Problem 1. Refracting Surface (Modified from Pedrotti 2-2)

Part (a)

Fermat's principle requires that every ray that emanates from the object and passes through the image point must be isochronous (i.e., have equal optical path lengths).

The common optical path length (OPL_{common}) can be easily found using the ray that passes along the optical axis from the object to the image point, while taking into account the refractive indexes of air (n_o) and glass (n_i):

$$OPL_{common} = n_o s_o + n_i s_i$$

Let x and y be the horizontal and vertical coordinates, respectively, with the origin at the object point. The optical path length ($OPL_{arbitrary}$) of any arbitrary ray that emanates from the object at (0, 0) , enters the glass at coordinate (x, y), and then passes through the image point at ($s_i + s_o$, 0), can be found from the in-air propagation distance (d_o) and in-glass propagation distance (d_i), while taking into account the indexes of refraction:

$$d_o = \sqrt{x^2 + y^2}$$
$$d_i = \sqrt{(s_o + s_i - x)^2 + y^2}$$

$$OPL_{arbitrary} = n_o d_o + n_i d_i$$
$$OPL_{arbitrary} = n_o \sqrt{x^2 + y^2} + n_i \sqrt{(s_o + s_i - x)^2 + y^2}$$

The correct refracting surface corresponds to all (x, y) such that $OPL_{arbitrary}$ equals OPL_{common}.

$$OPL_{arbitrary} = OPL_{common}$$
$$n_o \sqrt{x^2 + y^2} + n_i \sqrt{(s_o + s_i - x)^2 + y^2} = n_o s_o + n_i s_i$$

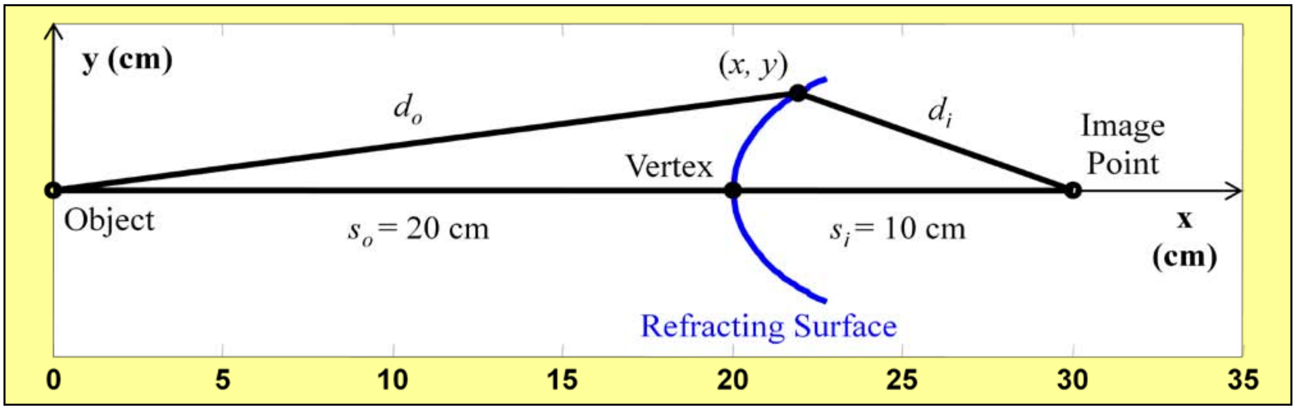
Plugging in n_o = 1, n_i = 1.5, s_o = 20 cm, and s_i = 10 cm:

$$\boxed{\sqrt{x^2 + y^2} + 1.5\sqrt{(30 - x)^2 + y^2} = 35}$$

Note: This equation is only 2-dimensional. The 3-dimensional solution would be a radial-symmetric surface, where each cross-section (through the vertex) follows the 2D equation.

Part (b)

Below is a plot of the refracting surface using labels from Part (a).



