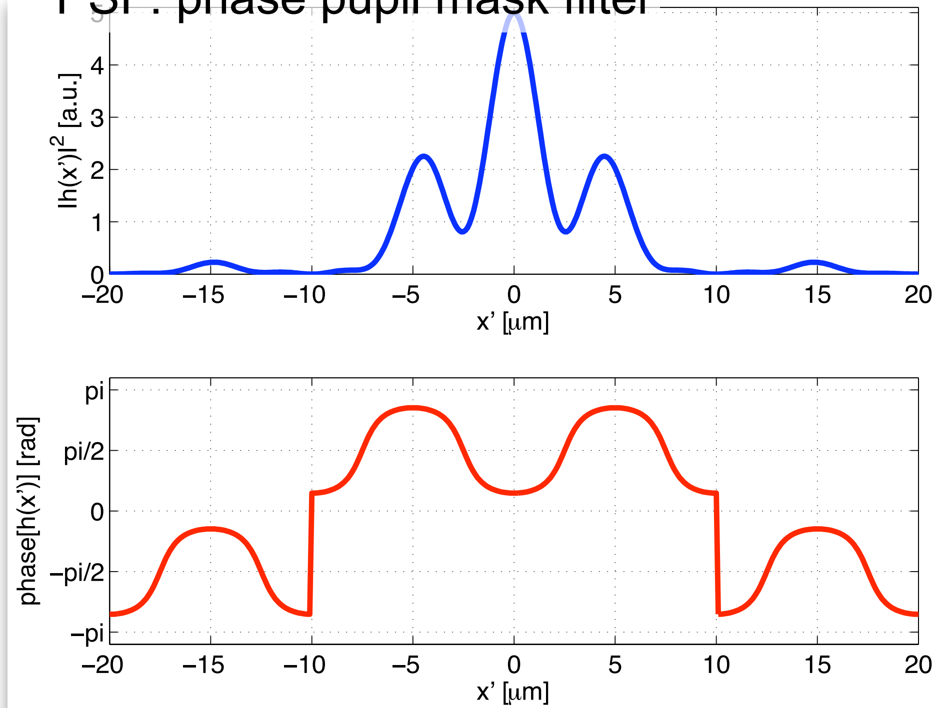
$$
\begin{aligned}
h(x') &= G_{\mathrm{PM}}\left(\frac{x'}{\lambda f}\right) = (3\ \mathrm{cm})\,\mathrm{sinc}\left(\frac{x'}{3.33\mu\mathrm{m}}\right) + (i-1)\times(1\ \mathrm{cm})\,\mathrm{sinc}\left(\frac{x'}{1\mu\mathrm{m}}\right); \\
|h(x')|^2 &= \left[3\,\mathrm{sinc}\left(\frac{x'}{3.33}\right) - \mathrm{sinc}\left(\frac{x'}{1}\right)\right]^2 + \left[\mathrm{sinc}\left(\frac{x'}{1}\right)\right]^2 \\
\angle h(x') &= \arctan\frac{\mathrm{sinc}\,(x')}{3\,\mathrm{sinc}\,(0.3\,x') - \mathrm{sinc}\,(x')}.
\end{aligned}
$$

Comparison: low-pass filter vs phase pupil mask filter

PSF: low-pass filter



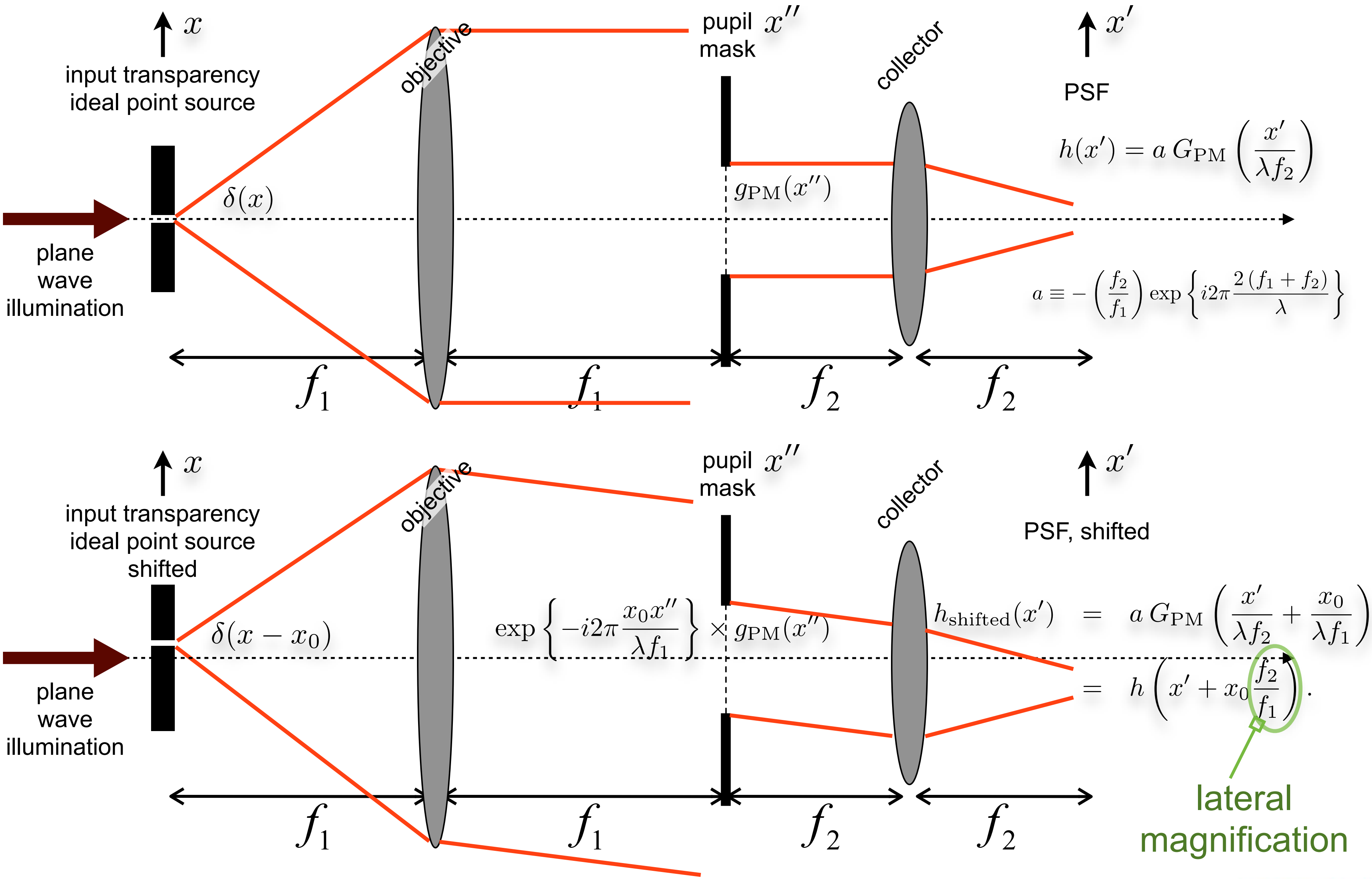
PSF: phase pupil mask filter



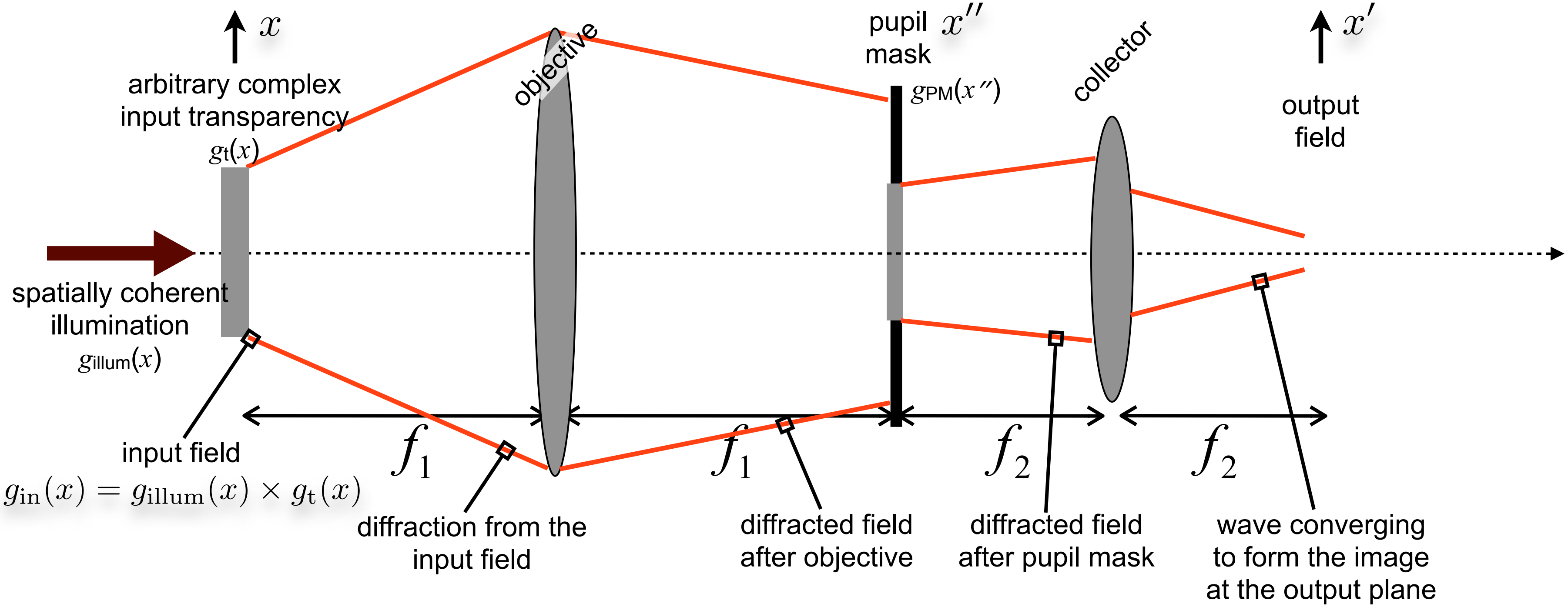
$$
\begin{aligned}
h(x') &= (3\ \text{cm})\,\text{sinc}\left(\frac{x'}{3.33\mu m}\right); \\
|h(x')|^2 &= 9\,\text{sinc}^2\left(\frac{x'}{3.33}\right) \\
\angle h(x') &= \begin{cases} 0, & \text{if sinc}\,(0.3\,x') > 0 \\ \pi, & \text{if sinc}\,(0.3\,x') < 0 \end{cases}.
\end{aligned}
$$

$$
\begin{aligned}
h(x') &= (3\ \text{cm})\,\text{sinc}\left(\frac{x'}{3.33\mu m}\right) + (i-1)\times(1\ \text{cm})\,\text{sinc}\left(\frac{x'}{1\mu m}\right); \\
|h(x')|^2 &= \left[3\,\text{sinc}\left(\frac{x'}{3.33}\right) - \text{sinc}\left(\frac{x'}{1}\right)\right]^2 + \left[\text{sinc}\left(\frac{x'}{1}\right)\right]^2 \\
\angle h(x') &= \arctan\frac{\text{sinc}\,(x')}{3\,\text{sinc}\,(0.3\,x') - \text{sinc}\,(x')}.
\end{aligned}
$$

Shift invariance of the 4F system



The significance of the PSF



Finally, consider the same 4F system with an input transparency whose transmittance is an arbitrary complex function $g_t(x)$ and the field $g_{in}(x)$ immediately to the right of the input transparency.

According to the sifting theorem for δ-functions, we may express $g_{in}(x)$ as $g_{in}(x) = \int_{-\infty}^{+\infty} g_{in}(x_1)\delta(x - x_1)\mathrm{d}x_1$

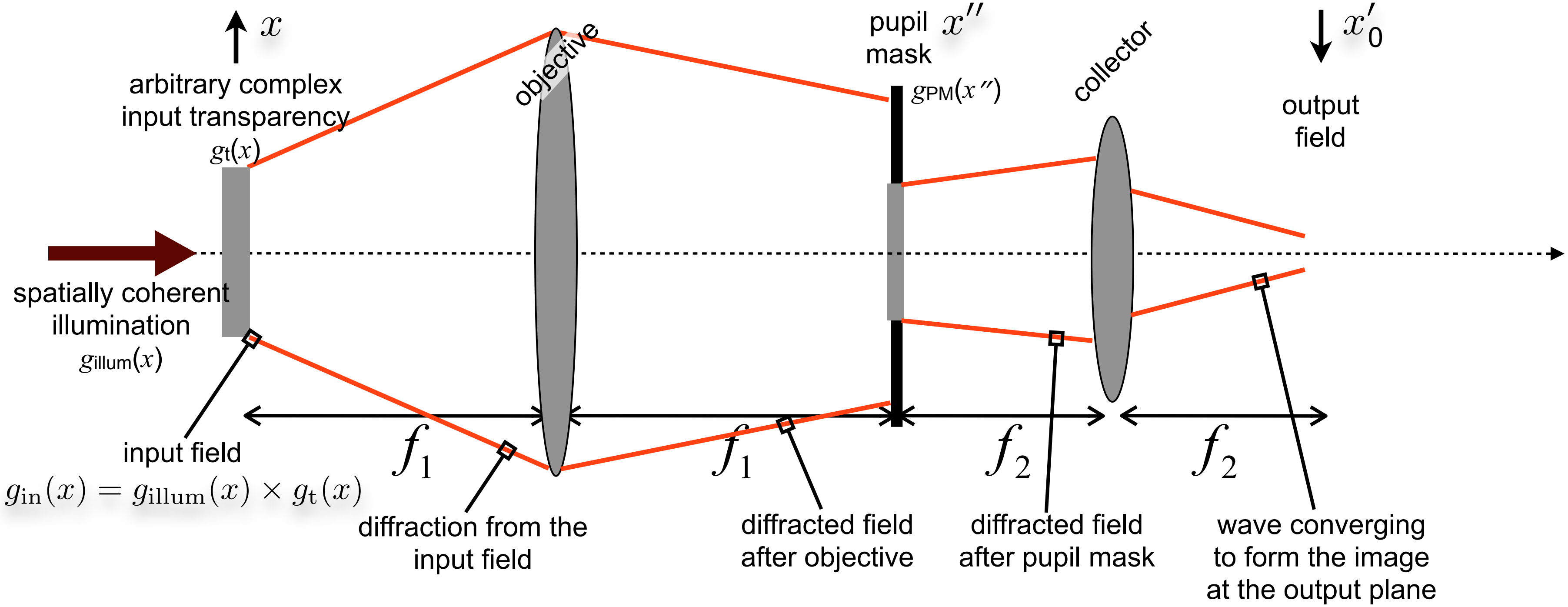
Physically, this expresses a superposition of point sources weighted by the input field, i.e. Huygens' principle.

