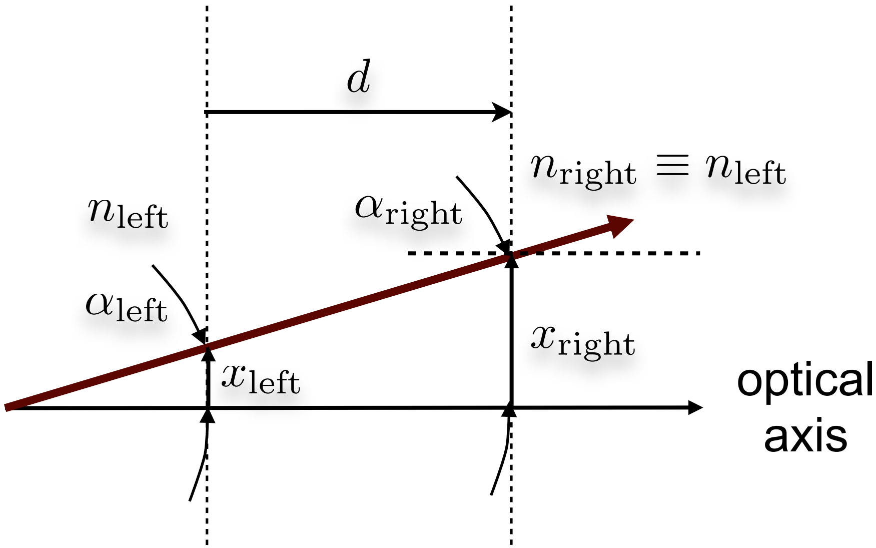
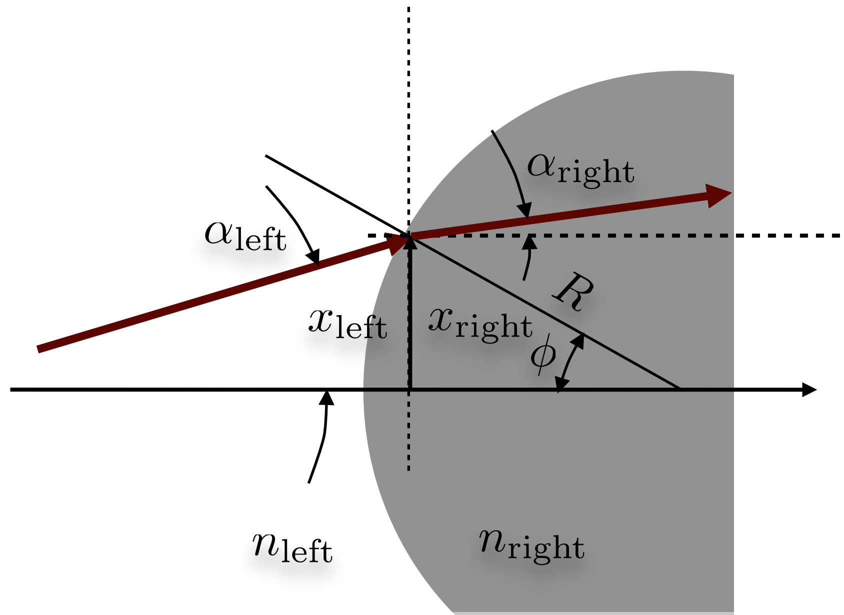
Overview

- Last lecture:
  - spherical and plane waves
  - perfect focusing and collimation elements:
    - paraboloidal mirrors, ellipsoidal and hyperboloidal refractors
  - imperfect focusing: spherical elements
  - the paraxial approximation
  - ray transfer matrices
- Today:
  - paraxial ray tracing using the matrix approach
  - thin lenses
  - focal length and power of optical elements
  - real and virtual images

Ray transfer matrices





$$\begin{pmatrix} n_{\text{right}}\alpha_{\text{right}} \\ x_{\text{right}} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \dfrac{d}{n_{\text{left}}} & 1 \end{pmatrix} \begin{pmatrix} n_{\text{left}}\alpha_{\text{left}} \\ x_{\text{left}} \end{pmatrix}$$