I found (x, y) coordinates on the surface by substituting d_o back into the surface equation, as follows:

$$\sqrt{x^2+y^2}+1.5\sqrt{(30-x)^2+y^2}=35$$
$$\sqrt{x^2+y^2}+1.5\sqrt{900-60x+(x^2+y^2)}=35$$
$$d_o+1.5\sqrt{900-60x+d_o^2}=35$$
$$1.5^2(900-60x+d_o^2)=(35-d_o)^2$$

In this form, x can easily be found from d_o, and y can be found from x and d_o:

$$x=\frac{1.5^2(900+d_o^2)-(35-d_o)^2}{60\cdot 1.5^2}$$

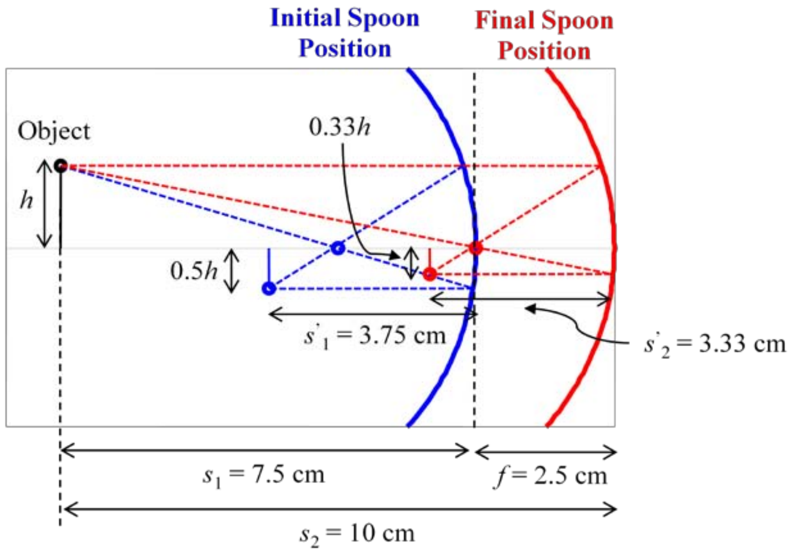
$$y=\sqrt{d_o^2-x^2}$$

By choosing a series of values for d_o, I can calculate (x, y) coordinates that satisfy the original surface equation, as follows:

d_o	x	y
20	20	0
21	20.90	±2.07
22	21.81	±2.85
23	22.75	±3.38

After calculating many closely-spaced (x, y) coordinates in this way, I was able to plot the refracting surface above with Matlab®.

Problem 2. Ice-cream Spoon



First, let's define some variables:

- Let s_1 and s_2 be the initial and final object distances, respectively.
- Let s'_1 and s'_2 be the initial and final image distances, respectively.
- Let m_1 and m_2 be the initial and final magnifications, respectively.
- Let d be the distance moved by the mirror (initial to final)

The described problem can be quantified by 5 equations with 5 unknowns $(s_1, s_2, s_1', s_2', f)$**:**

Mirror equation at the initial object distance	$\frac{1}{s_1}+\frac{1}{s_1'}=\frac{1}{f}$	**Eq. 1**
Mirror equation at the final object distance	$\frac{1}{s_2}+\frac{1}{s_2'}=\frac{1}{f}$	**Eq. 2**
Magnification at the initial object distance	$m_1=-\frac{s_1'}{s_1}$	**Eq. 3**
Magnification at the final object distance	$m_2=-\frac{s_2'}{s_2}$	**Eq. 4**
Difference between initial and final distances	$s_2=s_1+d$	**Eq. 5**

The system of equations can be solved for the unknowns as follows:

Combine Eq. 1 and Eq. 2 $$\frac{1}{s_1}+\frac{1}{s_1'}=\frac{1}{s_2}+\frac{1}{s_2'}$$

Use Eq. 3 and Eq. 4 to remove s'_1 and s'_2

$$\frac{1}{s_1}-\frac{1}{m_1 s_1}=\frac{1}{s_2}-\frac{1}{m_2 s_2}$$

Use Eq. 5 to remove s_2

$$\frac{1}{s_1}-\frac{1}{m_1 s_1}=\frac{1}{s_1+d}-\frac{1}{m_2(s_1+d)}$$

Solve for s_1

$$s_1=\frac{d\cdot(m_1 m_2-m_2)}{m_2-m_1}$$

Solve for remaining unknowns

$$s_2=s_1+d \qquad s'_1=-m_1 s_1$$

$$s'_2=-m_2 s_2 \qquad f=\frac{s_1 s'_1}{s_1+s'_1}$$

In this problem, we are given $|d|$ = 2.5 cm, $|m_1|$ = 0.5, and $|m_2|$ = 1/3, but we are not given their signs (+ or -). Fortunately, we can safely (but arbitrarily) set d = +2.5 cm. After all, the sign of d should be inconsequential, since the solutions (to the system of equations) for $d = \pm 2.5$ will simply be the reverse of one another (left becomes right, right becomes left). However, the solution to the system of equations will, indeed, depend on the signs of m_1 and m_2 (i.e., whether the images are inverted or not)