Using the shift invariance property, we find that the field produced at the output plane by each Huygens point source is the PSF, shifted by $-x_1 \times f_2/f_1$, (note: $-f_2/f_1$ is the lateral magnification) and weighted by $g_{in}(x_1)$; i.e.,

the output field is *almost* a convolution (within a sign & a scaling factor) of the input field with the PSF:

$$g_{out}(x') = \int_{-\infty}^{+\infty} g_{in}(x_1) h\left(x' + x_1 \frac{f_2}{f_1}\right) \mathrm{d}x_1$$

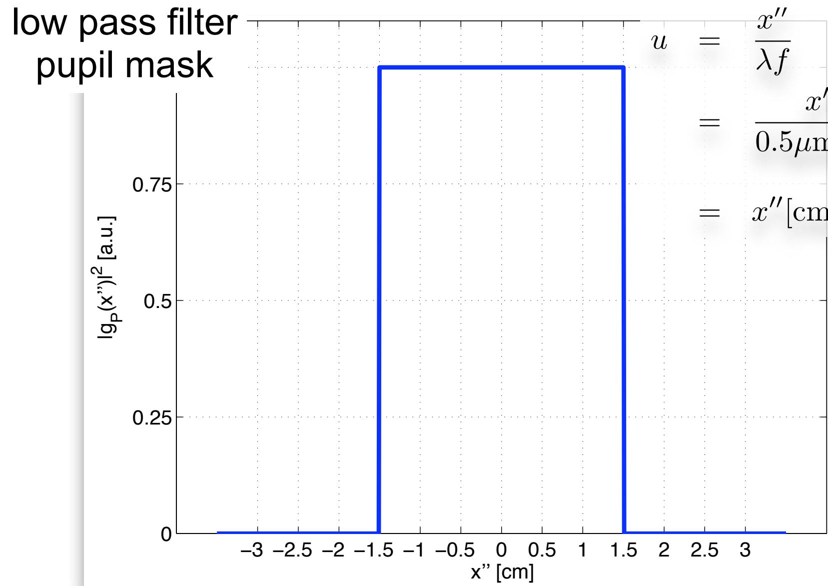
The Amplitude Transfer Function (ATF)



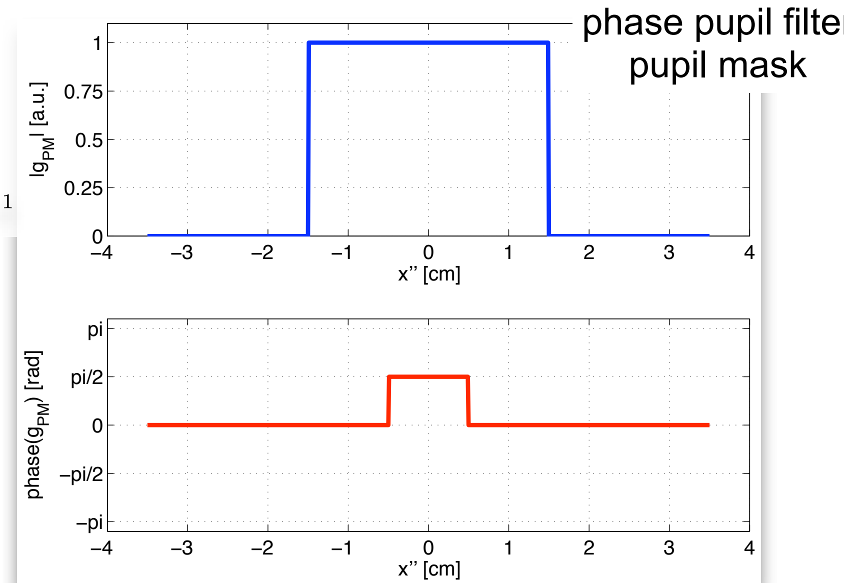
To avoid the mathematical complications of inversion and lateral magnification at the output plane, we define the “reduced” output coordinate $x'_0 = -\frac{f_1}{f_2}x'$. Inverted and re-sampled in the reduced coordinate, the output field is expressed simply as a convolution $g_{out}(x'_0) = \int_{-\infty}^{+\infty} g_{in}(x) h(x'_0 - x)\, dx$.

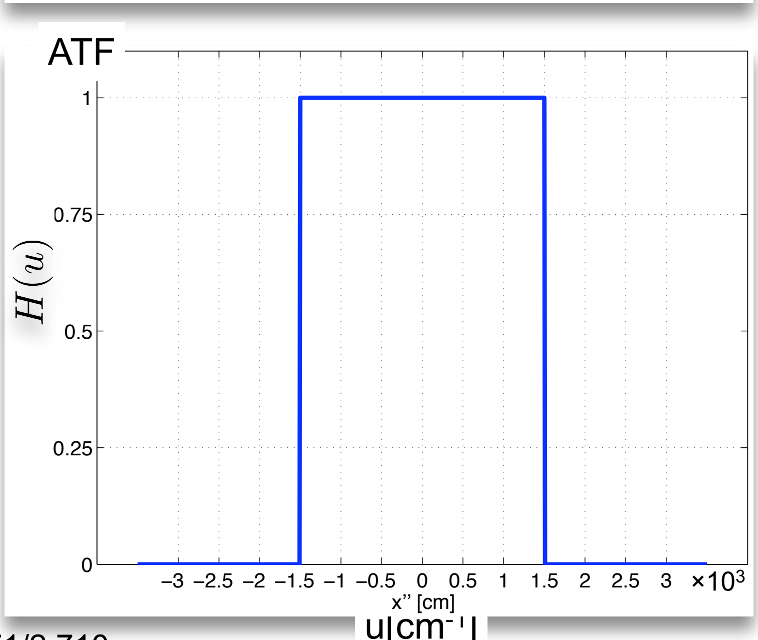
Using the convolution theorem of Fourier transforms, we can re-express this input-output relationship in the spatial frequency domain as $G_{out}(u) = G_{in}(u) H(u)$, where $H(u)$ is the **amplitude transfer function (ATF)**.
Inverting the Fourier transform relationship between $g_{PM}(x'')$ and the PSF, we obtain $H(u) = a\, g_{PM}(\lambda f_1 u)$
That is, the ATF of the 4F system is obtained directly from the pupil mask, via a *coordinate scaling transformation*.

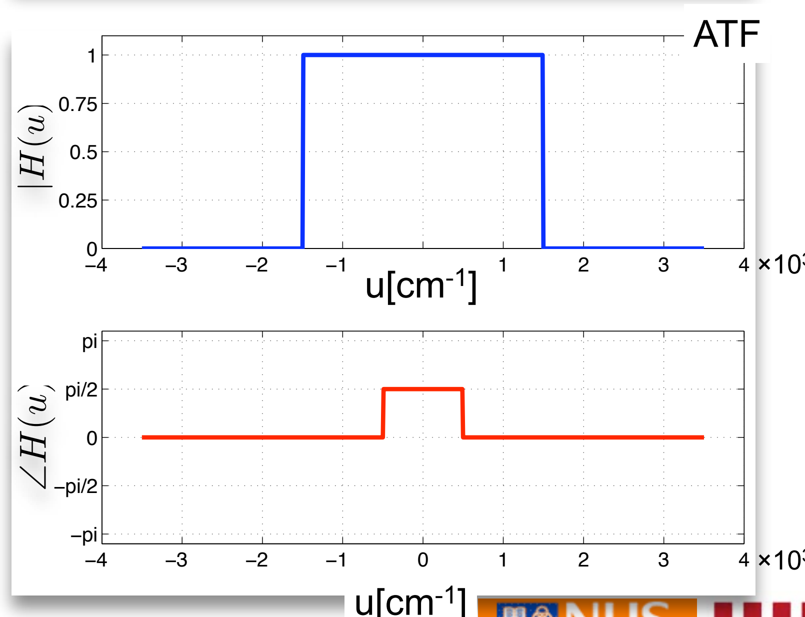
Example: ATFs of the low-pass filter and the phase pupil mask



$$u = \frac{x''}{\lambda f} = \frac{x''[\mathrm{cm}]}{0.5\mu\mathrm{m} \times 20\mathrm{cm}} = x''[\mathrm{cm}] \times 10^3\ \mathrm{cm}^{-1}$$







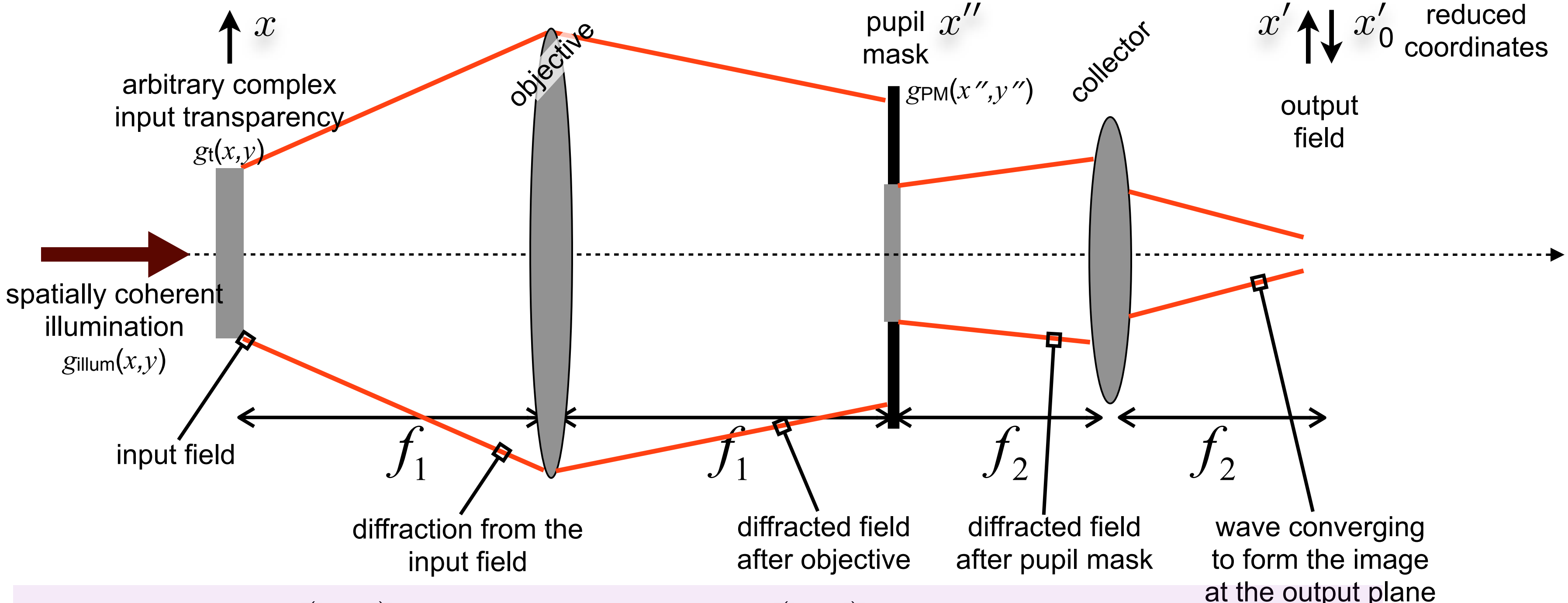
Today

- Lateral and angular magnification
- The Numerical Aperture (NA) revisited
- Sampling the space and frequency domains, and the Space-Bandwidth Product (SBP)
- Pupil engineering

next two weeks

- Depth of focus and depth of field
- The angular spectrum
- “Non-diffracting” beams
- Temporal and spatial coherence
- Spatially incoherent imaging

4F imaging as a linear shift invariant system



Illumination: $g_{\text{illum}}(x, y)$ Input transparency: $g_{\text{t}}(x, y)$

Field to the right of the input transparency (input field): $g_{\text{in}}(x, y) = g_{\text{illum}}(x, y) \times g_{\text{t}}(x, y)$

Pupil mask: $g_{\text{PM}}(x'', y'')$ Amplitude transfer function (ATF): $H(u, v) \propto g_{\text{PM}}(\lambda f_1 u, \lambda f_1 v)$

Point Spread Function (PSF):

$$h(x', y') \propto G_{\text{PM}}\left(\frac{x'}{\lambda f_2}, \frac{y'}{\lambda f_2}\right)$$ in actual coordinates

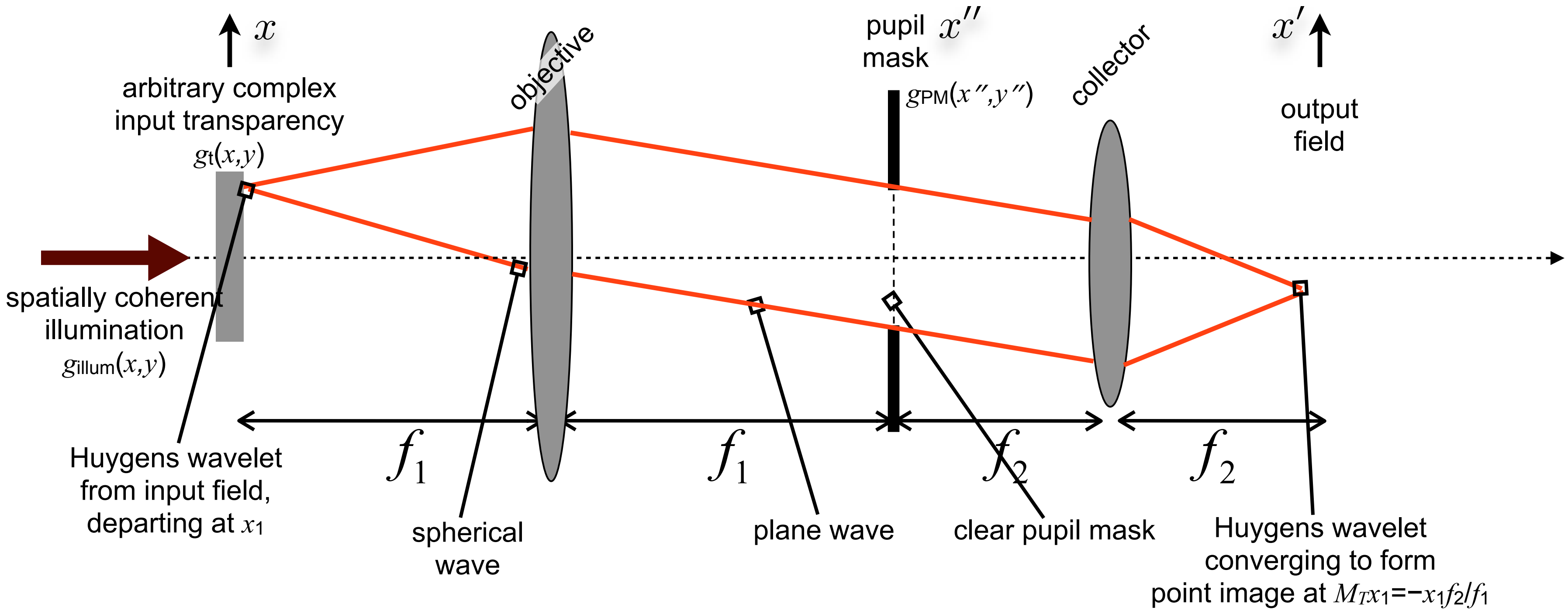
$$h(x'_0, y'_0) \propto G_{\text{PM}}\left(-\frac{x'_0}{\lambda f_1}, -\frac{y'_0}{\lambda f_1}\right)$$ in reduced coordinates

Output field:

$$g_{\text{out}}(x', y') = \iint g_{\text{in}}(x, y) h\left(x' + \frac{f_2}{f_1}x, y' + \frac{f_2}{f_1}y\right) \mathrm{d}x\, \mathrm{d}y$$ in actual coordinates

$$g_{\text{out}}(x'_0, y'_0) \propto \iint g_{\text{in}}(x, y) h(x'_0 - x, y'_0 - y)\, \mathrm{d}x\, \mathrm{d}y$$ in reduced coordinates

Lateral magnification



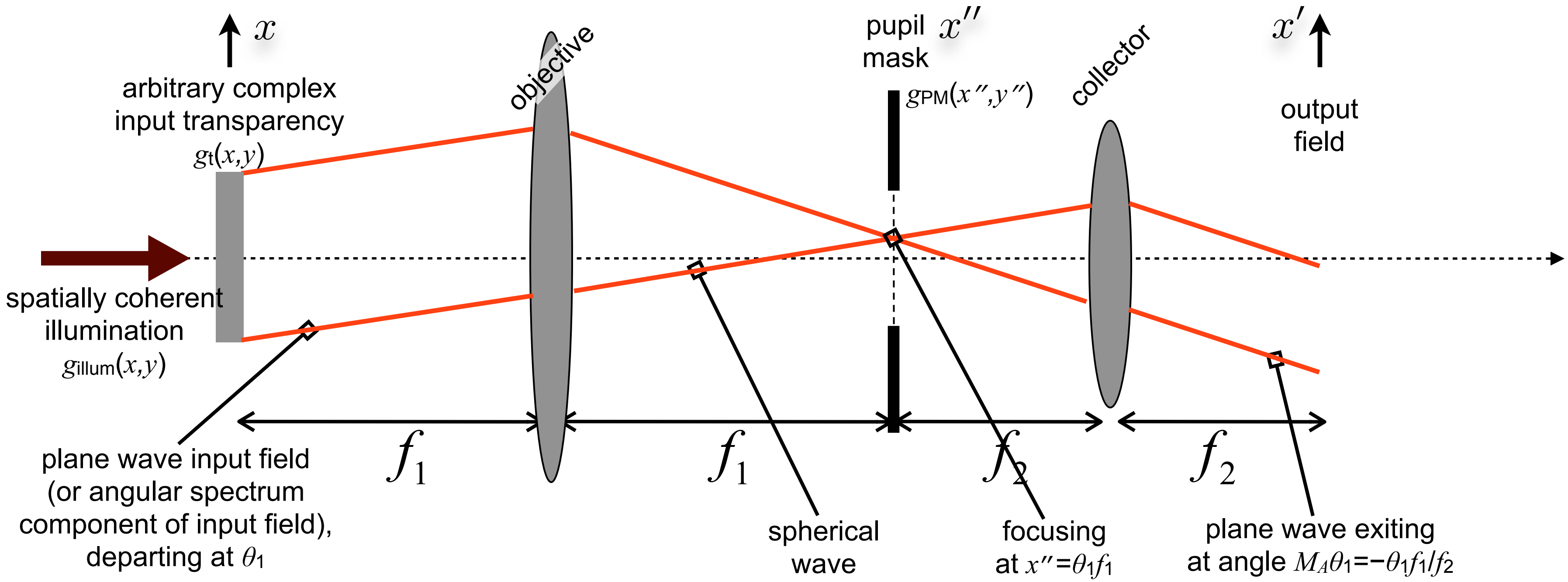
An ideal imaging system with a point source as input field should form a point image at the output plane. Therefore, the ideal PSF is a δ-function. This is of course the limit of Geometrical Optics and in practice unachievable because of diffraction; alternatively, it implies that the spatial bandwidth is infinite, or that the lateral extent of the pupil mask is infinite. Both conditions are non-physical; however, for the purpose of calculating the geometrical magnification, let us indeed assume that

$$h(x',y') = \delta(x',y') \Rightarrow g_{\text{out}}(x',y') = \iint g_{\text{in}}(x,y)\delta\left(x' + \frac{f_2}{f_1}x, y' + \frac{f_2}{f_1}y\right)\mathrm{d}x\mathrm{d}y \Rightarrow g_{\text{out}}(x',y') \boxed{\propto} g_{\text{in}}\left(-\frac{f_1}{f_2}x', -\frac{f_1}{f_2}y'\right)$$