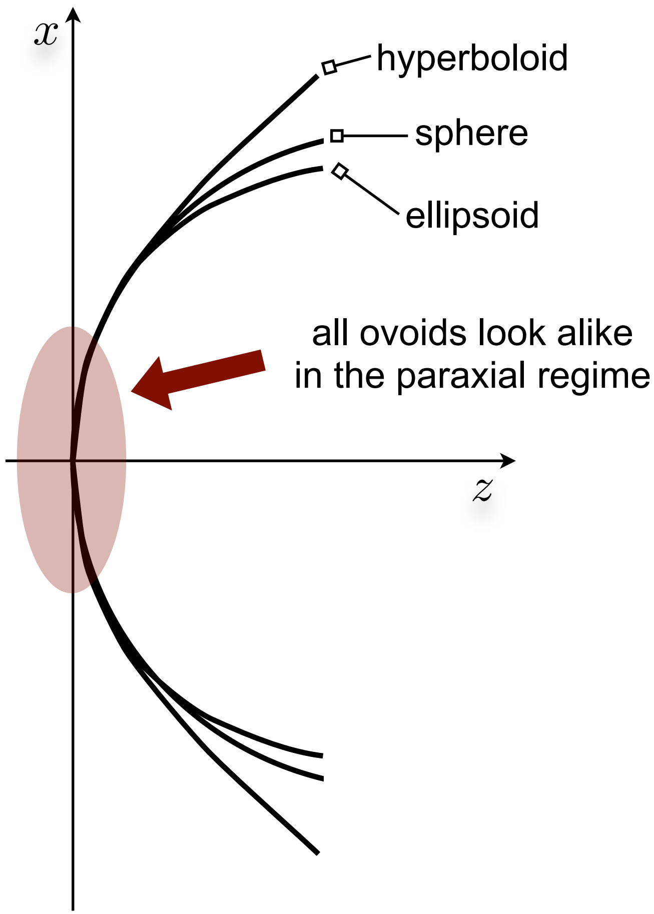
$$\begin{pmatrix} n_{\text{right}}\alpha_{\text{right}} \\ x_{\text{right}} \end{pmatrix} = \begin{pmatrix} 1 & -\dfrac{n_{\text{right}} - n_{\text{left}}}{R} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} n_{\text{left}}\alpha_{\text{left}} \\ x_{\text{left}} \end{pmatrix}$$

Propagation through uniform space:
distance d, index of refraction n_{left}

Refraction at spherical interface:
radius R, indices n_{left}, n_{right}

By using these elemental matrices, we may ray trace through an arbitrarily long cascade of optical elements *(provided the paraxial approximation remains valid throughout.)*

Spheres, ellipsoids, hyperboloids, and paraboloids in the paraxial approximation



The ovoid equations on the xz cross–sectional plane are of the following form:

sphere: $$x^2 + z^2 = r^2;$$

ellipsoid: $$\frac{x^2}{a^2} + \frac{z^2}{b^2} = 1;$$

hyperboloid: $$\frac{x^2}{a^2} - \frac{z^2}{b^2} = 1; \qquad \text{and}$$

paraboloid: $$z = cx^2.$$

Any surface with rotational symmetry around the z axis, including the ovoids, can be expressed in the Taylor–series form

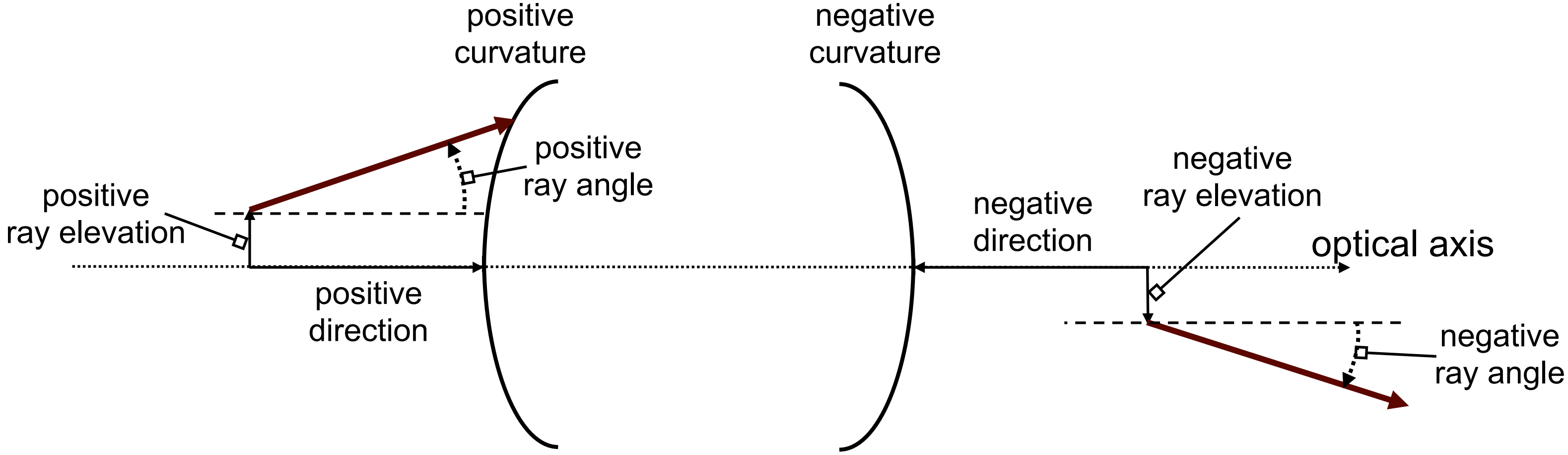
$$z = c_2x^2 + c_4x^4 + c_6x^6 + \dots$$