The table below shows the solutions to the system of equations for all 4 combinations of m_1 (+ or -) and m_2 (+ or -). The table reveals that, depending on the signs of m_1 and m_2, the mirror may take on a concave or convex geometry for the initial and/or final object-to-mirror distances. As highlighted below, only the last combination of m_1 (-0.5) and m_2 (-1/3) conforms to the premise of the problem (i.e., that the ice-cream spoon is being used as a concave mirror).

Knowns			**Solution to System of Equations (in centimeters)**					**Mirror Geometry**
d_o (cm)	**m_1**	**m_2.**	**s_1**	**s_2**	**s'_1**	**s'_2**	**f**	
2.5	0.5	1/3	2.5	5	-1.25	-1.67	-2.5	Convex mirror for initial and final distances
2.5	0.5	-1/3	-0.5	2	0.25	0.67	0.5	Convex for initial; concave for final
2.5	-0.5	1/3	-1.5	1	-0.75	-0.33	-0.5	Concave for initial; convex for final
2.5	-0.5	-1/3	7.5	10	3.75	3.33	2.5	Concave for initial and final distances

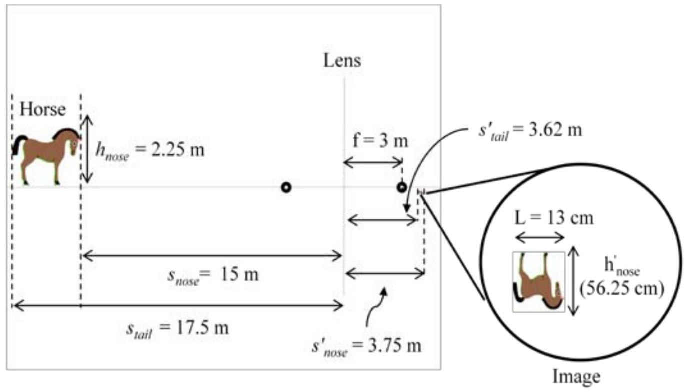
The position (p) of the image, relative to the object, at each object-to-mirror distance is:

$$p_1 = s_1 - s'_1 = 7.5\text{ cm} - 3.75\text{ cm} = 3.75\text{ cm}$$
$$p_2 = s_2 - s'_2 = 10\text{ cm} - 3.33\text{ cm} = 6.67\text{ cm}$$

Thus, the answers to the specific questions are:

- When the spoon moved 2.5 cm (i.e., $s_2 - s_1$) further away from the object, the image moved 2.92 cm (i.e., $p_2 - p_1$) further away from the object.
- The focal length of the spoon concave mirror is 2.5 cm.

Problem 3. Horse and Camera



Part (a)

The image distance of the horse's nose (s'_{nose}) can be found with the thin-lens equation, given the object distance of the horse's nose (s_{nose} = 15 m) and the focal length of the thin lens (f = 3 m):

$$\frac{1}{s_{nose}} + \frac{1}{s'_{nose}} = \frac{1}{f} \quad \rightarrow \quad s'_{nose} = \frac{s_{nose} \cdot f}{s_{nose} - f} = \frac{15 \cdot 3}{15-3} = \frac{45}{12} = 3.75$$

The height of the image of the horse's nose (h'_{nose}) can be found with the magnification equation, given the height of the horse's nose (h_{nose} = 2.25 m):

$$m = -\frac{s'_{nose}}{s_{nose}} = -\frac{3.75}{15} = -0.25$$

$$m = \frac{h'_{nose}}{h_{nose}} \quad \rightarrow \quad h'_{nose} = mh_{nose} = -0.25 \cdot 2.25 = -0.5625$$

Thus, the image of the horse's nose is located 3.75 m to the right of the lens and 56.25 cm below the optical axis.