So our calculation has reproduced the Geometrical Optics result for a telescope with finite conjugates:

$$M_T = -\frac{f_2}{f_1}$$

Angular magnification



Now let us consider a plane wave input field $\exp\left\{i2\pi\frac{\theta_1}{\lambda}x\right\} = \exp\{i2\pi u_1 x\};\qquad u_1 \equiv \frac{\theta_1}{\lambda}$ spatial frequency

Still assuming the ideal geometrical PSF, the output field is

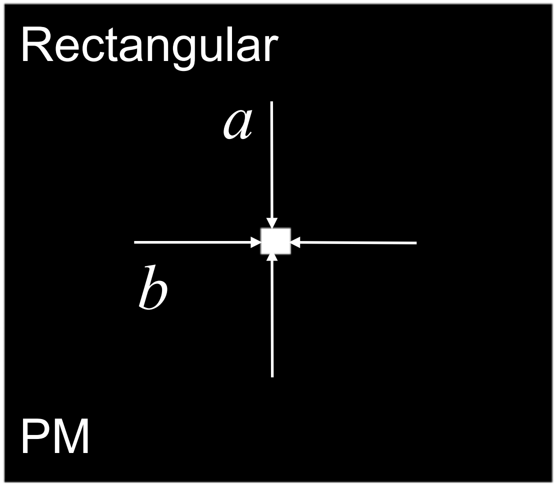
$$g_{\text{out}}(x') \boxed{\propto} g_{\text{in}}\left(-\frac{f_1}{f_2}x'\right) = \exp\left\{i2\pi\frac{\theta_1}{\lambda}\left(-\frac{f_1}{f_2}\right)x'\right\} = \exp\left\{i2\pi\frac{1}{\lambda}\left(-\frac{f_1}{f_2}\theta_1\right)x'\right\}$$

So the angular magnification is $M_A = -\frac{f_1}{f_2}$ again in agreement with Geometrical Optics.

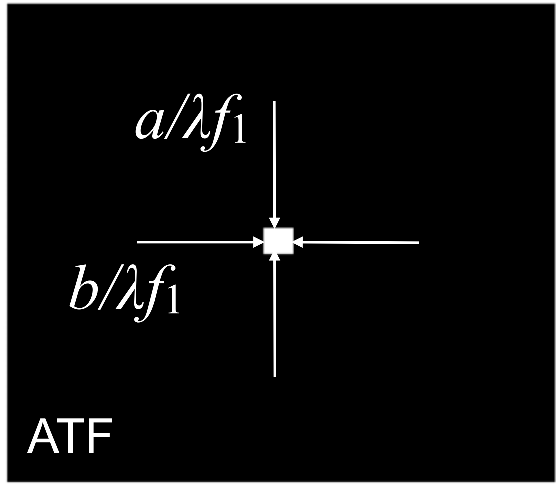
Based on the interpretation of propagation angle as spatial frequency, the magnification results are also in agreement with the scaling (similarity) theorem of Fourier transforms:

$$g_{\text{out}}(x',y') \boxed{\propto} g_{\text{in}}\left(-\frac{f_1}{f_2}x', -\frac{f_1}{f_2}y'\right) \Leftrightarrow G_{\text{out}}(u,v) \boxed{\propto} \left(\frac{f_2}{f_1}\right)^2 G_{\text{in}}\left(-\frac{f_2}{f_1}u, -\frac{f_2}{f_1}v\right)$$

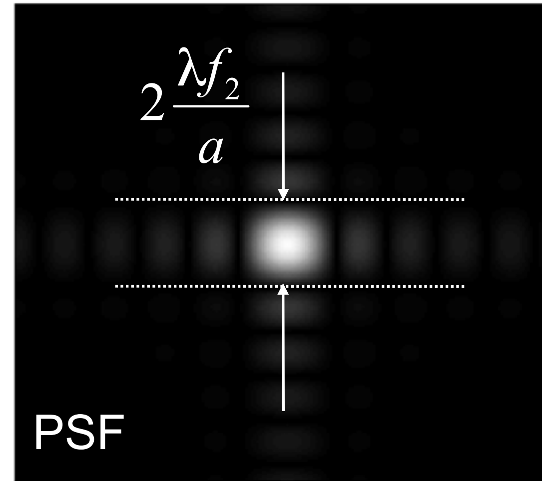
Example: PM, ATF and PSF for clear apertures



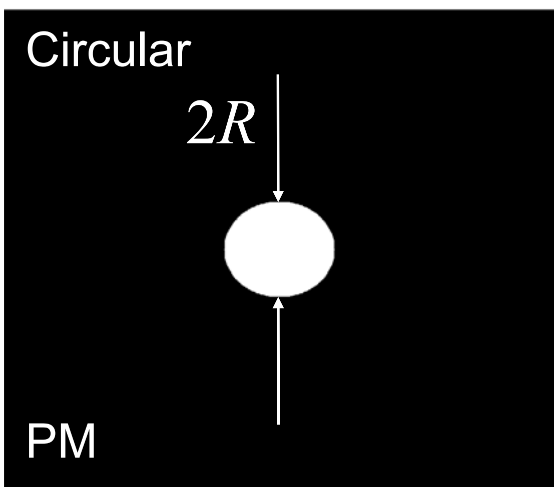
$$\mathrm{rect}\left(\frac{x}{a}\right)\mathrm{rect}\left(\frac{y}{b}\right)$$



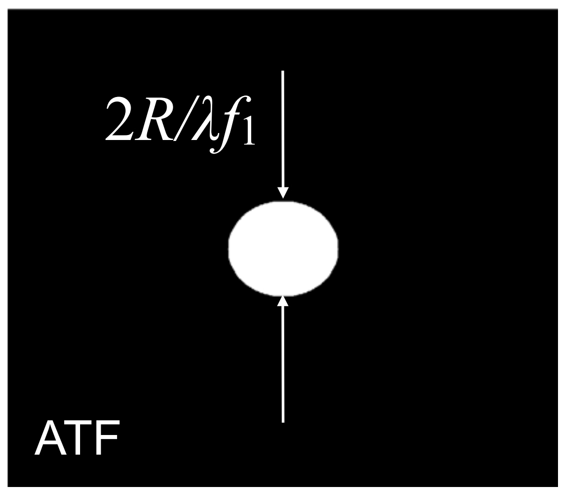
$$\mathrm{rect}\left(\frac{\lambda f_1 u}{a}\right)\mathrm{rect}\left(\frac{\lambda f_1 v}{b}\right)$$



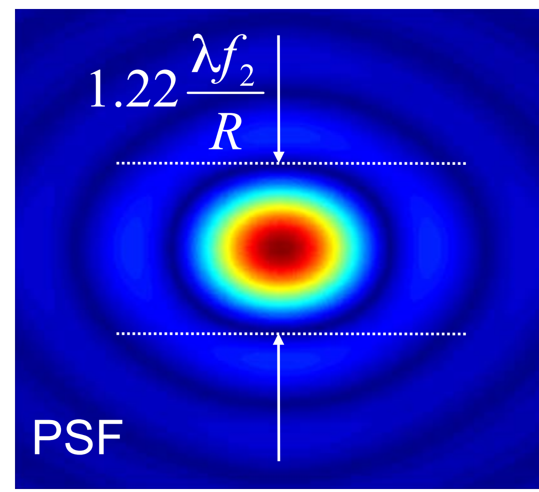
$$\mathrm{sinc}\left(\frac{ax'}{\lambda f_2}\right)\mathrm{sinc}\left(\frac{by'}{\lambda f_2}\right)$$



$$\mathrm{circ}\left(\frac{\sqrt{x^2+y^2}}{R}\right)$$

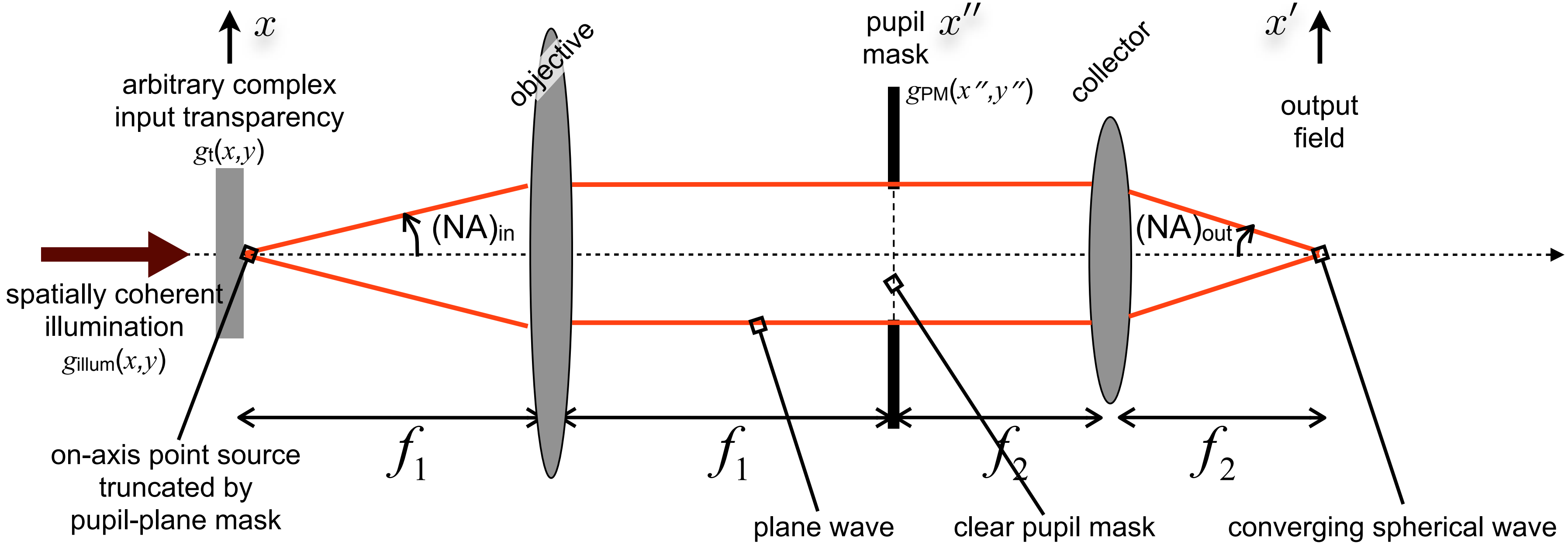


$$\mathrm{circ}\left(\frac{\lambda f_1\sqrt{u^2+v^2}}{R}\right)$$



$$\mathrm{jinc}\left(\frac{a\sqrt{x'^2+y'^2}}{\lambda f_2}\right)$$

Numerical aperture (NA) and PSF size

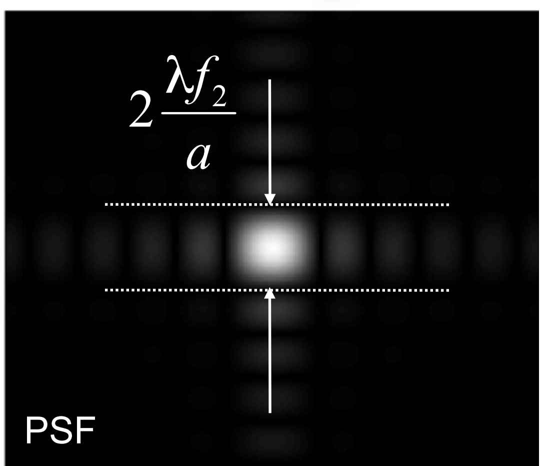


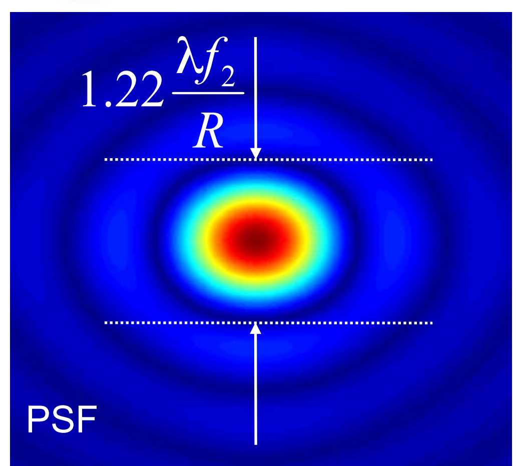
The pupil mask is the system aperture (assuming that the lenses are sufficiently large); therefore,

$$(\mathrm{NA})_{\mathrm{in}} = \frac{a}{f_1}; \qquad (\mathrm{NA})_{\mathrm{out}} = \frac{a}{f_2} = |M_A| \times (\mathrm{NA})_{\mathrm{in}}$$

Let $\Delta x'$ denote the half-size of the PSF main lobe; then,

$$\Delta x' = \frac{\lambda}{(\mathrm{NA})_{\mathrm{out}}}$$

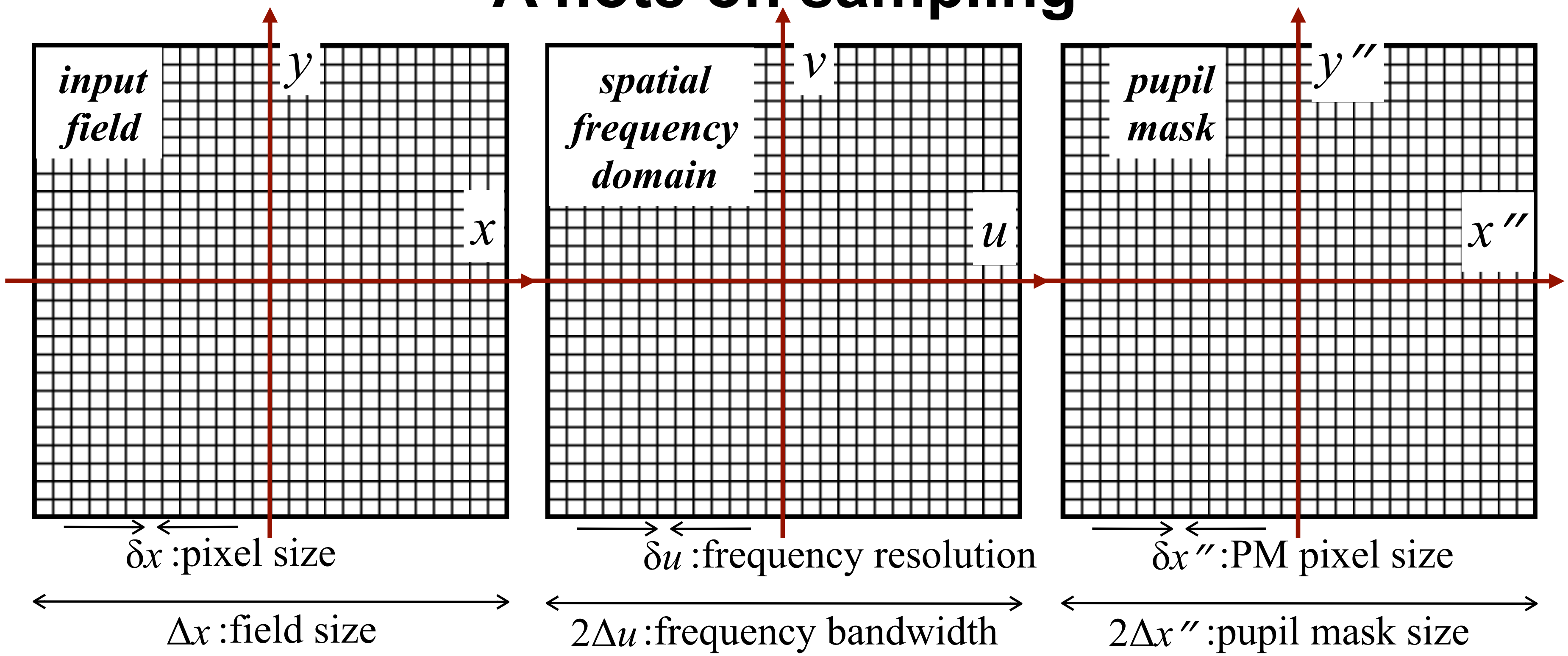




Let $\Delta r'$ denote the radius of the PSF main lobe; then,

$$\Delta r' = 0.61\frac{\lambda}{(\mathrm{NA})_{\mathrm{out}}}$$

A note on sampling



Nyquist sampling relationships

$$\Delta u = \frac{1}{2\delta x} \qquad \Delta x = \frac{1}{\delta u}$$

$$\Delta v = \frac{1}{2\delta y} \qquad \Delta y = \frac{1}{\delta v}$$

$$\Delta x'' = \frac{\lambda f_1}{2\delta x} \qquad \Delta x = \frac{\lambda f_1}{\delta x''}$$