Therefore, they are all well approximated as paraboloids for $|x|$ sufficiently small compared to the radii of curvature.

This observation reassures us that the equations that we derived are applicable to any rotationally symmetric surface, within the paraxial approximation.

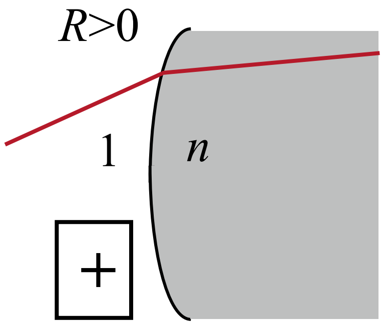
Sign conventions

- Light travels from left to right
- A radius of curvature is positive if the surface is convex towards the left
- Longitudinal distances are positive if pointing to the right
- Lateral distances are positive if pointing up
- Ray angles are positive if the ray direction is obtained by rotating the $+z$ axis counterclockwise through an acute angle



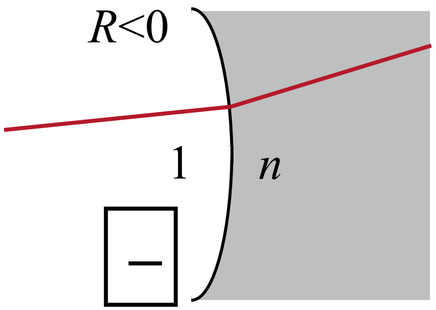
Types of refraction from spherical surfaces

- Positive power bends rays "inwards"



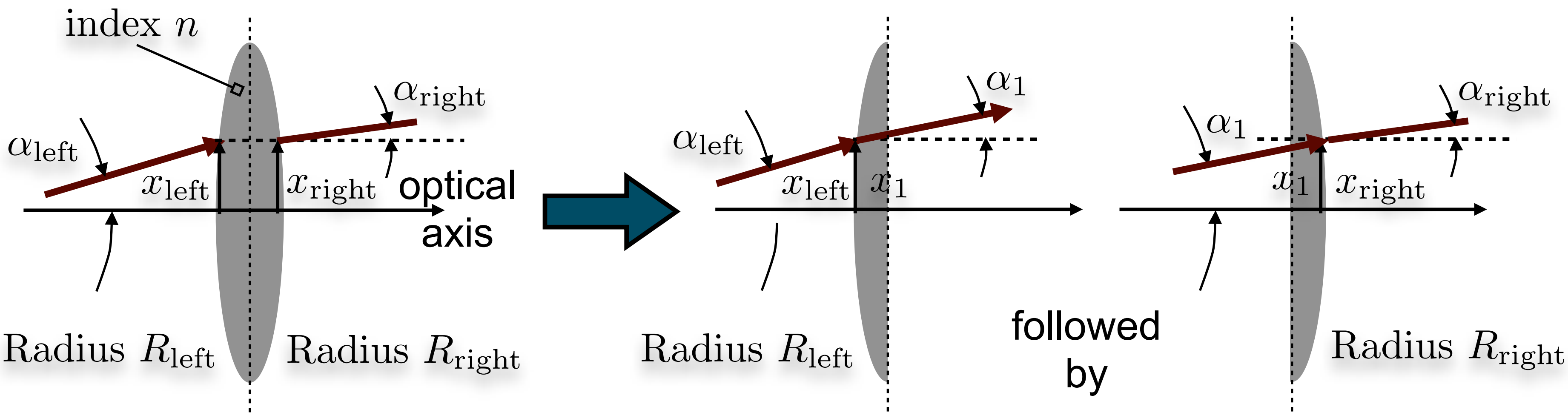
Positive power

- Negative power bends rays "outwards"