Part (b)

The magnification is m = -0.25. In the vertical direction, the image is inverted, as indicated by the negative magnification. In other words, the horse appears upside-down. In the horizontal direction (as calculated in Part (d)), the horse's tail comes to focus closer to the lens than the horse's nose does. In other words, the image of the horse's tail is to the left of the image of the horse's nose. The graph above shows the orientation of the image (but not to scale), as described here.

Part (c)

The image is 56.25 cm tall (but inverted), as calculated in Part (a).

Part (d)

The image distance of the horse's tail (s'_{tail}) can be found with the thin-lens equation, given the object distance of the horse's tail (s_{tail} = 17.5 m) and the focal length of the thin lens (f = 3 m):

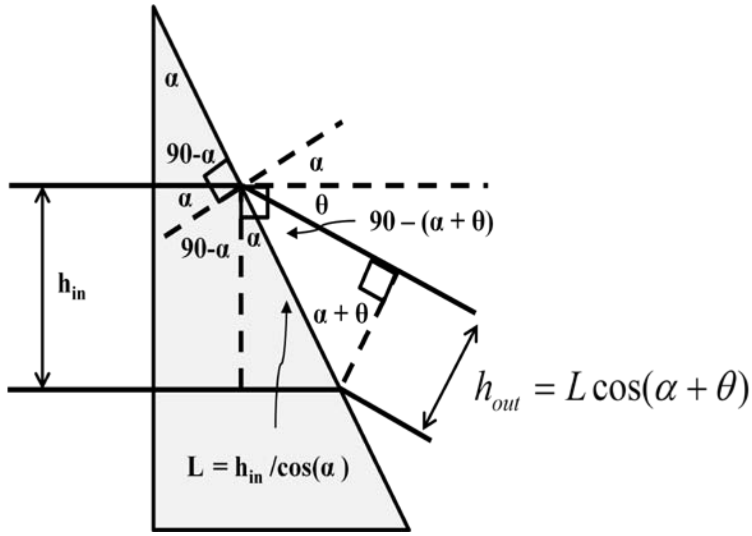
$$\frac{1}{s_{tail}} + \frac{1}{s'_{tail}} = \frac{1}{f} \rightarrow s'_{tail} = \frac{s_{tail} \cdot f}{s_{tail} - f} = \frac{17.5 \cdot 3}{17.5 - 3} = \frac{52.5}{14.5} \approx 3.62$$

The length of the image (L) can be calculated as follows:

$$L = s'_{nose} - s'_{tail} = 3.75 - 3.62 = 0.13$$

Thus, the image of the horse is 13 cm long (from nose to tail).

Problem 4. Prism Pair



Part (a)

At the first surface of the prism, the angle of incidence is zero, so no refraction occurs. At the second surface, as indicated in the drawing, the angle of incidence is α and the angle of refraction is $(\alpha + \theta)$. Snell's law then gives the relation among θ, α, and n (assuming the index of refraction of air is 1):

$$\boxed{n\sin(\alpha) = \sin(\alpha+\theta)} \quad \text{for } \alpha < \sin^{-1}\left(\frac{1}{n}\right)$$

If $\alpha \geq \sin^{-1}\left(\frac{1}{n}\right)$, then total internal reflection (TIR) occurs and the rays must cross the bottom interface of the prism to escape. Thus, the geometry of the original prism pair becomes invalid.

Part (b)

As indicated in the drawing, the length (L) over which the beam intersects the second surface of the prism is:

$$L = \frac{h_{in}}{\cos(\alpha)}$$

The width of the beam after the prism is:

$$h = L\cos(\alpha+\theta) = \frac{\cos(\alpha+\theta)}{\cos(\alpha)} h_{in}$$

Thus, the width of the beam is reduced by a factor (r) of:

$$r = \frac{\cos(\alpha+\theta)}{\cos(\alpha)} = \frac{\sqrt{1-\sin^2(\alpha+\theta)}}{\cos(\alpha)}$$

To remove θ from the equation, we can substitute in the relation from Part (a):

$$r = \frac{\sqrt{1-n^2\sin^2(\alpha)}}{\cos(\alpha)}$$

The second prism simply repeats the same process, giving a total reduction factor of r^2.