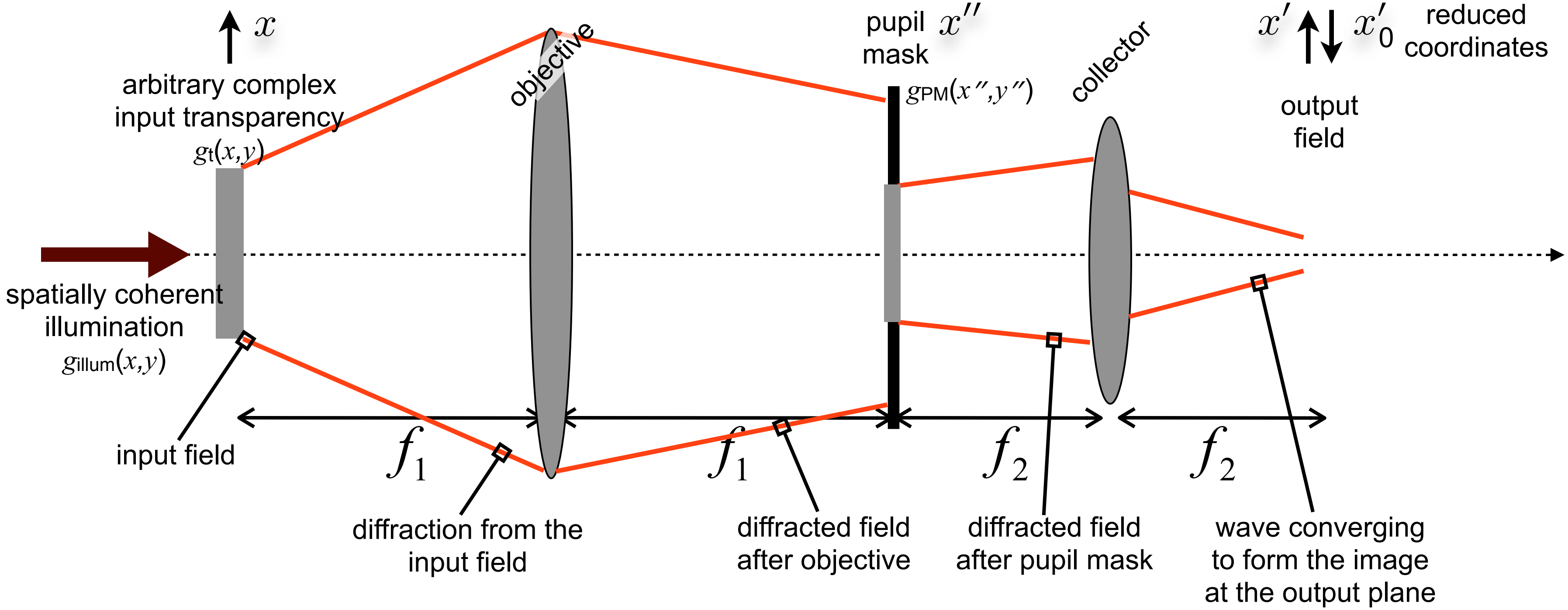
$$\Delta y'' = \frac{\lambda f_1}{2\delta y} \qquad \Delta y = \frac{\lambda f_1}{\delta y''}$$

$$\frac{\Delta x}{\delta x} = \frac{2\Delta u}{\delta u} = \frac{2\Delta x''}{\delta x''} \equiv N_x$$

$$\frac{\Delta y}{\delta y} = \frac{2\Delta v}{\delta v} = \frac{2\Delta y''}{\delta y''} \equiv N_y$$

Space-Bandwidth Product (SBP)

Pupil engineering



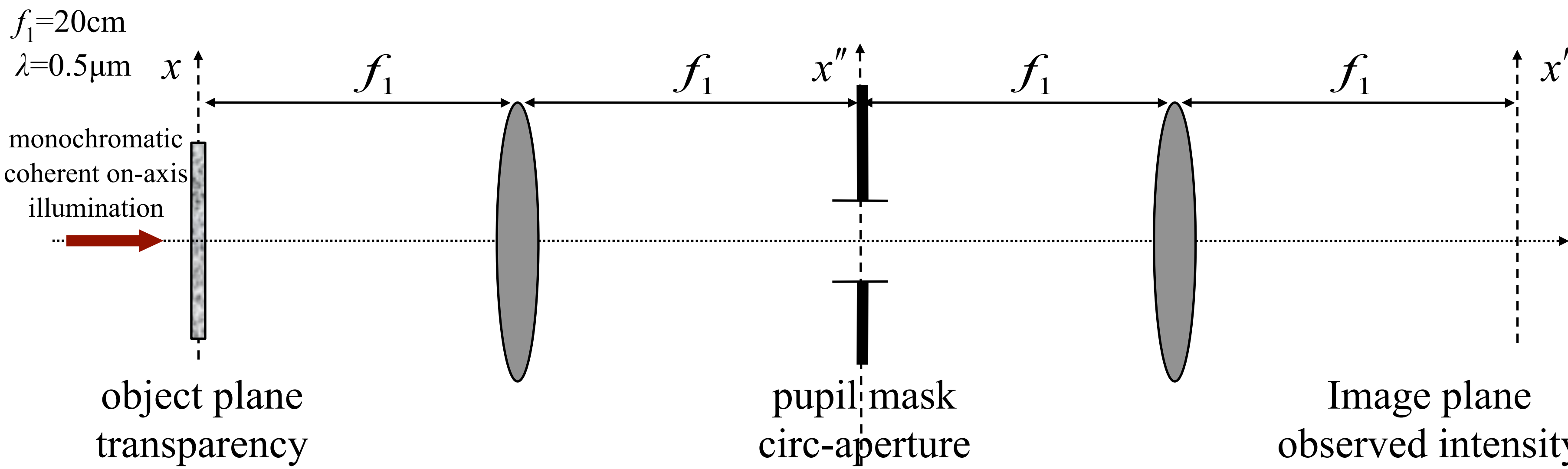
Amplitude transfer function (ATF): $H(u,v) \propto g_{PM}(\lambda f_1 u, \lambda f_1 v)$

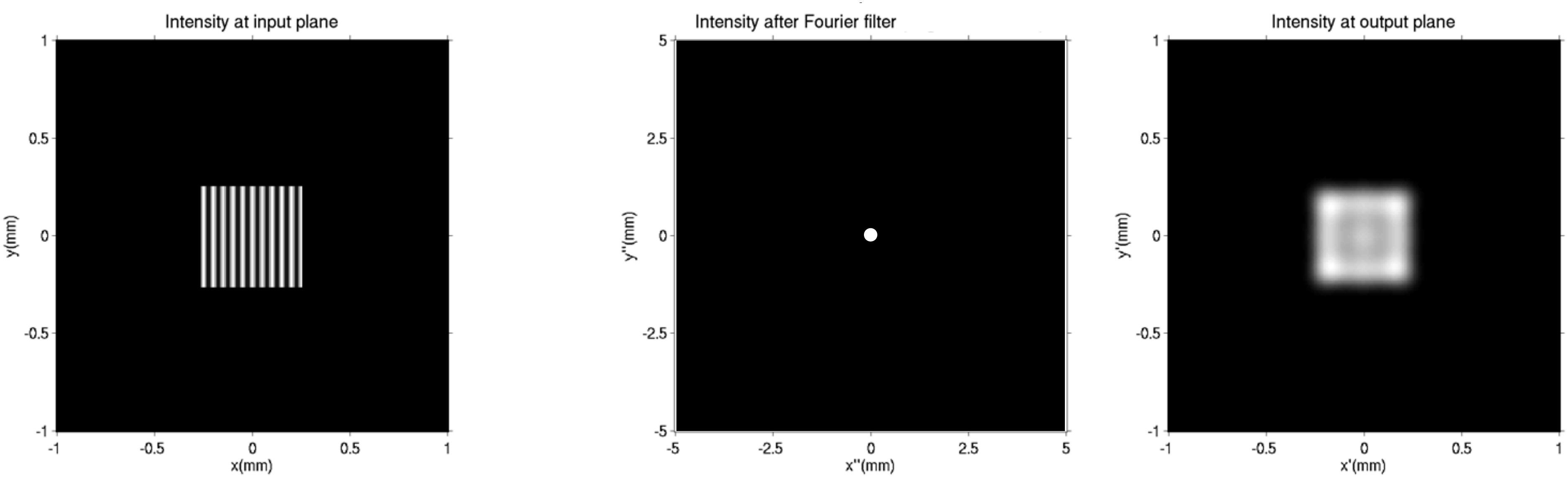
Point Spread Function (PSF): $h(x',y') \propto G_{PM}\left(\frac{x'}{\lambda f_2}, \frac{y'}{\lambda f_2}\right)$ in actual coordinates

Output field: $g_{out}(x',y') = \iint g_{in}(x,y) h\left(x' + \frac{f_2}{f_1}x, y' + \frac{f_2}{f_1}y\right) \mathrm{d}x\, \mathrm{d}y$ in actual coordinates

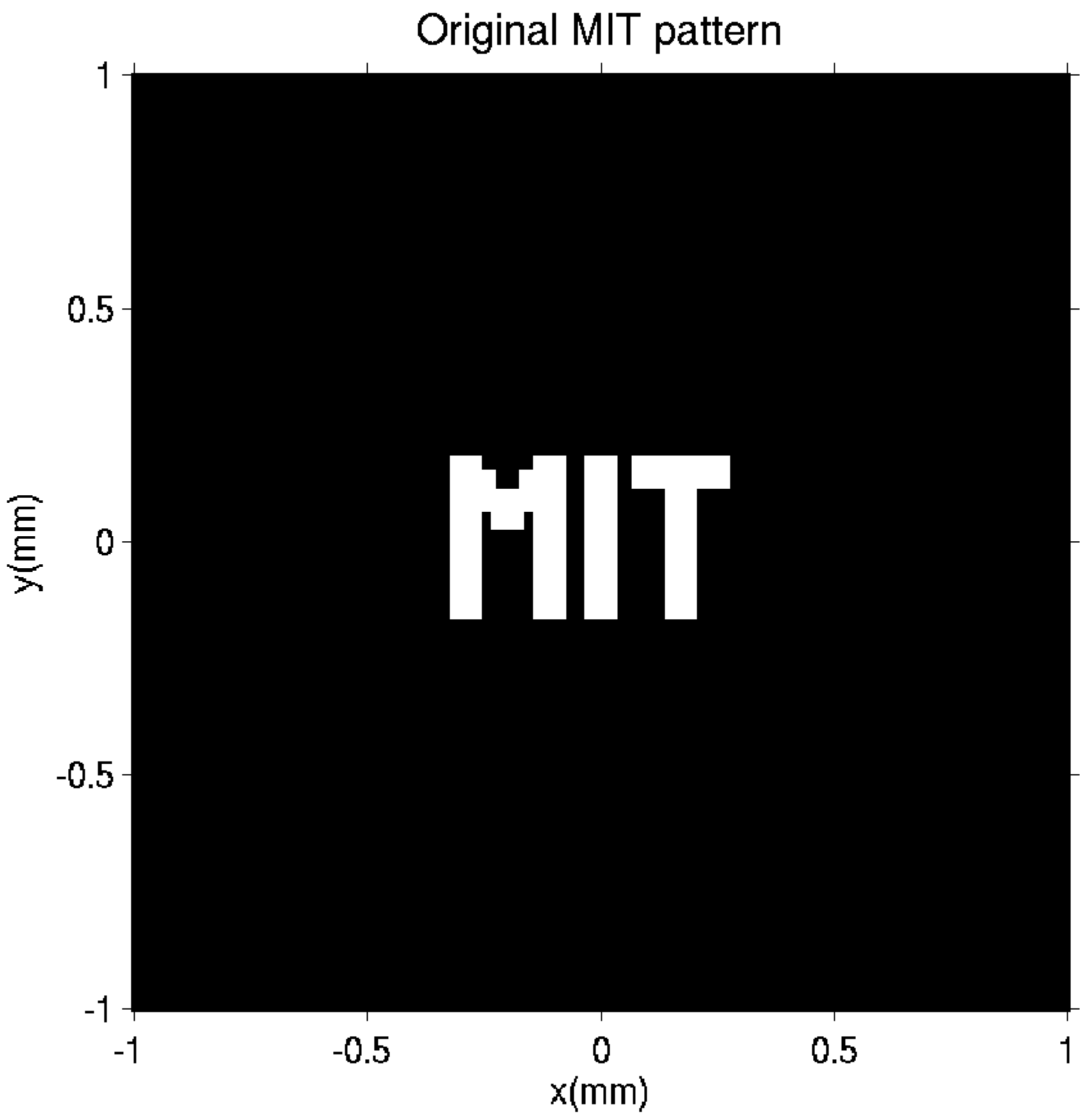
Pupil engineering is the design of a pupil mask $g_{PM}(x'',y'')$ such that the ATF, the PSF or the output field meet requirement(s) specified by the user of the imaging system