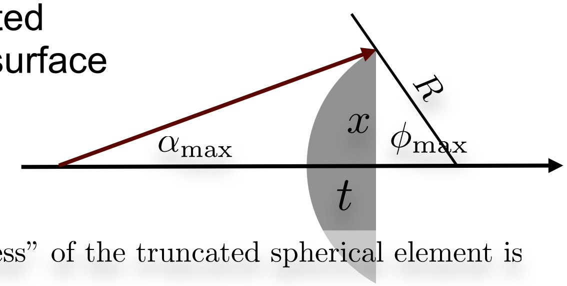


Negative power

Example: thin spherical lens in air



Thin or thick?



The "thickness" of the truncated spherical element is

$$t = R - R\cos\phi = 2R\sin^2\frac{\phi}{2}.$$

In the paraxial approximation,

$$\alpha_{\max} \ll 1 \Rightarrow \phi_{\max} \ll 1 \quad \text{and} \quad t \sim \mathcal{O}\left(\phi^2\right) \Rightarrow t \approx 0.$$

That is, the thickness is negligible.

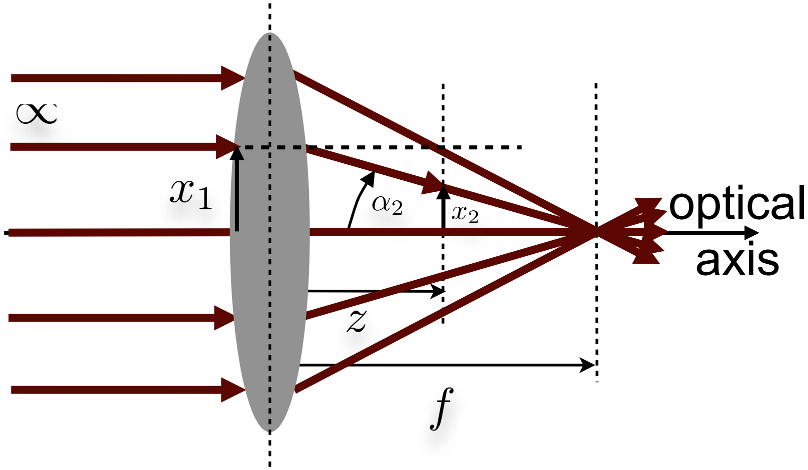
$$\begin{pmatrix}\alpha_{\text{right}}\\ x_{\text{right}}\end{pmatrix} = \begin{pmatrix}1 & -\dfrac{1-n}{R_{\text{right}}}\\ 0 & 1\end{pmatrix}\begin{pmatrix}n\alpha_1\\ x_1\end{pmatrix}$$

$$\begin{pmatrix}n\alpha_1\\ x_1\end{pmatrix} = \begin{pmatrix}1 & -\dfrac{n-1}{R_{\text{left}}}\\ 0 & 1\end{pmatrix}\begin{pmatrix}\alpha_{\text{left}}\\ x_{\text{left}}\end{pmatrix}$$

$$\Rightarrow \begin{pmatrix}\alpha_{\text{right}}\\ x_{\text{right}}\end{pmatrix} = \begin{pmatrix}1 & -\dfrac{1-n}{R_{\text{right}}}\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & -\dfrac{n-1}{R_{\text{left}}}\\ 0 & 1\end{pmatrix}\begin{pmatrix}\alpha_{\text{left}}\\ x_{\text{left}}\end{pmatrix}$$

$$= \begin{pmatrix}1 & -(n-1)\left(\dfrac{1}{R_{\text{left}}} - \dfrac{1}{R_{\text{right}}}\right)\\ 0 & 1\end{pmatrix}\begin{pmatrix}\alpha_{\text{left}}\\ x_{\text{left}}\end{pmatrix}$$