Thus, the reduction factors (r_1 and r_2) after the first and second prism, relative to the initial beam width, are:

$$\boxed{r_1 = r = \frac{\sqrt{1-n^2\sin^2(\alpha)}}{\cos(\alpha)}} \quad \text{for } \alpha < \sin^{-1}\left(\frac{1}{n}\right)$$

$$\boxed{r_2 = r^2 = \frac{1-n^2\sin^2(\alpha)}{\cos^2(\alpha)}} \quad \text{for } \alpha < \sin^{-1}\left(\frac{1}{n}\right)$$

Note: For sufficiently small α, then sin(α) ≈ α, sin(α + θ) ≈ α + θ, and cos(α) ≈ 1. The answers to the problem could then possibly be approximated as follows:

$$n\alpha \approx \alpha + \theta \quad \rightarrow \quad \theta \approx (n-1)\cdot\alpha$$

$$r_1 = r \approx \sqrt{1-n^2\alpha^2}$$
$$r_2 = r^2 \approx 1-n^2\alpha^2$$

Problem 5. Two Thin Lenses

The focal length of a combination of two thin lenses (f_1 and f_2), separated by distance d, is:

$$\frac{1}{f} = \frac{1}{f_1} + \frac{1}{f_2} - \frac{d}{f_1 f_2} \rightarrow f = \frac{f_1 f_2}{f_1 + f_2 - d}$$

Plugging in f_1 = 20 cm, f_2 = -8 cm, and d = 15 cm:

$$f = \frac{20 \cdot (-8)}{20 - 8 - 15} = 53.33 \text{ cm}$$

The distance from the negative lens to the film plane should equal the distance past the negative lens at which a horizontal, collimated beam would come to focus. This can be determined as follows. The matrix (M_{system}) representation of the two-lens system is

$$M_{system} = \begin{bmatrix} 1 & 0 \\ -1/f_2 & 1 \end{bmatrix} \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1/f_1 & 1 \end{bmatrix} = \begin{bmatrix} 1 - d/f_1 & d \\ d/f_1 f_2 - 1/f_1 - 1/f_2 & 1 - d/f_2 \end{bmatrix}$$

The vector (x_{in}) representation of a ray that has a height of 1 and is parallel to the optical axis (i.e., corresponding to a horizontal collimated beam) is:

$$\vec{x}_{in} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

If x_{in} is sent into the two-lens system, the output ray (x_{out}) would be:

$$\vec{x}_{out} = M_{system} \vec{x}_{in} = \begin{bmatrix} 1 - d/f_1 \\ d/f_1 f_2 - 1/f_1 - 1/f_2 \end{bmatrix}$$

If x_{out} is allowed to propagate a distance (L) past the negative lens, the resulting ray (x_{focus}) would be:

$$\vec{x}_{focus} = \begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix} \vec{x}_{out} = \begin{bmatrix} 1 - d/f_1 + L \cdot \left(d/f_1 f_2 - 1/f_1 - 1/f_2 \right) \\ d/f_1 f_2 - 1/f_1 - 1/f_2 \end{bmatrix}$$

The distance from the negative lens to the film plane should equal the L at which the height of x_{focus} is zero (i.e., $x_{focus}(1) = 0$):

$$\vec{x}_{focus}(1) = 0 \quad \rightarrow \quad L \cdot \left(\frac{d}{f_1 f_2} - \frac{1}{f_1} - \frac{1}{f_2} \right) = \frac{d}{f_1} - 1$$

$$L = \frac{\frac{d}{f_1} - 1}{\left(\frac{d}{f_1 f_2} - \frac{1}{f_1} - \frac{1}{f_2} \right)}$$

$$L = \frac{f_2 (d - f_1)}{d - (f_1 + f_2)}$$

Plugging in f_1 = 20 cm, f_2 = -8 cm, and d = 15 cm:

$$L = \frac{-8 \cdot (15 - 20)}{15 - (20 - 8)} = 13.33 \text{ cm}$$

If a distant object subtends 2 degrees at the camera, then that means:

$$\frac{h_o}{s_o} = \tan(2^\circ)$$

The magnification formula says:

$$m = \frac{h_i}{h_o} = -\frac{s_i}{s_o} \rightarrow h_o = -\frac{h_i}{s_i} s_o$$

Plugging in $s_o = f$ = 53.33 cm (since the film plane is at the focus of the system):

$$h_o = -\tan(2^\circ) \cdot 53.33 = -1.86$$

Thus, the image of a 2 degree distant object is 1.86 cm tall.

2.71 / 2.710 Optics
Spring 2014