Example: spatial frequency clipping

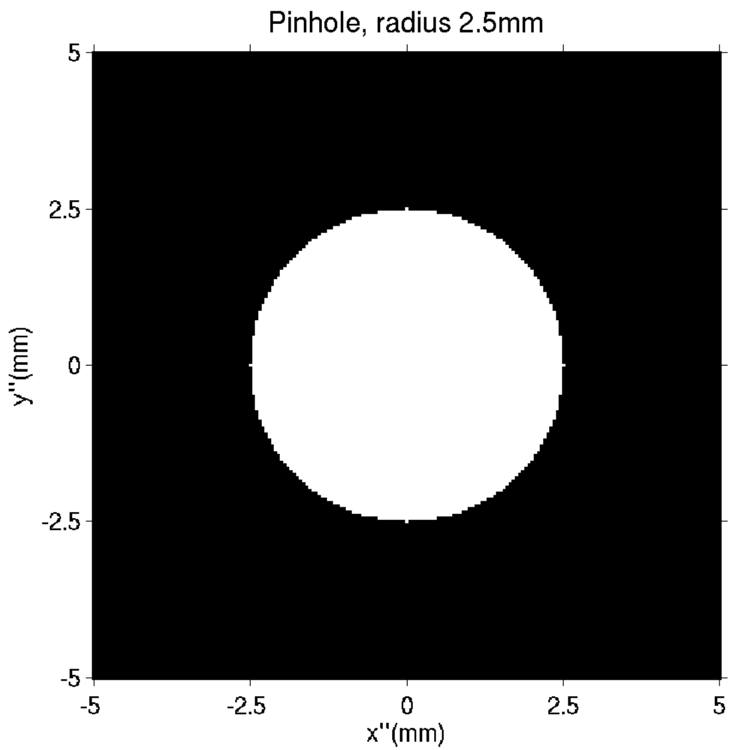


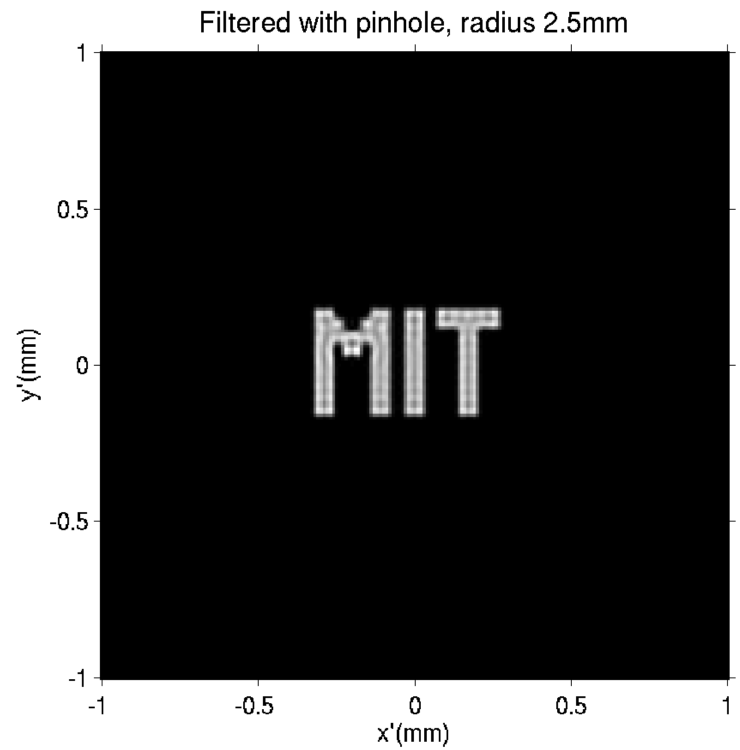


Spatial filtering examples: the amplitude MIT pattern



Weak low-pass filtering





f_1=20cm

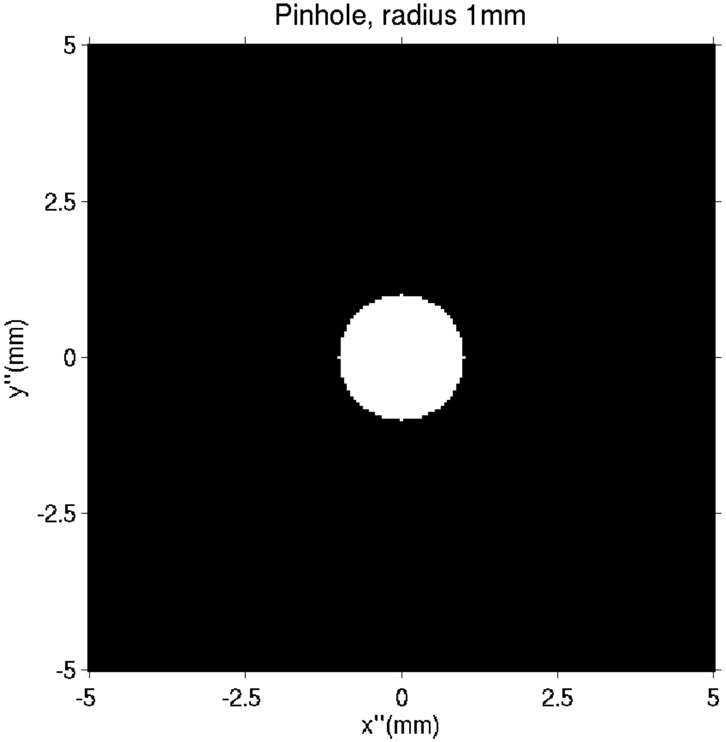
λ=0.5µm

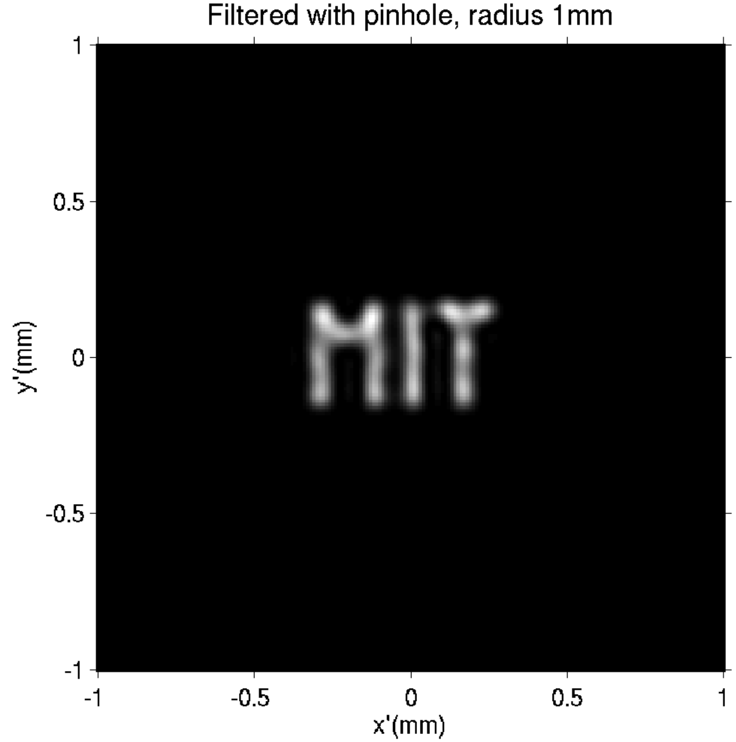
pupil mask

Intensity @ image plane

Moderate low-pass filtering

(moderate blurring)





f_1=20cm

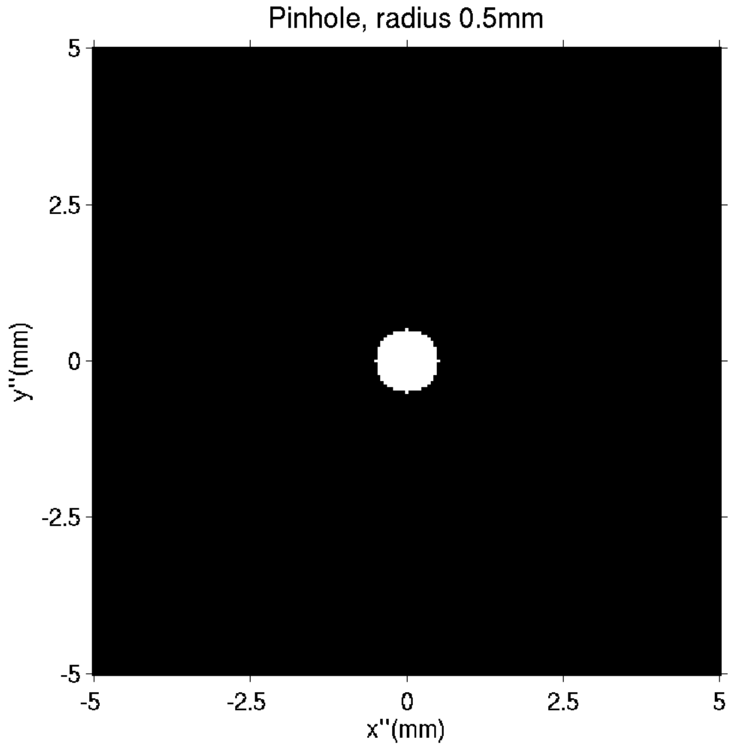
λ=0.5μm

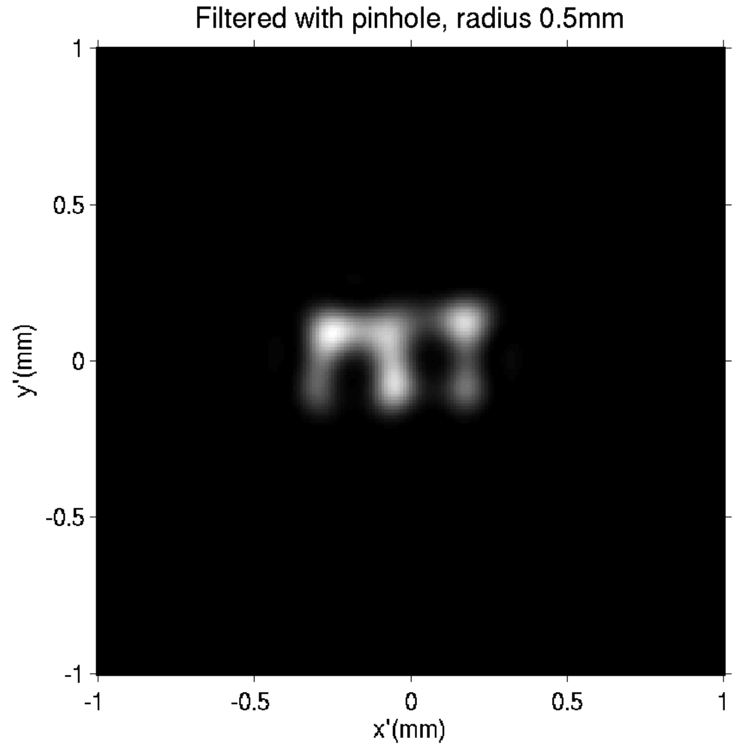
pupil mask

Intensity @ image plane

Strong low-pass filtering

(strong blurring)





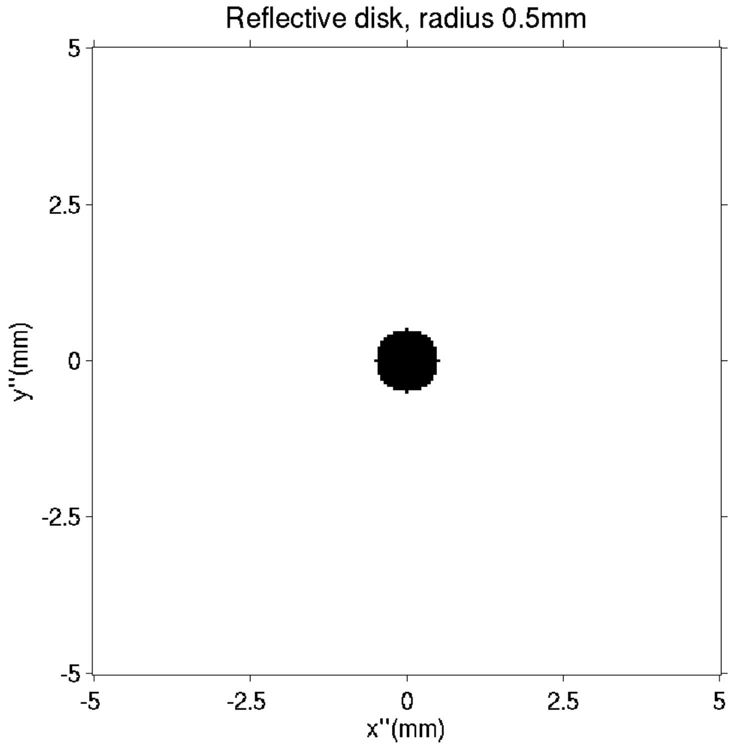
f_1=20cm

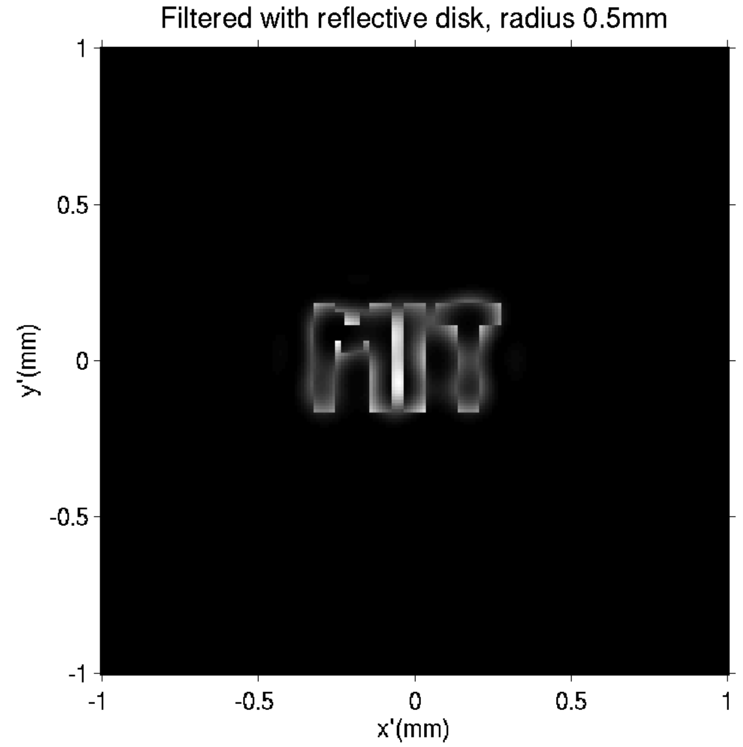
λ=0.5µm

pupil mask

Intensity @ image plane

Moderate high-pass filtering





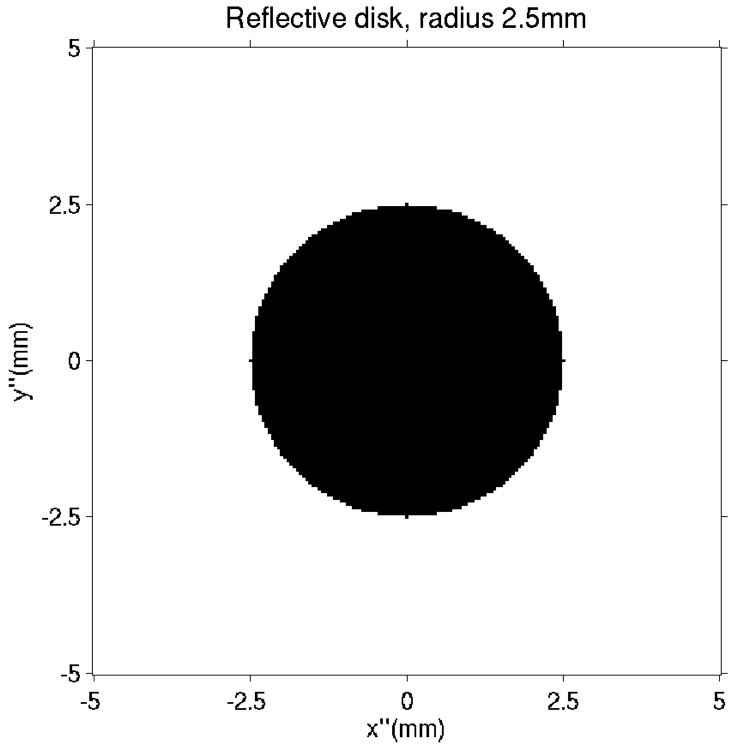
f_1=20cm
λ=0.5μm

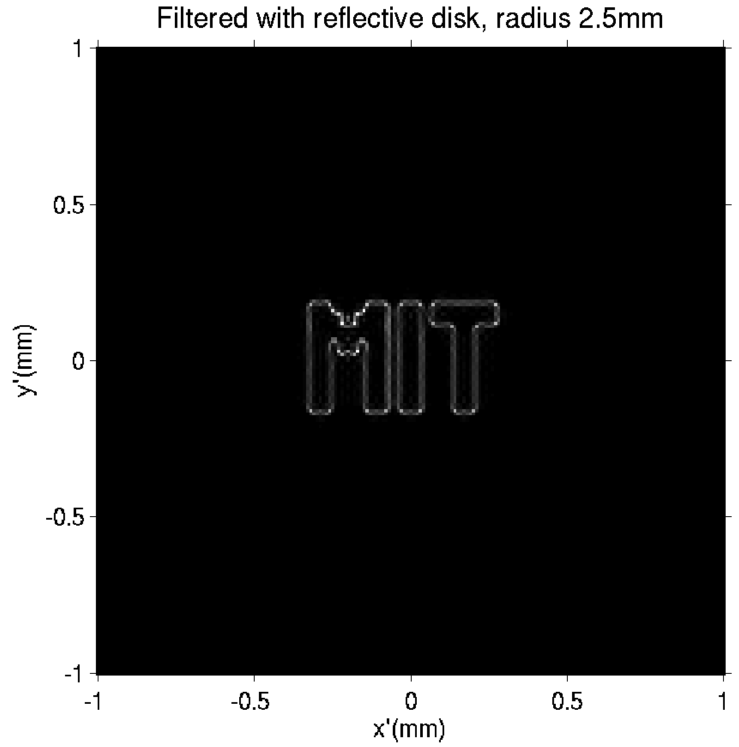
pupil mask

Intensity @ image plane

Strong high-pass filtering

(edge enhancement)