Example: thin spherical lens in air



$$\begin{pmatrix} \alpha_{\text{right}} \\ x_{\text{right}} \end{pmatrix} = \begin{pmatrix} 1 & -\frac{1}{f} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha_{\text{left}} \\ x_{\text{left}} \end{pmatrix} \quad \text{where}$$

$$P \equiv \frac{1}{f} = (n-1)\left(\frac{1}{R_{\text{left}}} - \frac{1}{R_{\text{right}}}\right) \quad \text{Lens maker's Equation}$$

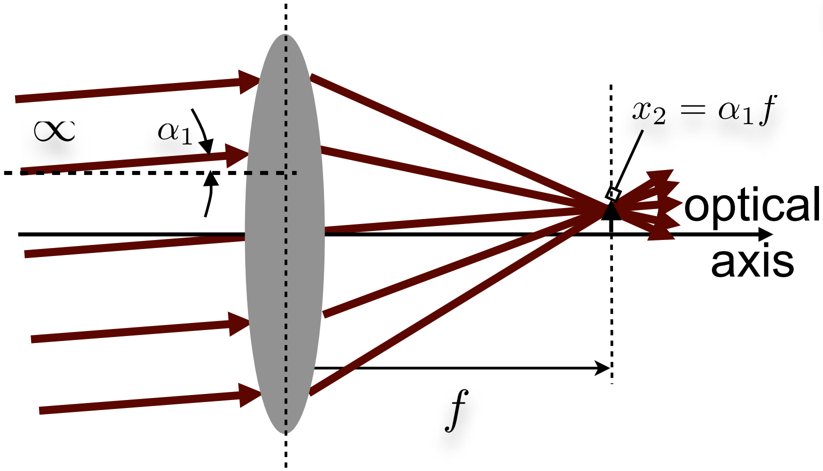
Consider a ray arriving from infinity at angle $\alpha_1 = 0$ (*i.e.*, parallel to the optical axis) and at elevation x_1. The ray is refracted by the thin lens and propagates a further distance z to the right of the lens. We seek to determine its elevation x_2 and angle of propagation α_2 as function of z. We use the matrix approach:

$$\begin{pmatrix} \alpha_2 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix} \begin{pmatrix} 1 & -\frac{1}{f} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ x_1 \end{pmatrix} = \begin{pmatrix} 1 & -\frac{1}{f} \\ z & 1-\frac{z}{f} \end{pmatrix} \begin{pmatrix} \alpha_1 \\ x_1 \end{pmatrix}$$

$$\Rightarrow x_2 = \alpha_1 z + x_1\left(1 - \frac{z}{f}\right) = x_1\left(1 - \frac{z}{f}\right) \qquad (\text{since } \alpha_1 = 0.)$$

We observe that at $z = f \Rightarrow x_2 = 0$ for all x_1; *i.e.*, all the rays from infinity converge to the optical axis independent of the elevation x_1 at arrival. We say that the plane wave from infinity comes to a **focus** at $z = f$, and f is referred to as **focal length** of the thin lens.

$P \equiv \frac{1}{f}$ is the **lens power**, measured in Diopters $[\mathrm{m}^{-1}]$.

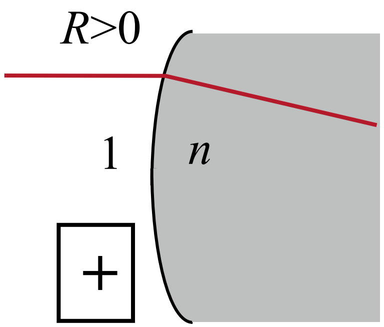


We can easily see that if $\alpha_1 \neq 0$, the rays still come to a focus at distance $z = f$ to the right of the lens, at elevation $x_2 = \alpha_1 f$. This is the **image** of the (off–axis) source at infinity.

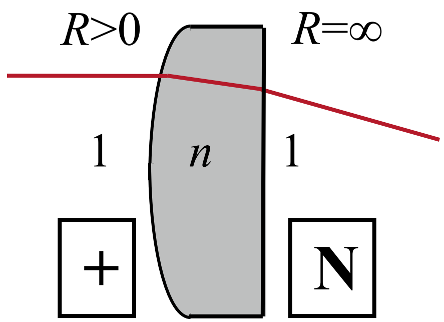
Note that, if the indeces of refraction to the left and right of the lens are the same, then a ray going through the optical center of the lens emerges parallel to the incident direction.

Types of lenses

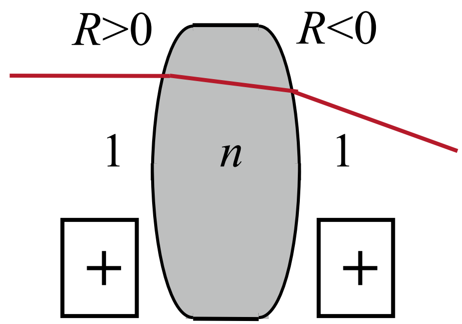
- Positive lenses have positive power ⇔ positive focal length



Simple spherical refractor (positive)

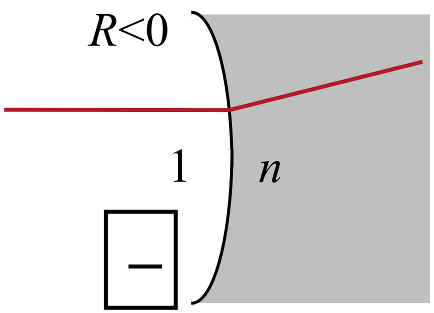


Plano-convex lens

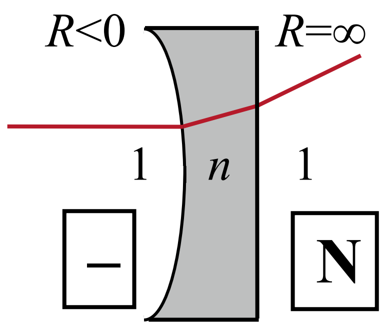


Bi-convex lens

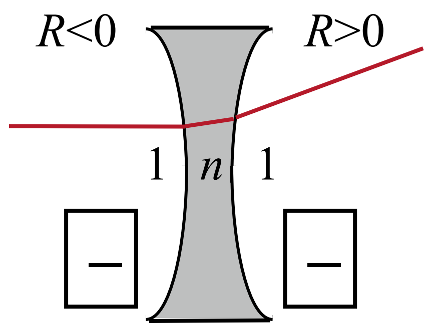
- Negative lenses have negative power ⇔ negative focal length



Simple spherical refractor (negative)