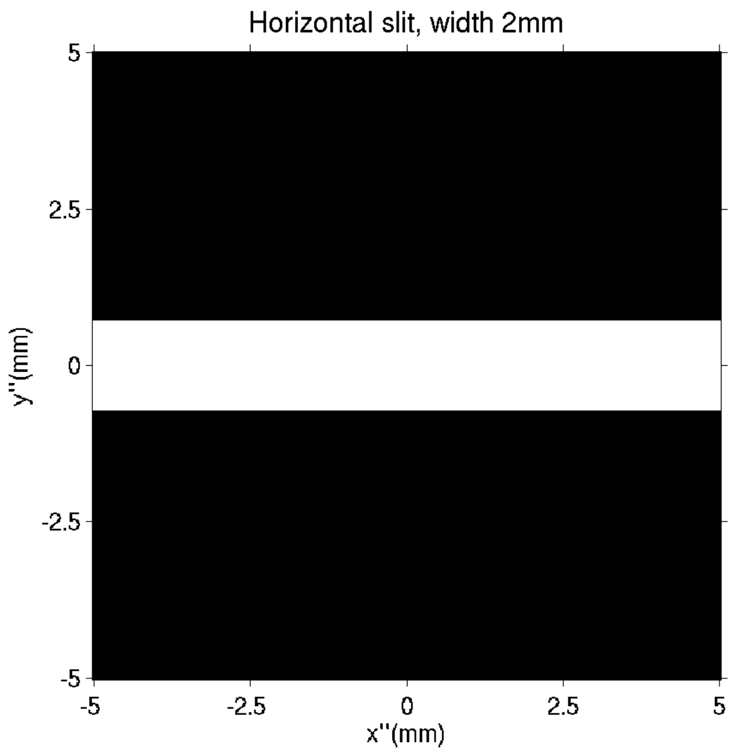
f_1=20cm

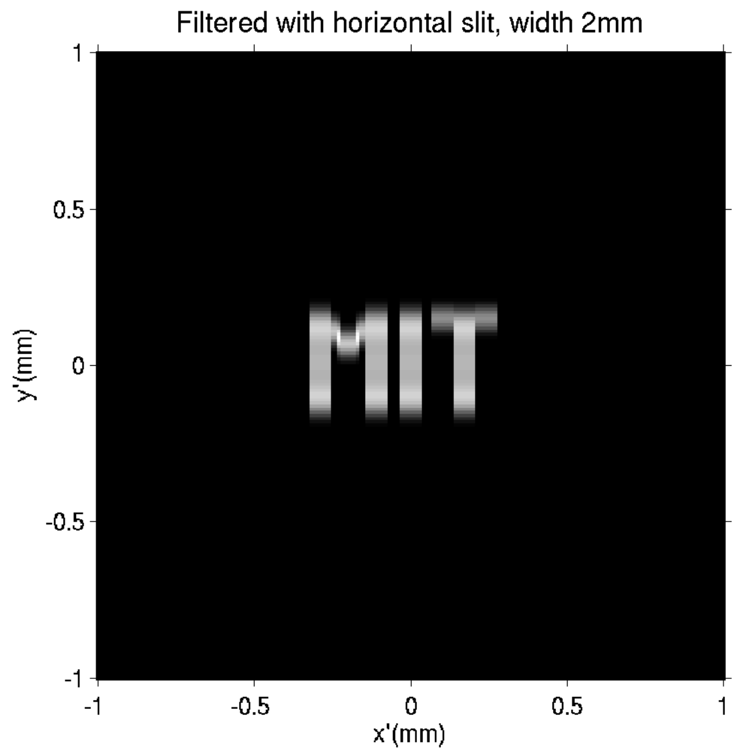
λ=0.5µm

pupil mask

Intensity @ image plane

One-dimensional (1D) blur: vertical



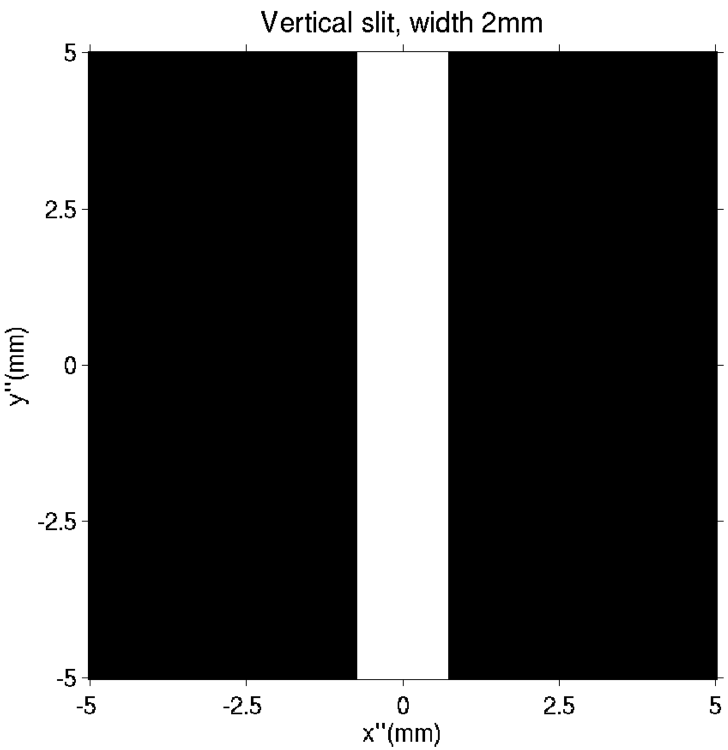


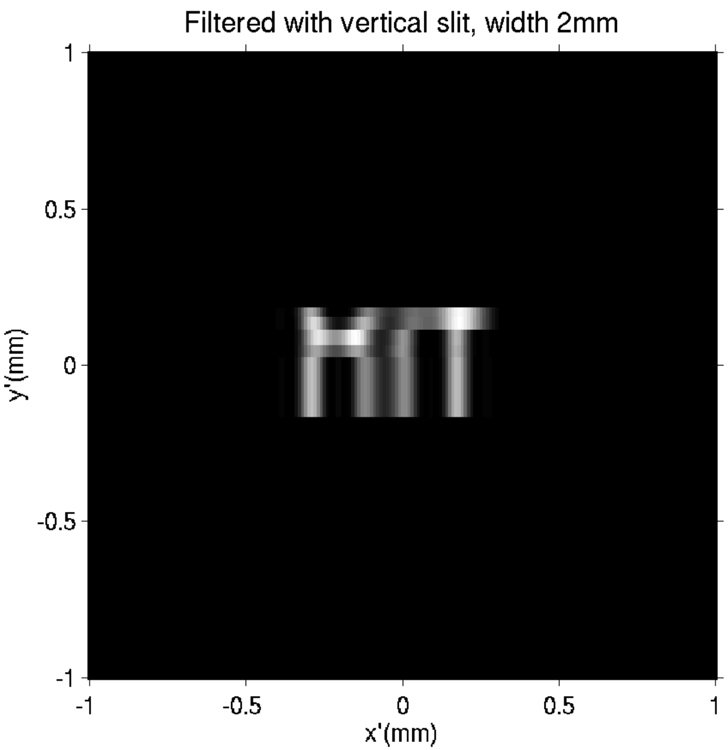
f_1=20cm
λ=0.5μm

pupil mask

Intensity @ image plane

One-dimensional (1D) blur: horizontal





f_1=20cm

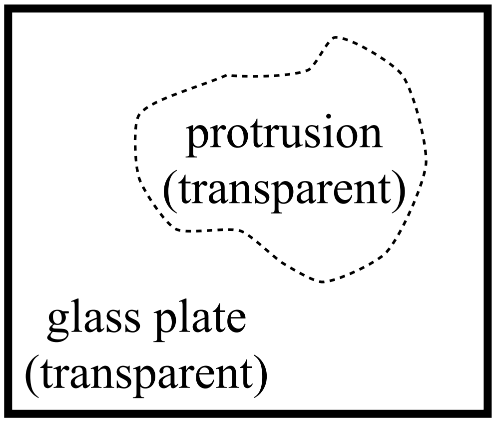
λ=0.5μm

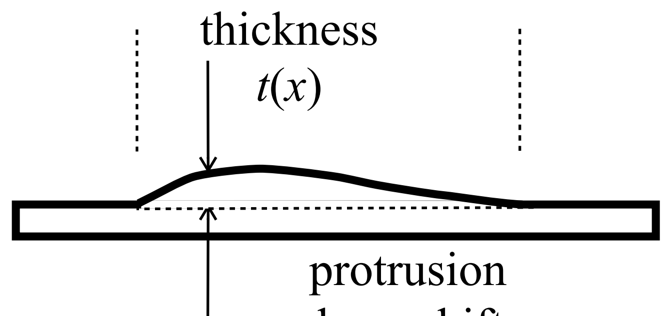
pupil mask

Intensity @ image plane

Phase objects

TOP VIEW





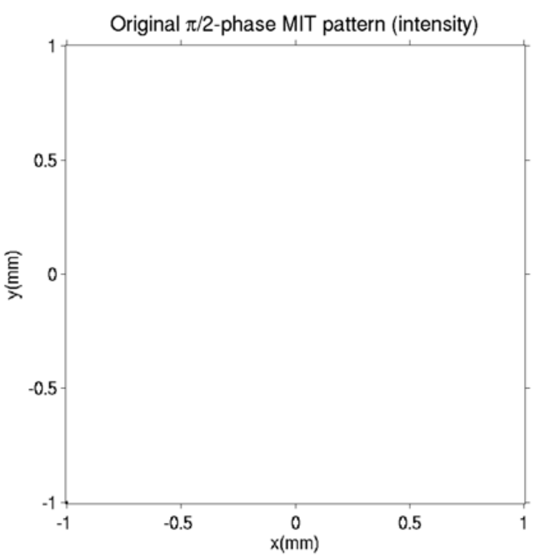
CROSS SECTION

protrusion phase-shifts coherent illumination by amount $\varphi(x) = 2\pi(n-1)t(x)/\lambda$

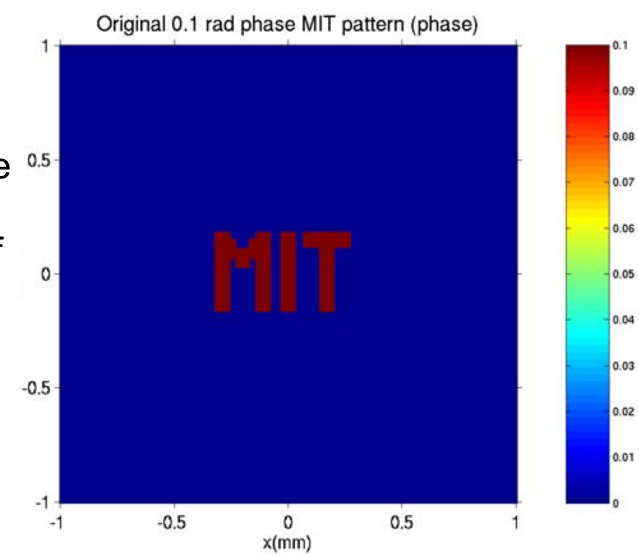
This model is often useful for imaging biological objects (cells, etc.)

$$g_{\text{in}}(x,y) = g_{\text{illum}}(x,y) \times \exp\{i\phi(x,y)\}$$

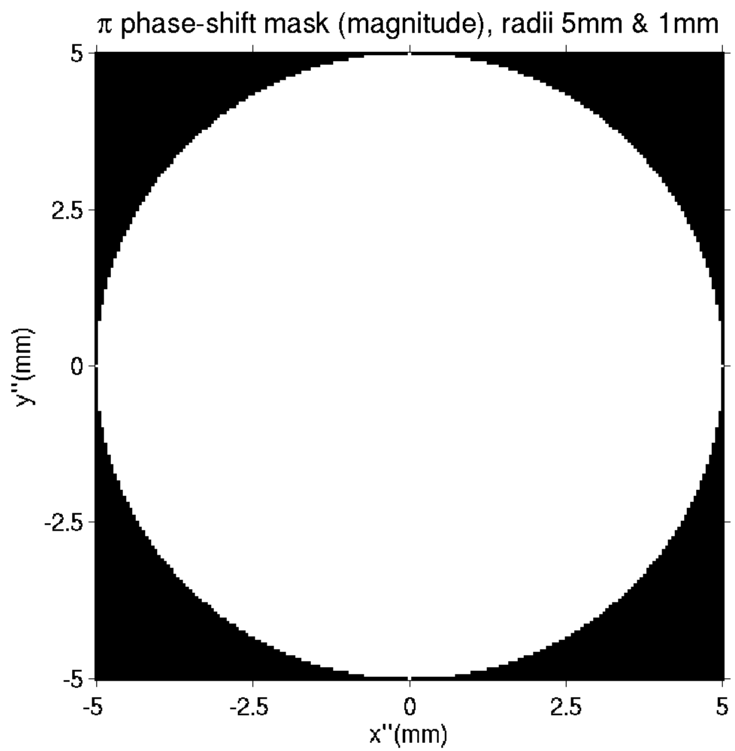
If we illuminate this object uniformly (e.g., with a plane wave) the resulting intensity $|g_{\text{in}}(x)|^2$ is also uniform, i.e. the object is invisible.

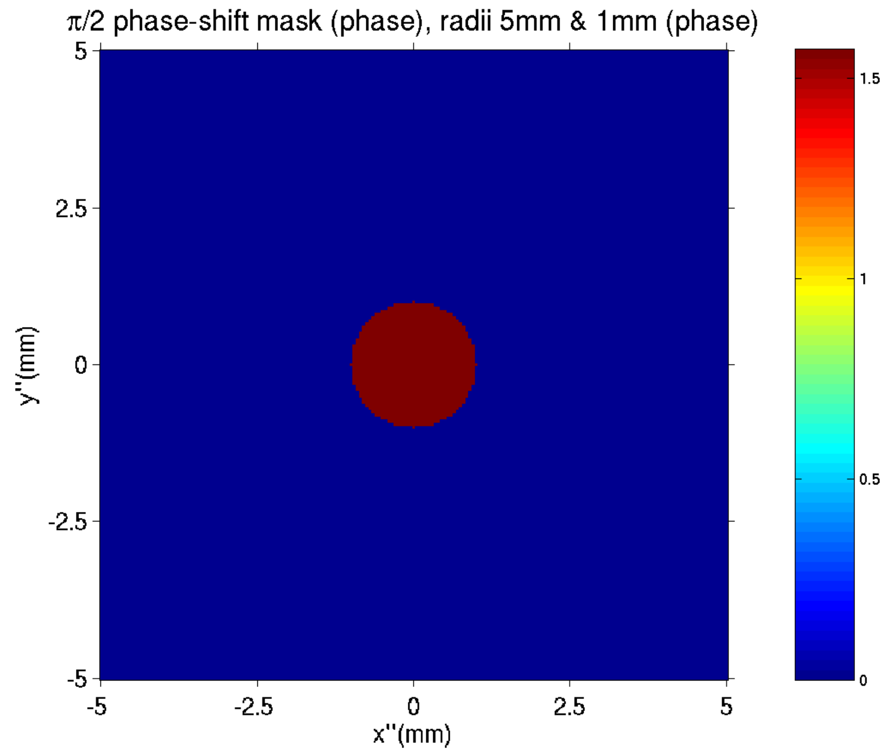


To visualize phase objects in subsequent slides, we use the color representation of phase as shown here. Physically, phase may be measured with interferometry.

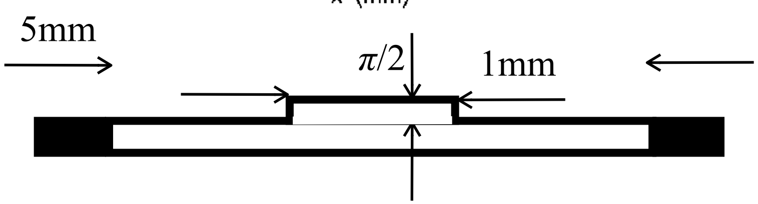


The Zernike phase pupil mask



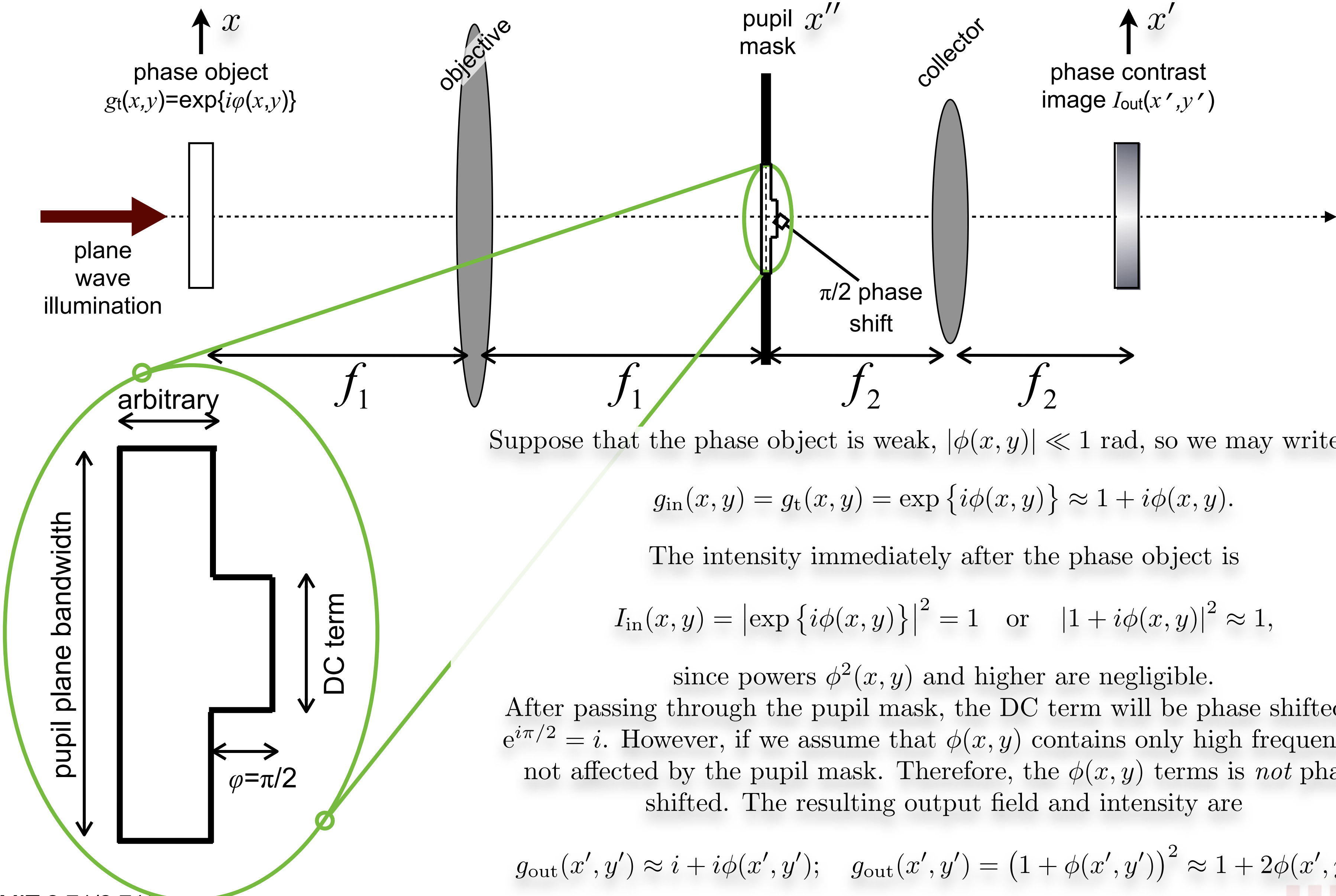


The Zernike mask is a phase pupil mask used often to visualize input transparencies that are themselves phase objects. The Zernike mask imparts phase delay of π/2 near the center of the pupil plane, i.e. at the lower spatial frequencies. The result at the output plane is *intensity contrast* at the edges of the phase object.