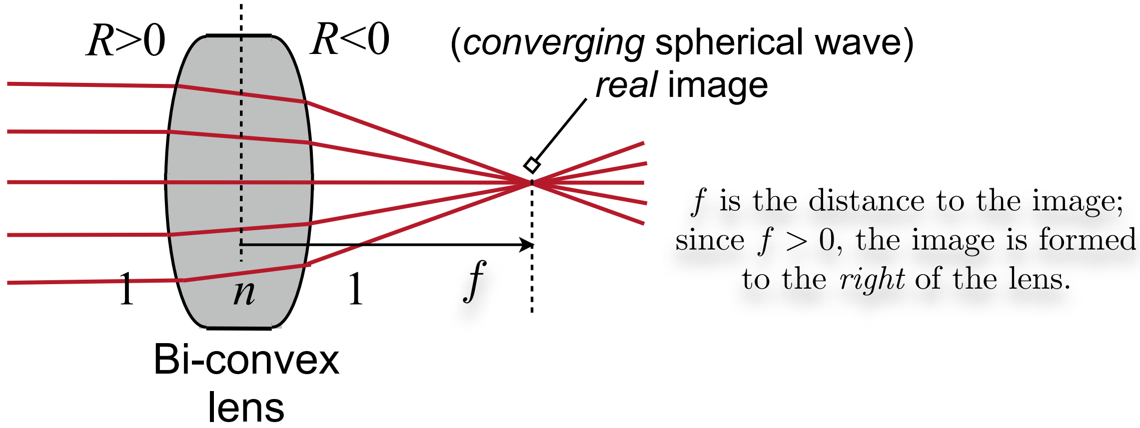
Plano-concave lens



Bi-concave lens

Real and virtual images of a source at infinity

- A *positive* lens creates a *real* image of an object at infinity



- A *negative* lens creates a *virtual* image of an object at infinity

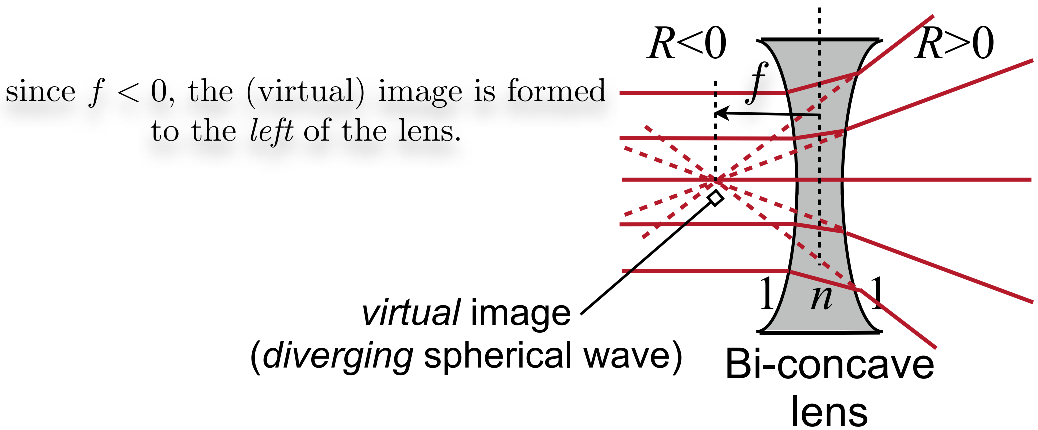
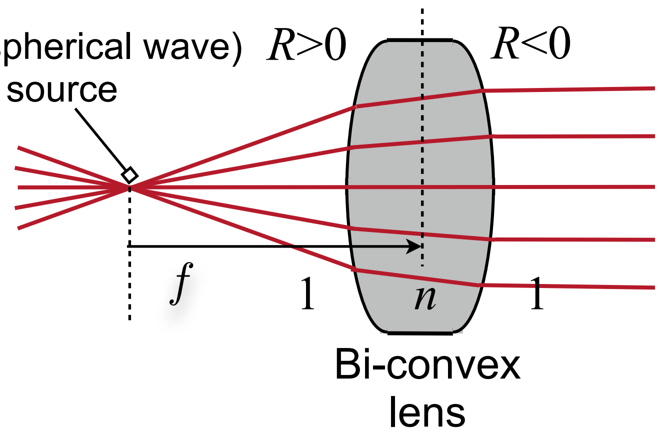
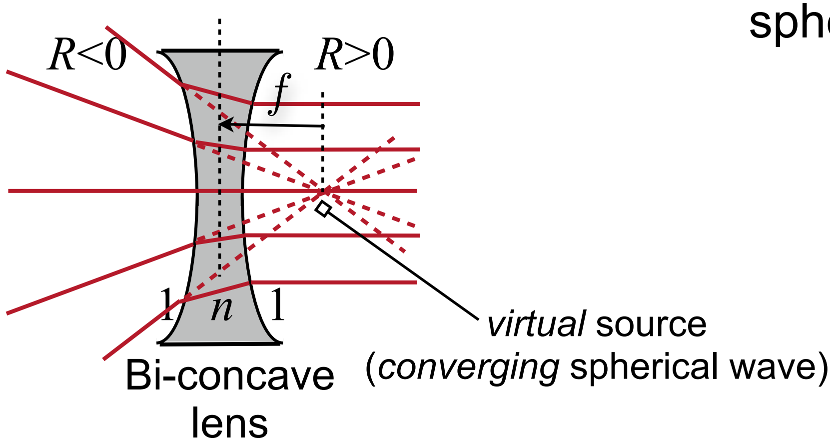


Image *at* infinity of real and virtual sources

- A *positive* lens will image a *real* object at infinity (collimate a diverging spherical wave)