Use of the Zernike phase mask is also called **phase contrast imaging.**

Phase contrast (Zernike mask) imaging



Suppose that the phase object is weak, $|\phi(x,y)| \ll 1$ rad, so we may write it as

$$g_{\text{in}}(x,y) = g_{\text{t}}(x,y) = \exp\left\{i\phi(x,y)\right\} \approx 1 + i\phi(x,y).$$

The intensity immediately after the phase object is

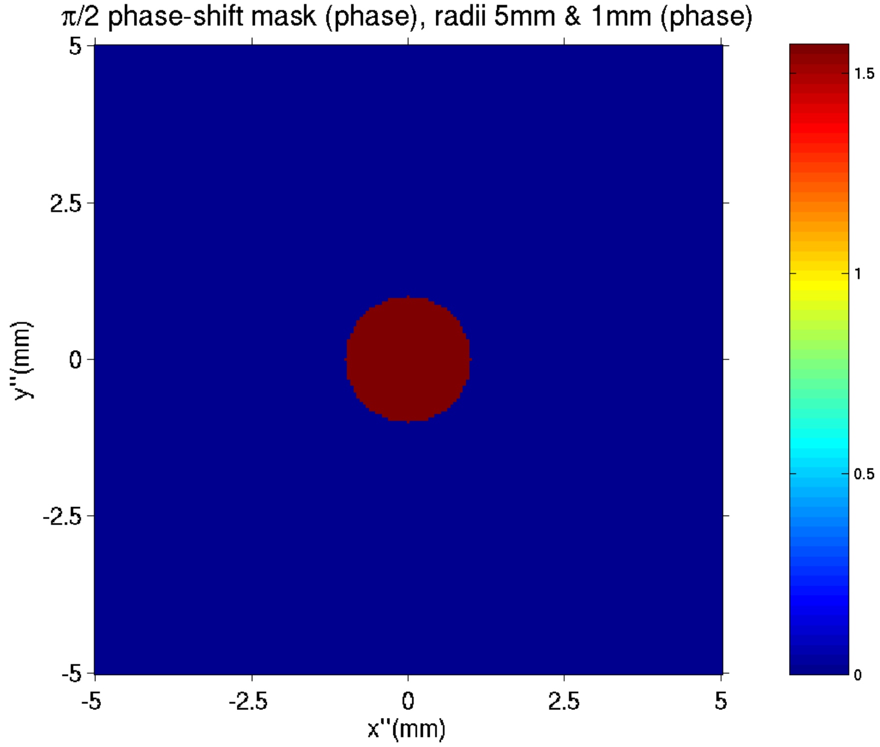
$$I_{\text{in}}(x,y) = \left|\exp\left\{i\phi(x,y)\right\}\right|^2 = 1 \quad \text{or} \quad \left|1+i\phi(x,y)\right|^2 \approx 1,$$

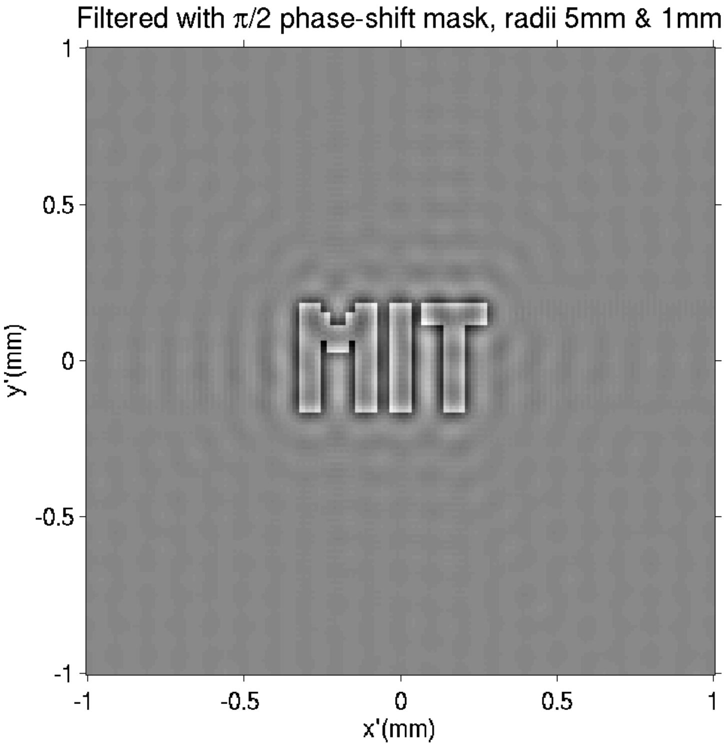
since powers $\phi^2(x,y)$ and higher are negligible.
After passing through the pupil mask, the DC term will be phase shifted by $e^{i\pi/2} = i$. However, if we assume that $\phi(x,y)$ contains only high frequencies, not affected by the pupil mask. Therefore, the $\phi(x,y)$ terms is *not* phase shifted. The resulting output field and intensity are

$$g_{\text{out}}(x',y') \approx i + i\phi(x',y'); \quad g_{\text{out}}(x',y') = \left(1+\phi(x',y')\right)^2 \approx 1 + 2\phi(x',y').$$

The term $2\phi(x,y)$ in the output intensity provides the *phase contrast*.

Phase contrast imaging the MIT phase object





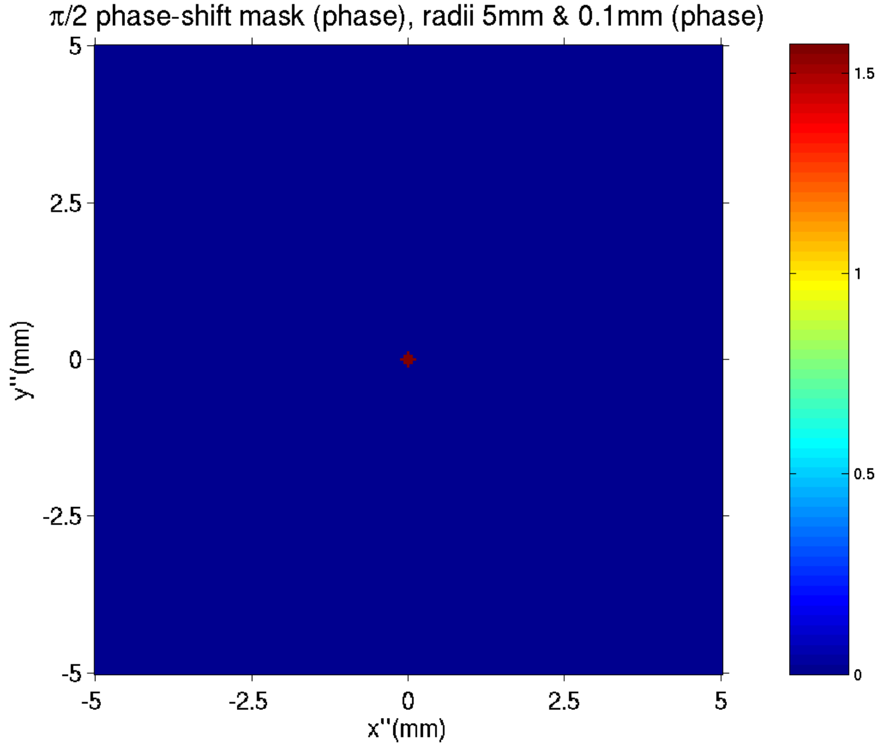
f_1=20cm

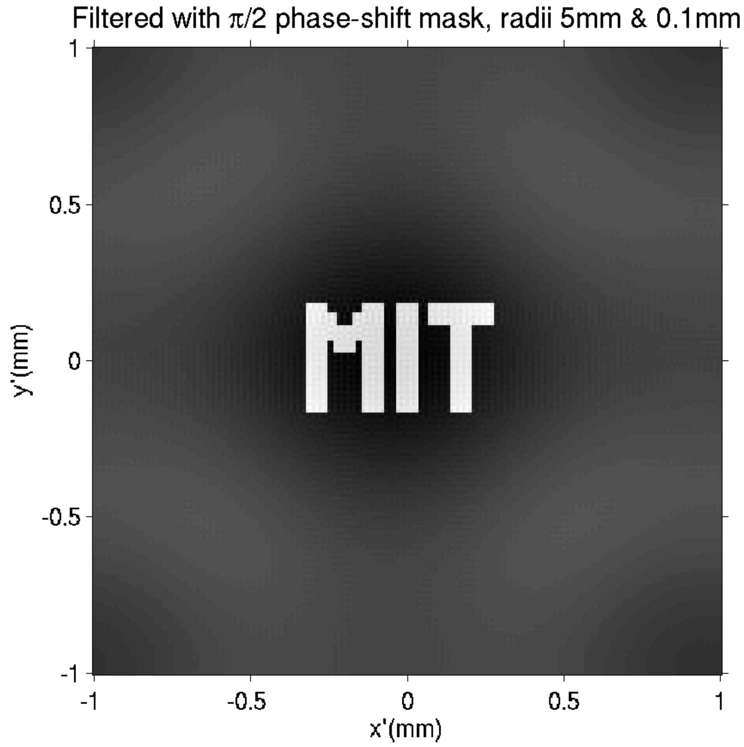
λ=0.5μm

pupil mask

Intensity @ image plane

Phase contrast imaging the MIT phase object





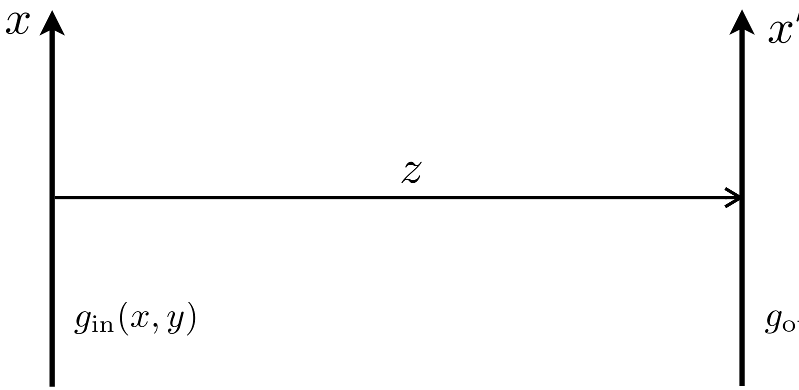
f_1=20cm

λ=0.5μm

pupil mask

Intensity @ image plane

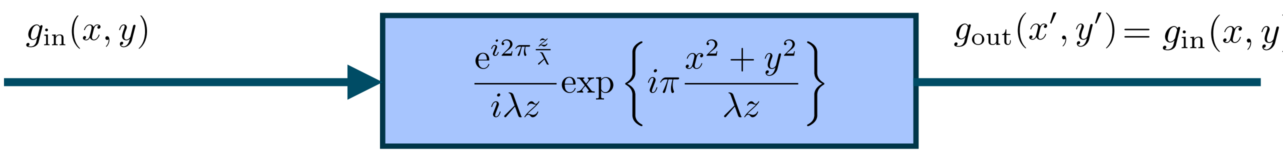
The transfer function of Fresnel propagation



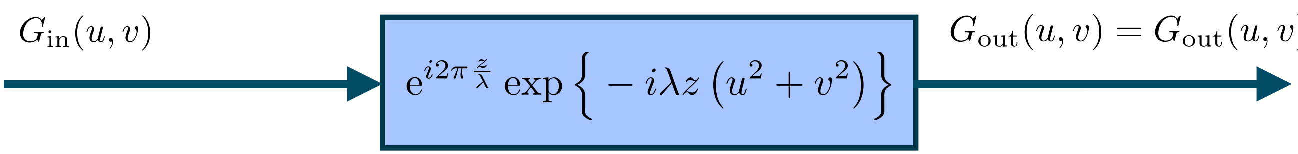
Fresnel (free space) propagation
may be expressed as a
convolution integral

$$g_{\rm out}(x',y') = g_{\rm in}(x,y) \star \left(\frac{e^{i2\pi\frac{z}{\lambda}}}{i\lambda z} \exp\left\{ i\pi \frac{x^2+y^2}{\lambda z} \right\} \right)$$

$$g_{\rm out}(x',y';z) = \frac{1}{i\lambda z} \exp\left\{ i2\pi \frac{z}{\lambda} \right\} \iint g_{\rm in}(x,y) \exp\left\{ i\pi \frac{(x'-x)^2+(y'-y)^2}{\lambda z} \right\} dx dy$$



$$g_{\rm out}(x',y') = g_{\rm in}(x,y) \star \left(\frac{e^{i2\pi\frac{z}{\lambda}}}{i\lambda z} \exp\left\{ i\pi \frac{x^2+y^2}{\lambda z} \right\} \right)$$

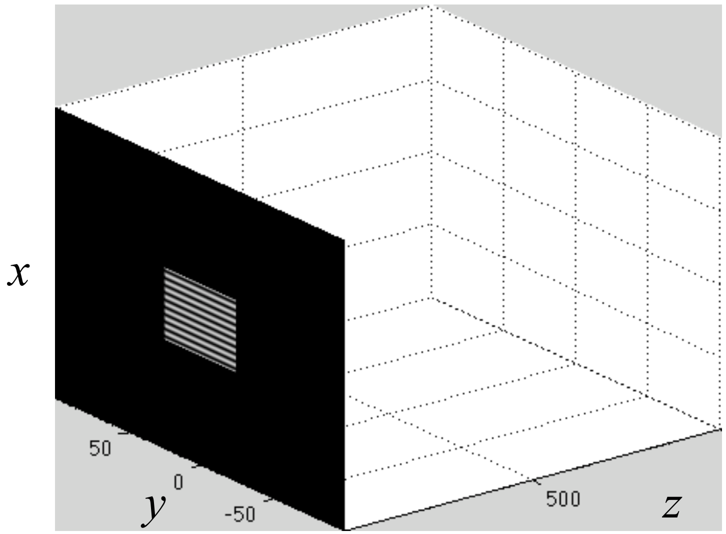


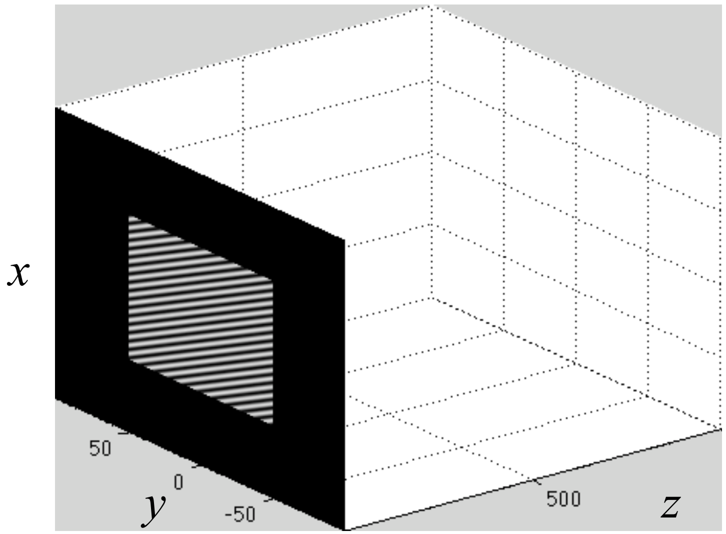
$$G_{\rm out}(u,v) = G_{\rm out}(u,v) \cdot \left(e^{i2\pi\frac{z}{\lambda}} \exp\left\{ -i\lambda z \left(u^2+v^2\right) \right\} \right)$$

$$h(x,y) = \frac{e^{i2\pi\frac{z}{\lambda}}}{i\lambda z} \exp\left\{ i\pi \frac{x^2+y^2}{\lambda z} \right\}$$ Point–Spread Function (Impulse Response)

$$H(u,v) = e^{i2\pi\frac{z}{\lambda}} \exp\left\{ -i\lambda z \left(u^2+v^2\right) \right\}$$ Transfer Function

The Talbot effect





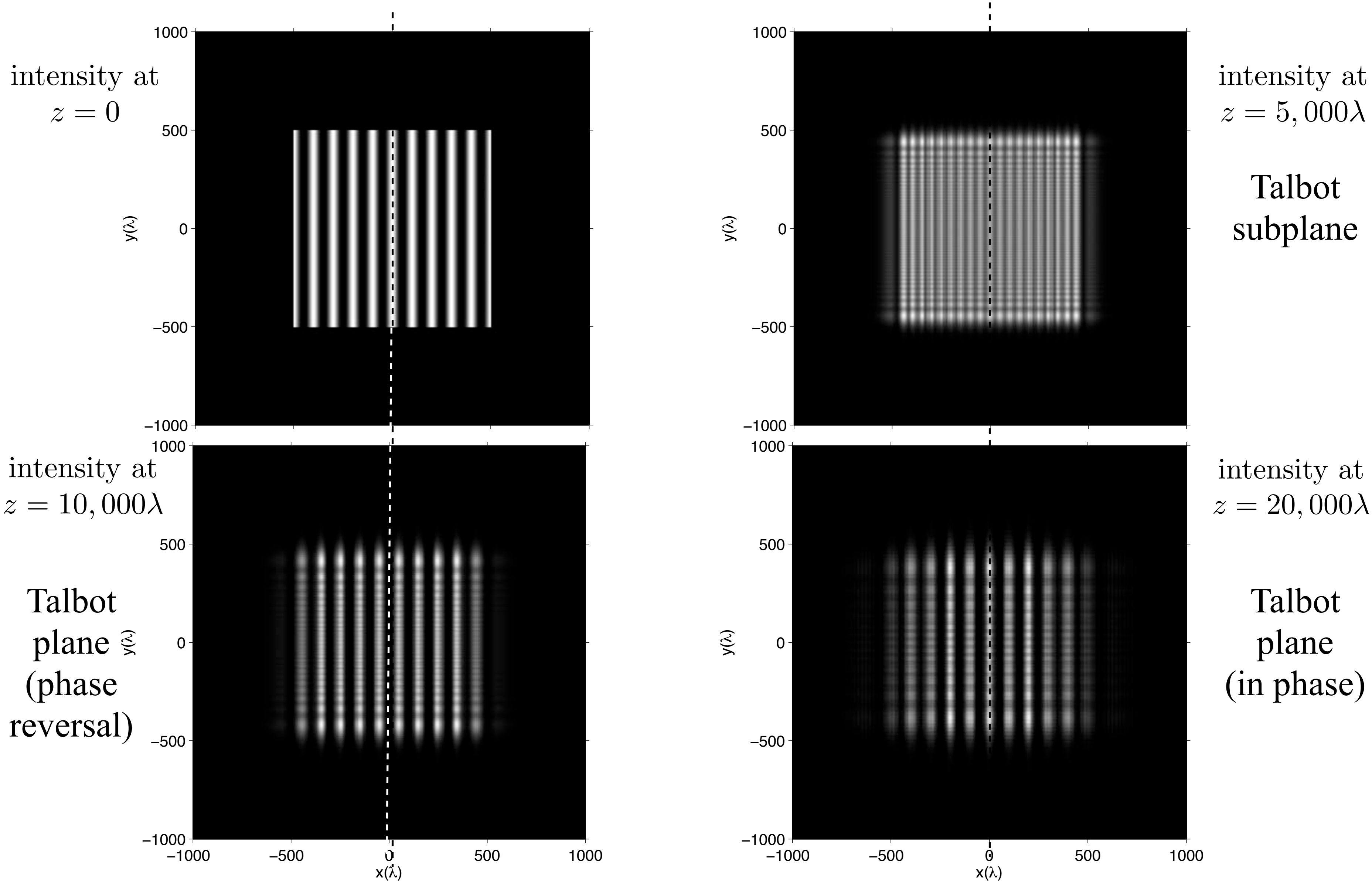
$$g_t(x) = \frac{1}{2}\left[1 + m\cos\left(2\pi\frac{x}{\Lambda} + \phi\right)\right]$$

$$\begin{aligned} \Lambda &= 5\lambda \\ m &= 1.0 \\ \phi &= 0. \end{aligned}$$

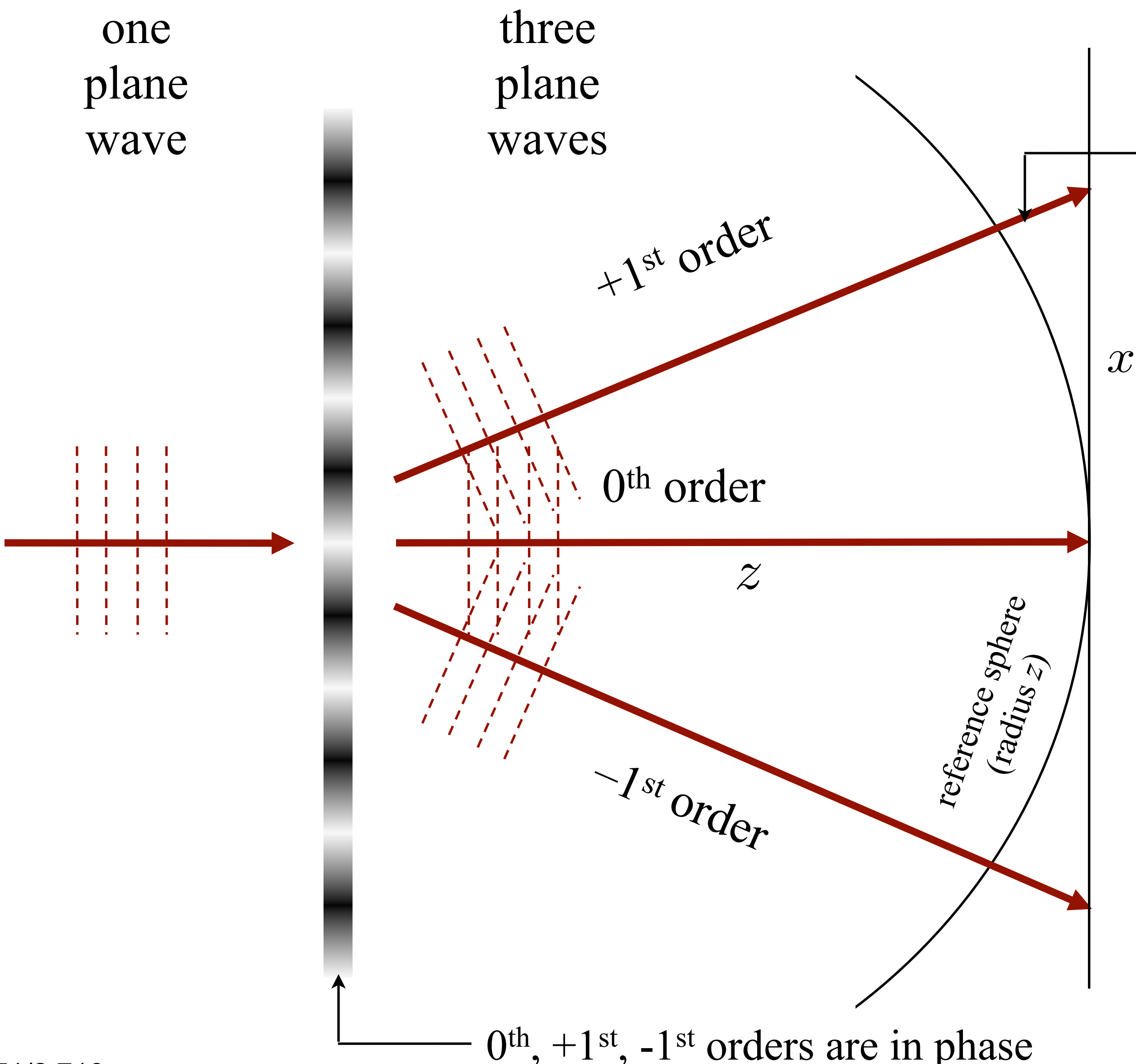
$$g_t(x,y) = \frac{1}{2}\left[1 + m\cos\left(2\pi\frac{x\sin\psi + y\cos\psi}{\Lambda} + \phi\right)\right]$$

$$\begin{aligned} \Lambda &= 5\lambda \\ m &= 1.0 \\ \phi &= 0 \\ \psi &= 30^\circ. \end{aligned}$$

Talbot effect: still shots



Physical explanation of the Talbot effect



$$\Delta r \equiv \sqrt{z^2 + x^2} - z \approx \frac{x^2}{2z}$$

$$x = z \tan\theta \approx \theta \approx \frac{\lambda}{\Lambda}$$

phase delay

$$\Delta\phi = 2\pi \frac{\Delta r}{\lambda} \approx \pi \frac{\lambda z}{\Lambda^2}.$$

$\Delta\phi = 0, \pm 2\pi, \pm 4\pi, \ldots$
replicates field at grating

$\Delta\phi = \pm\pi, \pm 3\pi, \ldots$
$\frac{1}{2}$ period shift wrt grating

$\Delta\phi = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots$
$+1^{st}$, -1^{st} orders interfere **destructively**

Mathematical derivation of the Talbot effect /1

- Field immediately past grating

$$g(x;0)=\frac{1}{2}\left\{1+m\cos\left(2\pi\frac{x}{\Lambda}\right)\right\}$$

  - also expressed as $g(x;0)=\frac{1}{2}\left\{1+\frac{m}{2}\exp\left(i2\pi\frac{x}{\Lambda}\right)+\frac{m}{2}\exp\left(-i2\pi\frac{x}{\Lambda}\right)\right\}$

- Fourier transform (spatial spectrum) of field past grating

$$G(u;0)=\frac{1}{2}\left\{\delta(0)+\frac{m}{2}\delta\left(u-\frac{1}{\Lambda}\right)+\frac{m}{2}\delta\left(u+\frac{1}{\Lambda}\right)\right\}$$

- Fourier transform (spatial spectrum) of Fresnel propagation kernel

$$H(u,v;z)=\exp\left\{-i\pi\lambda z\left(u^2+v^2\right)\right\}$$

  - value at grating's spatial frequencies $H(\pm\frac{1}{\Lambda},0)=\exp\left\{-i\pi\frac{\lambda z}{\Lambda^2}\right\}$

Mathematical derivation of the Talbot effect /2

- Fourier transform (spatial spectrum) of field propagated to distance z,

$$
\begin{aligned}
G(u,v;z) &= G(u,v;0)\times H(u,v;z)\\
&= \frac{1}{2}\left\{\delta(0)+\frac{m}{2}\delta\left(u-\frac{1}{\Lambda}\right)+\frac{m}{2}\delta\left(u+\frac{1}{\Lambda}\right)\right\}\times H(u,v;z)\\
&= \frac{1}{2}\left\{H(0,0;z)\delta(0)+\frac{m}{2}H\left(\frac{1}{\Lambda},0;z\right)\delta\left(u-\frac{1}{\Lambda}\right)\right.\\
&\quad\left.+\frac{m}{2}H\left(-\frac{1}{\Lambda},0;z\right)\delta\left(u+\frac{1}{\Lambda}\right)\right\}\\
&= \frac{1}{2}\left\{\delta(0)+\frac{m}{2}\exp\left(-i\pi\frac{\lambda z}{\Lambda^2}\right)\delta\left(u-\frac{1}{\Lambda}\right)\right.\\
&\quad\left.+\frac{m}{2}\exp\left(-i\pi\frac{\lambda z}{\Lambda^2}\right)\delta\left(u+\frac{1}{\Lambda}\right)\right\}
\end{aligned}
$$

- Inverse Fourier transforming,

$$
g(x;z)=\frac{1}{2}\left\{1+\frac{m}{2}\exp\left(-i\pi\frac{\lambda z}{\Lambda^2}\right)\exp\left(i2\pi\frac{x}{\Lambda}\right)+\frac{m}{2}\exp\left(-i\pi\frac{\lambda z}{\Lambda^2}\right)\exp\left(-i2\pi\frac{x}{\Lambda}\right)\right\}
$$

Mathematical derivation of the Talbot effect /3

- Our previous result

$$g(x;z)=\frac{1}{2}\left\{1+\frac{m}{2}\exp\left(-i\pi\frac{\lambda z}{\Lambda^2}\right)\exp\left(i2\pi\frac{x}{\Lambda}\right)+\frac{m}{2}\exp\left(-i\pi\frac{\lambda z}{\Lambda^2}\right)\exp\left(-i2\pi\frac{x}{\Lambda}\right)\right\}$$

  - also expressed as

$$g(x;z)=\frac{1}{2}\left\{1+m\exp\left(-i\pi\frac{\lambda z}{\Lambda^2}\right)\cos\left(2\pi\frac{x}{\Lambda}\right)\right\}$$

  - intensity is

$$\begin{aligned} I(x;z) &= |g(x;z)|^2 \\ &= \frac{1}{4}\left\{1+m^2\cos^2\left(2\pi\frac{x}{\Lambda}\right)+2m\cos\left(\pi\frac{\lambda z}{\Lambda^2}\right)\cos\left(2\pi\frac{x}{\Lambda}\right)\right\} \end{aligned}$$

  - note intensity immediately after grating

$$\begin{aligned} I(x;0) &= \left\{\frac{1}{2}\left[1+m\cos\left(2\pi\frac{x}{\Lambda}\right)\right]\right\}^2 \\ &= \frac{1}{4}\left\{1+m^2\cos^2\left(2\pi\frac{x}{\Lambda}\right)+2m\cos\left(2\pi\frac{x}{\Lambda}\right)\right\} \end{aligned}$$

Mathematical derivation of the Talbot effect /4

- Special cases of propagation distance

$$\frac{\lambda z}{\Lambda^2} = 2p \quad (p \text{ integer}) \Rightarrow \qquad \text{Talbot plane}$$

$$\begin{aligned} I\left(x; \frac{2p\Lambda^2}{\lambda}\right) &= \frac{1}{4}\left\{1 + m^2\cos^2\left(2\pi\frac{x}{\Lambda}\right) + 2m\cos\left(2\pi\frac{x}{\Lambda}\right)\right\} \\ &= I(x,0) \quad \text{[replica of original grating]} \end{aligned}$$

$$\frac{\lambda z}{\Lambda^2} = 2p+1 \quad (p \text{ integer}) \Rightarrow \qquad \text{(shifted) Talbot plane}$$

$$\begin{aligned} I\left(x; \frac{(2p+1)\Lambda^2}{\lambda}\right) &= \frac{1}{4}\left\{1 + m^2\cos^2\left(2\pi\frac{x}{\Lambda}\right) - 2m\cos\left(2\pi\frac{x}{\Lambda}\right)\right\} \\ &= \frac{1}{4}\left\{1 - m\cos\left(2\pi\frac{x}{\Lambda}\right)\right\}^2 \\ &= \text{[original grating shifted by half period]} \end{aligned}$$

$$\frac{\lambda z}{\Lambda^2} = (2p+1)\times\frac{1}{2} \quad (p \text{ integer}) \Rightarrow \qquad \text{Talbot subplane}$$

$$\begin{aligned} I\left(x; \frac{(2p+1)\Lambda^2}{2\lambda}\right) &= \frac{1}{4}\left\{1 + m^2\cos^2\left(2\pi\frac{x}{\Lambda}\right)\right\} \\ &= \text{[period doubling]} \end{aligned}$$

The Talbot planes