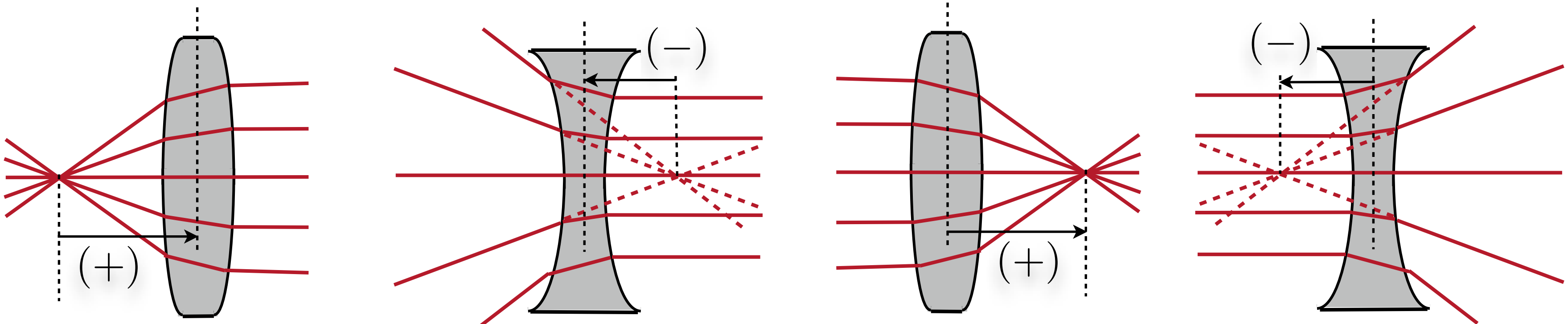


- A *negative* lens will image a *virtual* object at infinity (collimate a converging spherical wave)



How to make sense of the sign conventions

- <u>Recall:</u> **light propagates from left to right;** therefore:
- if an *object* is to the *left* of the optical element
  - then the distance from the object to the element is *positive*;
- if an *object* is to the *right* of the optical element
  - then the distance from the object to the element is *negative*;
- if an *image* is to the *right* of the optical element
  - then the distance from the element to the image is *positive*;
- if an *image* is to the *left* of the optical element
  - then the distance from the element to the image is *negative*;



object to the *left* of the element

object to the *right* of the element

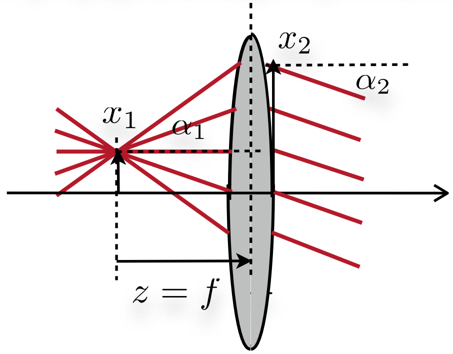
image to the *right* of the element

image to the *left* of the element

Sign conventions and off-axis objects

Consider an off–axis object at infinity, generating a plane wave with propagation angle α_1 wrt the optical axis. In slide #7, we derived the expression $x_2 = \alpha_1 f$ for the lateral coordinate of the image.

Now consider an off–axis object placed at distance $z = f$ to the left of the lens so the image is at infinity. We seek the propagation angle α_2 of the exiting rays.



$$\begin{pmatrix} \alpha_2 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 & -\frac{1}{f} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ x_1 \end{pmatrix}$$

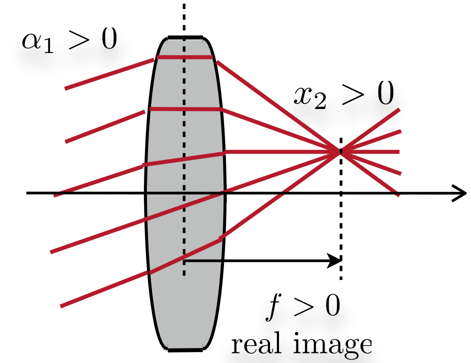
$$= \begin{pmatrix} 1 - \frac{z}{f} & -\frac{1}{f} \\ z & 1 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ x_1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} \alpha_2 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 & -\frac{1}{f} \\ f & 1 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ x_1 \end{pmatrix}$$

$$\Rightarrow \alpha_2 = -\frac{x_1}{f}.$$

object at infinity

$$x_2 = \alpha_1 f$$



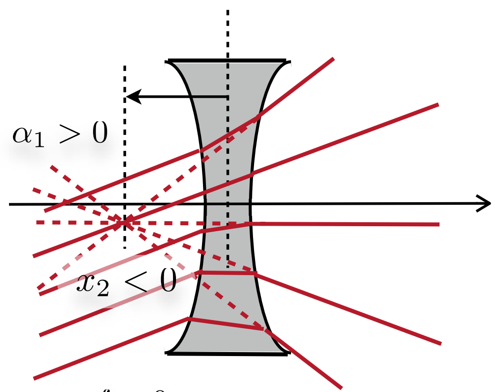


image at infinity

$$\alpha_2 = -\frac{x_1}{f}$$